PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Plan for Lecture 7:

Continue reading Chapters 2 & 3

1. **Methods of images -- planes, spheres**
2. **Solution of Poisson equation in for other geometries -- cylindrical**

Physics Colloquium

Thursday

January 26, 2023

Manufacturing Opportunities for Our Energy Security

The U.S. Department of Energy has a rich history of funding key battery innovations to lower the cost of rechargeable batteries to less than $60/kWh and increasing their energy density to more than 350 Wh/kg, being self-reliant on domestic critical materials, enhancing the nation's electrification system and achieving higher degrees of decarbonization in the next decade. These investments continue to be crucial for positioning the United States as the hub for the most cumulative battery R&D programs worldwide. However, major breakthroughs in establishing a domestic supply chain for lithium-based batteries are the linchpin to enable an electrified future evolving to predominantly incorporate rechargeable lithium-ion batteries. This will require a strong battery manufacturing base that will not only enable cutting-edge advancements in materials and manufacturing sciences but will also allow the U.S. to become the world leader in energy storage.

Ilias Belharouak, Ph.D.

Corporate Fellow,

Oak Ridge National Laboratory

4:00 pm - Olin 101*

Note: For additional information on the seminar, contact wfuphys@wfu.edu

Reception at 3:30pm - Olin Entrance

Course schedule for Spring 2023

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/9/2023	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/13/2023
2	Wed: 01/11/2023	Chap. 1	Electrostatic energy calculations	#2	01/18/2023
3	Fri: 01/13/2023	Chap. 1 & 3	Electrostatic energy calculations thanks to Ewald	#3	01/18/2023
	Mon: 01/16/2023		MLK Holiday -- no class		
4	Wed: 01/18/2023	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/20/2023
5	Fri: 01/20/2023	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/23/2023
6	Mon: 01/23/2023	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/25/2023
7	Wed: 01/25/2023	Chap. 2 & 3	Image charge constructions	#7	01/30/2023
8	Fri: 01/27/2023	Chap. 2 & 3	Cylindrical and spherical geometries		

PHY 712 -- Assignment #7

January 25, 2023

Continue reading Chap. 2 in **Jackson**.

1. Eq. 2.5 was derived as the surface change density on a sphere of radius a due to a charge q placed at a radius $y > a$ outside the sphere. Determine the total surface charge on the sphere surface.
2. Now consider the same system except assume $y < a$ representing the charge q being placed inside the sphere. What is the surface charge density and the total surface charge in this case?

Your questions –

From Paul -- Could you explain the + and - signs for the cylindrical coordinates?

From Zezhong -- So generally speaking this method of images utilizes some of the symmetry of certain shapes, so they are certain symmetrical cases that can be easier calculated?

From Banasree -- So method of images also depends on geometry? And Greens function or method of images, are they only applicable for drichlet boundary condition?

Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

1. Direct solution of differential equation
2. Solution by means of an integral equation; Green's function techniques
3. Orthogonal function expansions
4. Numerical methods (finite differences and finite element methods)
5. Method of images ← **today**

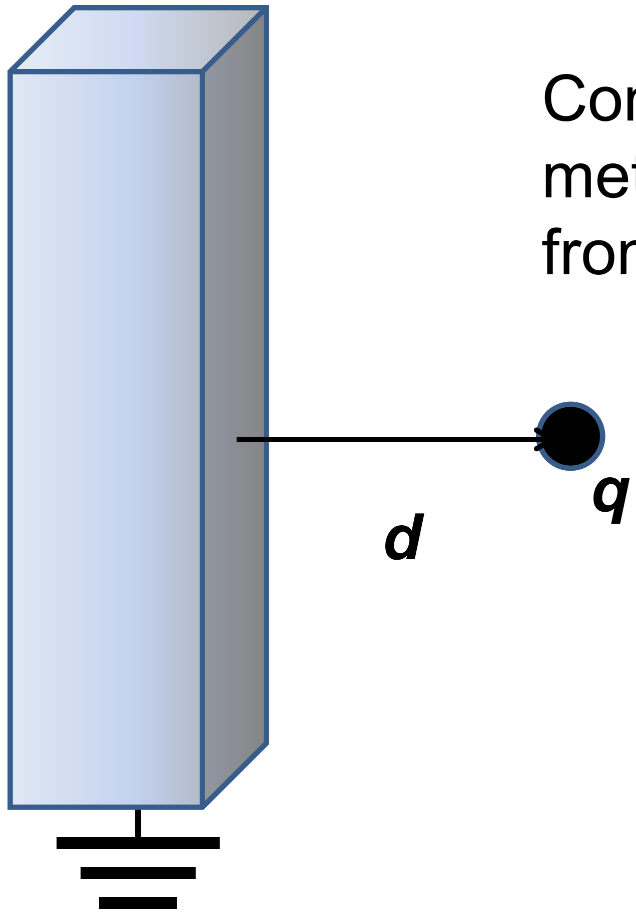
Depends on geometry; Cartesian, spherical, and cylindrical cases considered in textbook

