PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Class notes for Lecture 8:

Finish reading Chap. 2 and start Chap. 3

Solution of Poisson/Laplace equation for special geometries –

A. Cylindrical

B. Spherical

Course schedule for Spring 2023

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/9/2023	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/13/2023
2	Wed: 01/11/2023	Chap. 1	Electrostatic energy calculations	#2	01/18/2023
3	Fri: 01/13/2023	Chap. 1	Electrostatic energy calculations thanks to Ewald	#3	01/18/2023
	Mon: 01/16/2023		MLK Holiday -- no class		
4	Wed: 01/18/2023	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/20/2023
5	Fri: 01/20/2023	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/23/2023
6	Mon: 01/23/2023	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/25/2023
7	Wed: 01/25/2023	Chap. 2 & 3	Image charge constructions	#7	01/30/2023
8	Fri: 01/27/2023	Chap. 2 & 3	Cylindrical and spherical geometries		
9	Mon: 01/30/2023	Chap. 3 & 4	Spherical geometry and multipole moments		

Solution of the Poisson/Laplace equation in various geometries
➔cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

$$\Phi(\rho,\varphi) = \Phi(\rho,\varphi + m2\pi)$$

Assume: $\Phi(\rho,\phi) = f(\rho)g(\phi)$

Suppose $\frac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$

$g(\phi) = \sin(m\phi + \alpha_m) \qquad \Rightarrow m = \text{integer}$

Solution of the Poisson/Laplace equation in various geometries
➔cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Assume: $\Phi(\rho,\phi) = f(\rho)g(\phi)$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df_m(\rho)}{d\rho}\right) - \frac{m^2}{\rho^2}f_m(\rho) = 0$$

$$f_0(\rho) = \begin{cases} 1 \\ \ln\rho \end{cases} \qquad f_{m>0} = \rho^{\pm m}$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Phi(\rho,\phi)=\Phi(\rho,\phi+m2\pi)$$ ➔ *m*=integer

$\Rightarrow$ General solution of the Laplace equation in these coordinates :

$$\Phi(\rho,\phi)=\mathrm{A}_0+\mathrm{B}_0\ln(\rho)+\sum_{m=1}^{\infty}\left(A_m\rho^m+B_m\rho^{-m}\right)\sin(m\phi+\alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) G(\rho, \rho', \varphi, \varphi') =$$

$$-4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\varphi - \varphi')$$

It can be shown that the following form can be used:

$$G(\rho, \rho', \varphi, \varphi') = -\ln\left(\rho_>{}^2\right) + 2\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>} \right)^m \cos\left(m(\varphi - \varphi')\right)