Method of images

Clever trick for specialized geometries:

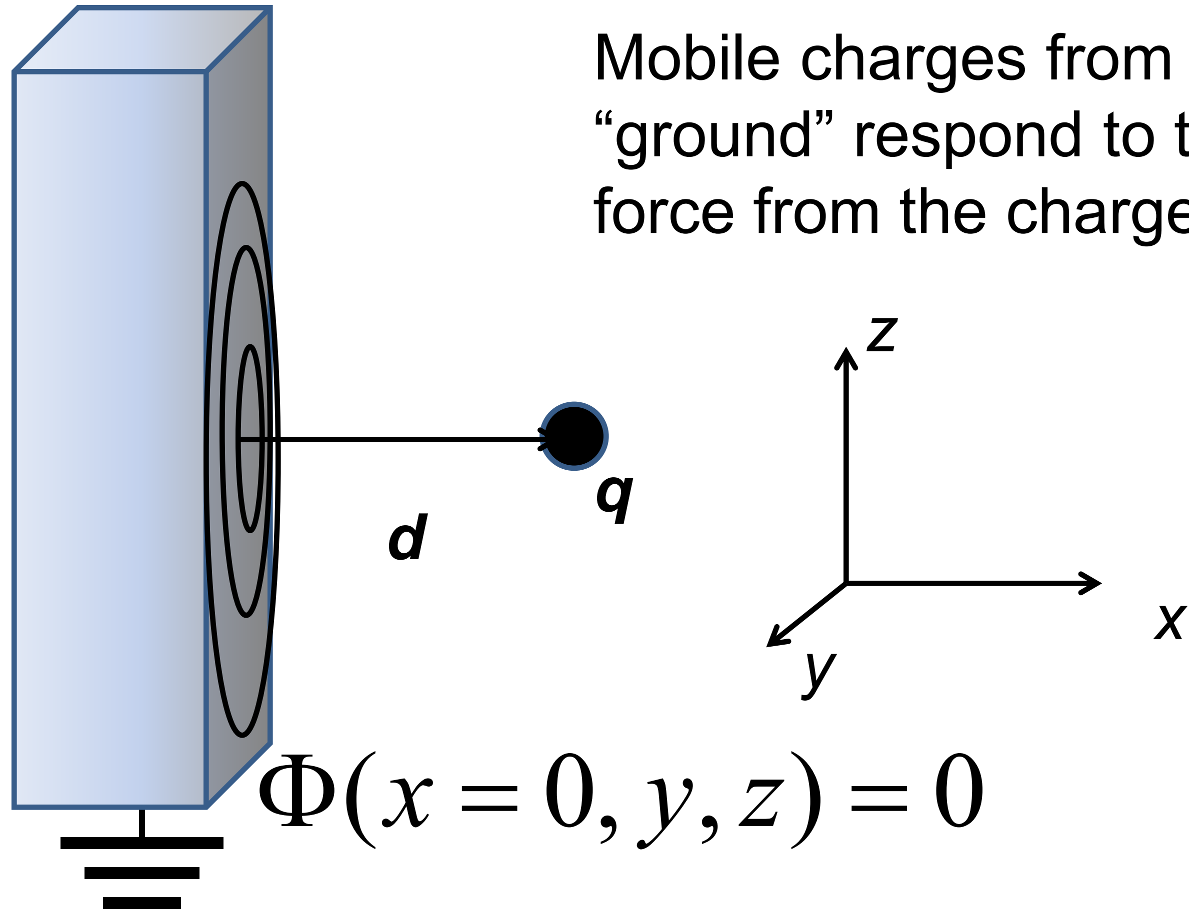
- Flat plane (surface)
- Sphere

Planar case:



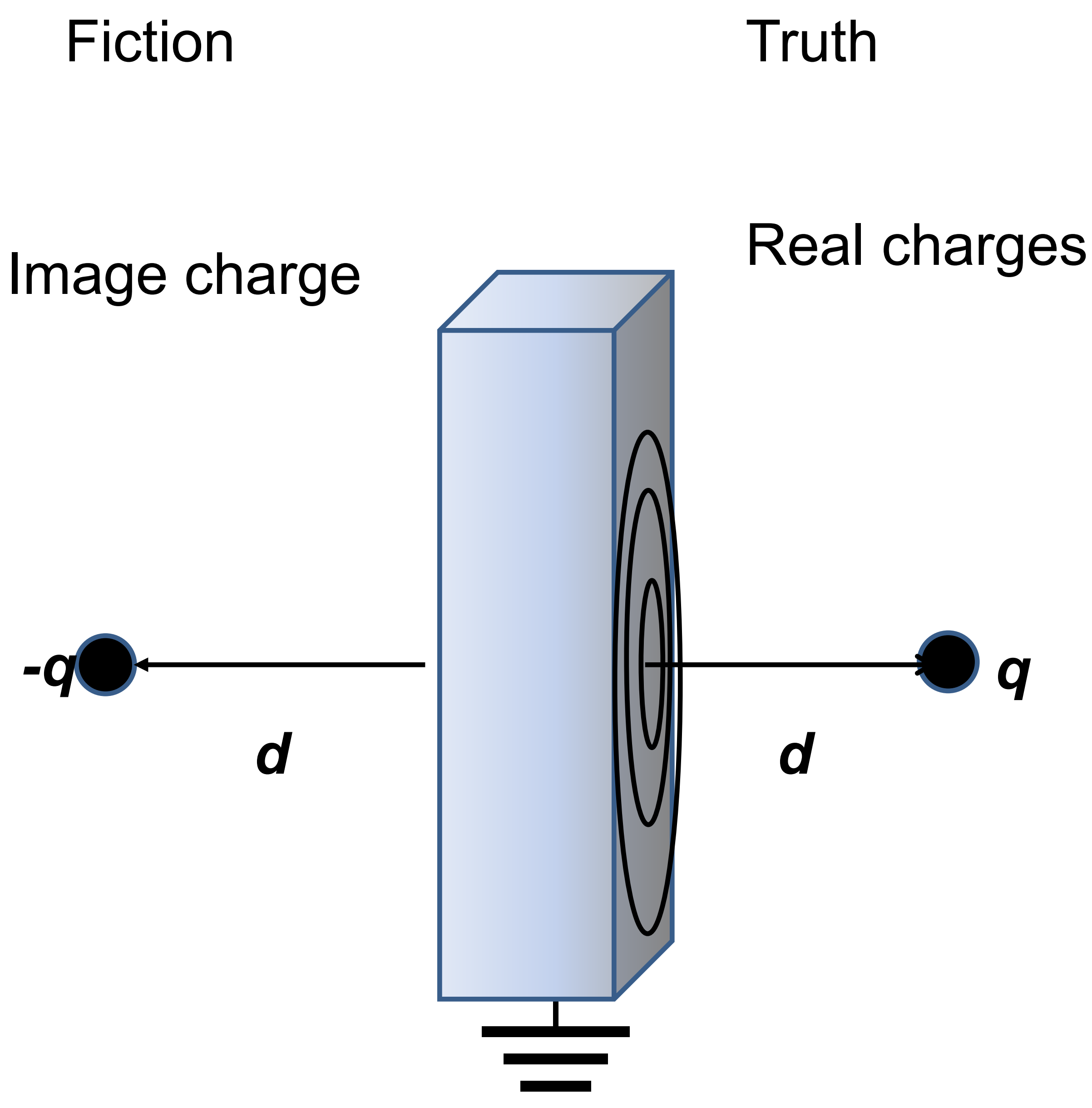
Consider a grounded metal sheet, a distance ***d*** from a point charge ***q***.

A grounded metal sheet, a distance ***d*** from a point charge ***q***.



Mobile charges from the "ground" respond to the force from the charge ***q***.

$$\Phi(x=0, y, z) = 0$$


Fiction
Truth
Real charges
Image charge
-q
q
d
d

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

$$\nabla^2\Phi = -\frac{q}{\varepsilon_0}\delta^3\left(\mathbf{r} - d\hat{\mathbf{x}}\right)$$

$$\Phi\left(x = 0, y, z\right) = 0$$

Trick for $x \geq 0$:

$$\Phi\left(x \geq 0, y, z\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\left|\mathbf{r} - d\hat{\mathbf{x}}\right|} - \frac{q}{\left|\mathbf{r} + d\hat{\mathbf{x}}\right|}\right)$$

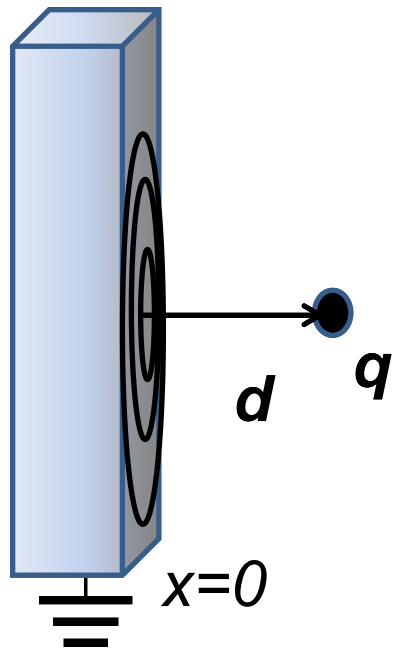
Surface charge density:

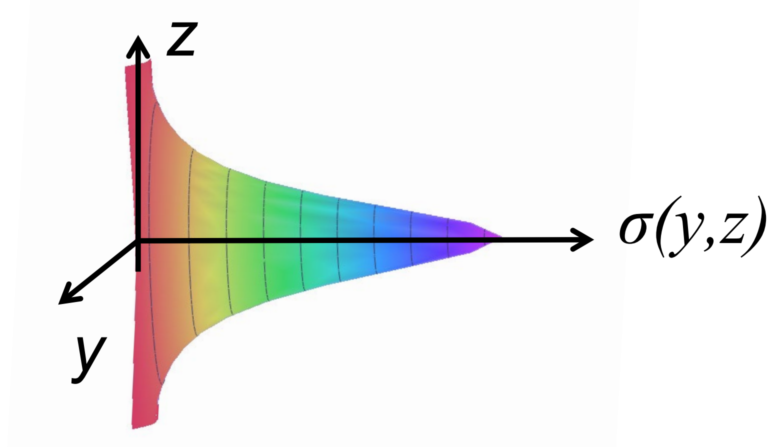
$$\sigma(y,z) = \varepsilon_0 E\left(0, y, z\right) = -\varepsilon_0 \frac{d\Phi\left(0, y, z\right)}{dx} = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2 + y^2 + z^2\right)^{3/2}}\right)$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

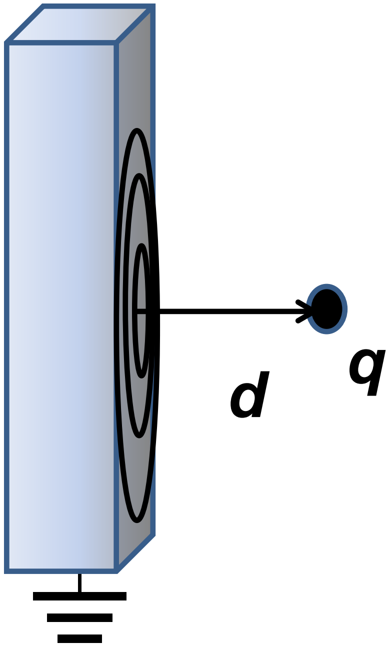
$$\text{Surface charge density:} \quad \sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

$$\text{Note:} \iint dydz \;\; \sigma(y,z) = -\frac{q2d}{4\pi}2\pi\int_0^\infty \frac{udu}{\left(d^2+u^2\right)^{3/2}} = -q$$





A grounded metal sheet, a distance ***d*** from a point charge ***q***.



Surface charge density :

$$\sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

Force between charge and sheet :

$$\mathbf{F} = \frac{-q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0\left(2d\right)^2}$$

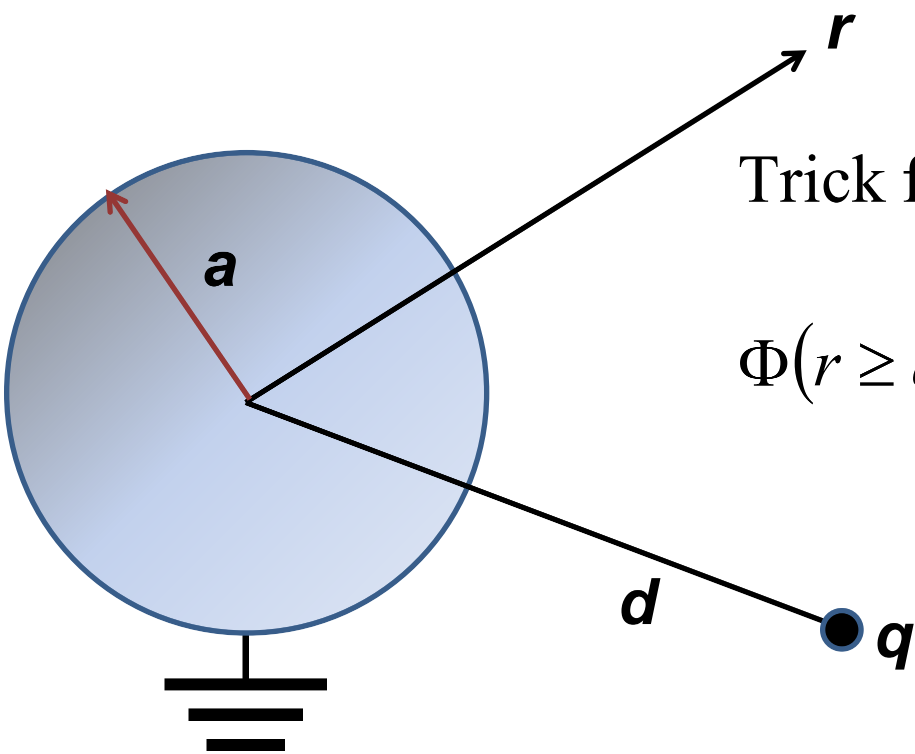
Image potential between charge and sheet at distance x :

$$V(x) = \frac{-q^2}{4\pi\varepsilon_0\left(4x\right)}$$

Note: this effect can be observed in photoemission experiments.

Image charge methods can be used in some other geometries --

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Trick for $r \geq a$:

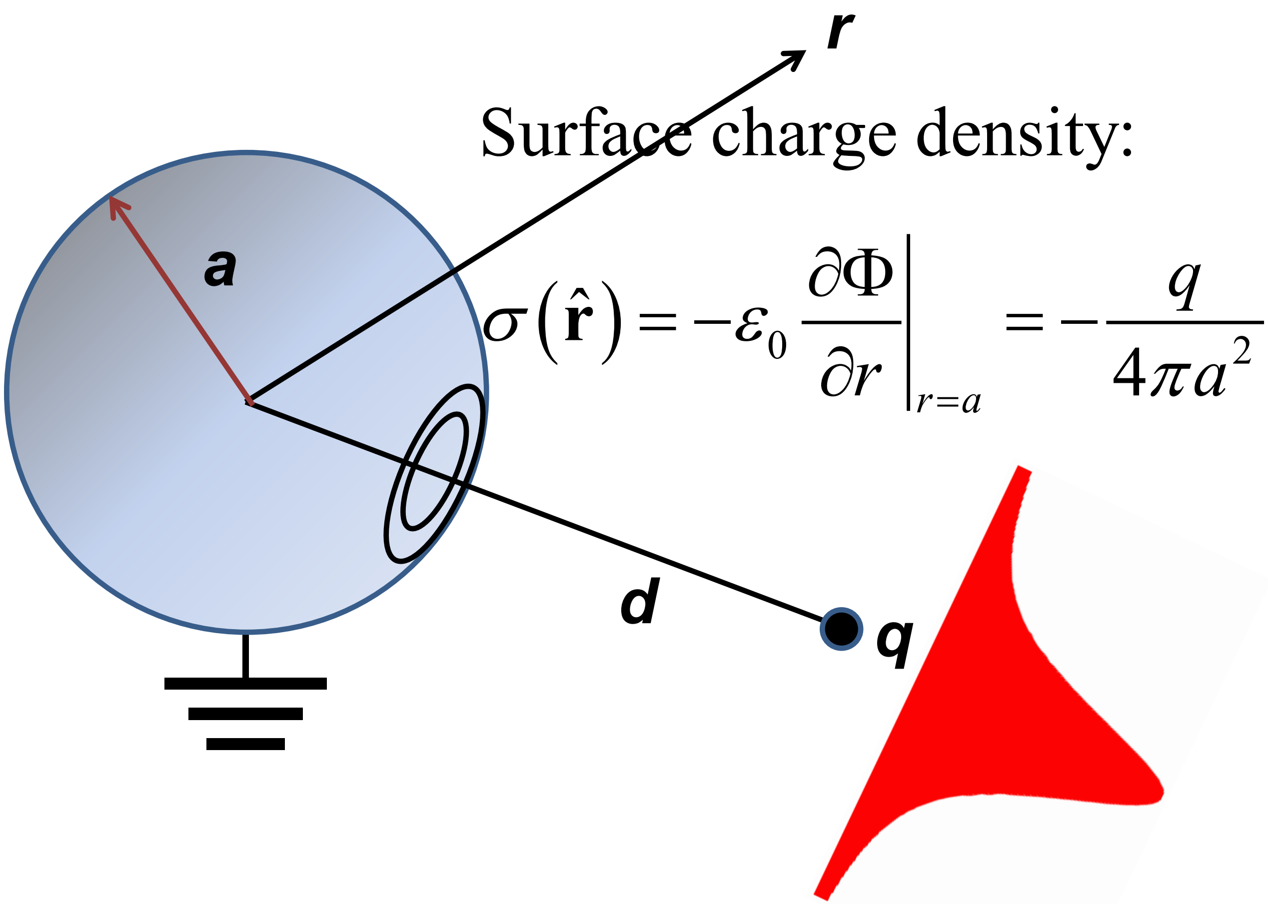
$$\Phi(r \geq a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{\frac{d}{a}\left|\mathbf{r}-\mathbf{d}\frac{a^2}{d^2}\right|}\right)$$

Interpreted as

Image charge of $q' = -q\dfrac{a}{d}$

Located along $\hat{\mathbf{d}}$ at $\hat{\mathbf{d}}a\dfrac{a}{d}$

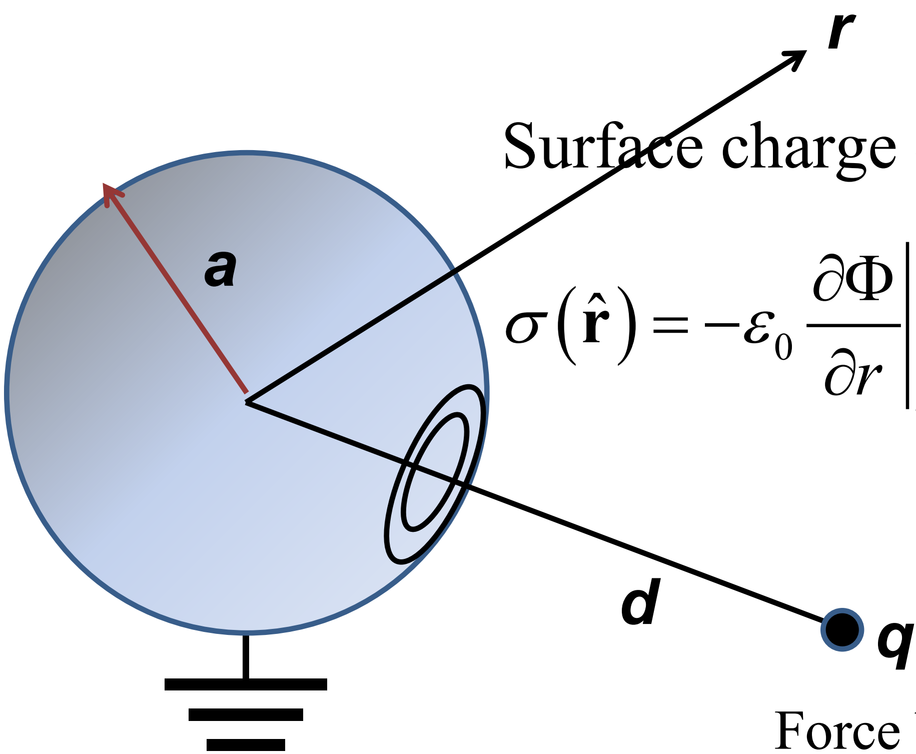
A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.

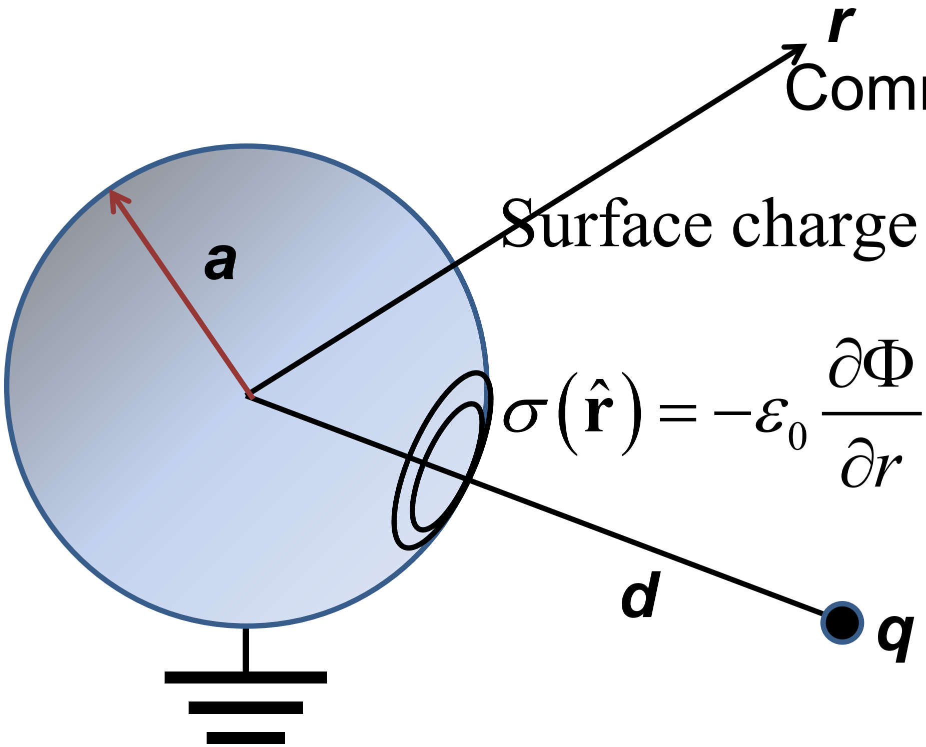


Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

Force between q and sphere

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2(a/d)}{\left(d-a^2/d\right)^2} = \frac{q^2}{4\pi\epsilon_0}\frac{ad}{\left(d^2-a^2\right)^2}$$



Comment on HW problem #7

Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

For #1, integrate the charge induced on the outer surface of the sphere due to the point charge q at the point $d > a$.

$$\int \sigma(\hat{\mathbf{r}})dS = -\int \frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}dS = -\frac{q}{4\pi a^2}\frac{a}{d}\left(1-\frac{a^2}{d^2}\right)2\pi a^2 \int \frac{d\cos\theta}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}}$$

For #2, the point charge q is located at a point $d < a$ and a similar analysis follows. Integrate the charge induced on the inner surface of the sphere.

(Answer to #2 should be different from that of #1.)

Use of image charge formalism to construct Green's function

Example:

Suppose we have a Dirichlet boundary value problem on a sphere of radius a:

$$\nabla^2 \Phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \Phi(r = a) = 0$$

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

$$\Rightarrow \text{For} \quad r, r' > a: \qquad G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{\frac{r'}{a}\left|\mathbf{r} - \frac{a^2}{r'^2}\mathbf{r}'\right|}$$

Analysis of Poisson/Laplace equation in various regular geometries

1. Rectangular geometries ➔ previous lectures
2. Cylindrical geometries ➔ now
3. Spherical geometries ➔ later

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

$$\Phi(\rho,\varphi) = \Phi(\rho,\varphi + m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation

in these coordinates:

$$\Phi(\rho,\varphi) = \mathrm{A}_0 + \mathrm{B}_0\ln(\rho) + \sum_{m=1}^{\infty}\left(A_m\rho^m + B_m\rho^{-m}\right)\sin(m\varphi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

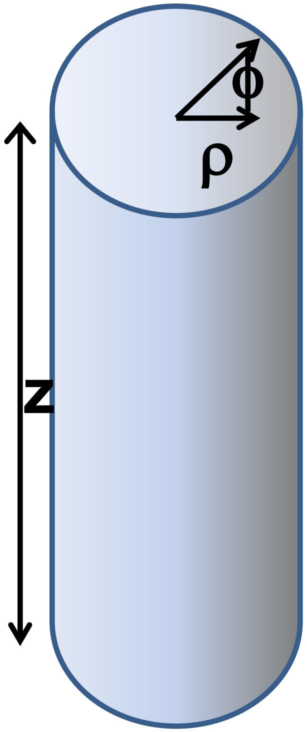
Note that here ρ means radial coordinate

Green's function appropriate for this geometry with boundary conditions at $\rho=0$ and $\rho=\infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>{}^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left(m(\phi-\phi')\right)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi(\rho,\varphi,z)=\Phi(\rho,\varphi+m2\pi,z)$$

$$\Phi(\rho,\varphi,z)=R(\rho)Q(\varphi)Z(z)$$

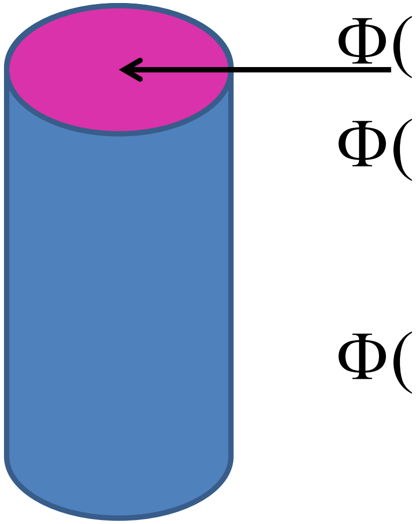
Cylindrical geometry continued:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry example:

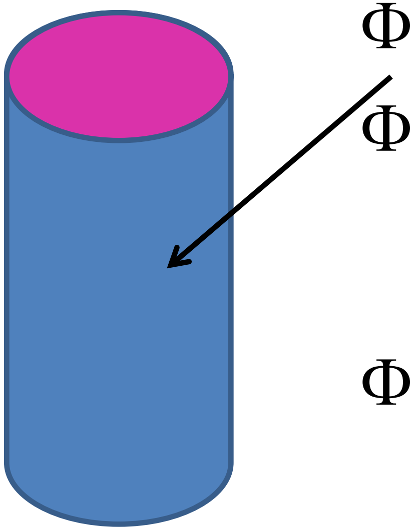


$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\phi+\alpha_{mn}\right)$$

Cylindrical geometry example:



$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

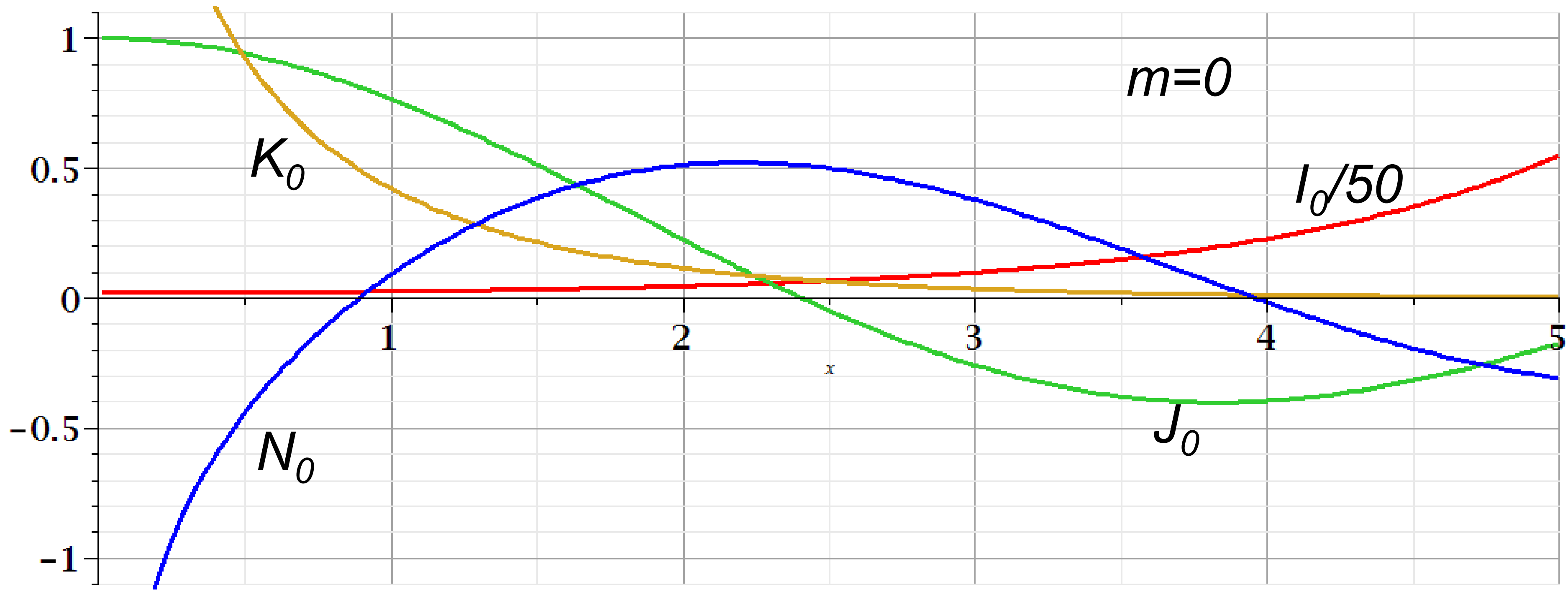
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \sin\left(m\phi + \alpha_{mn}\right)$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$



Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$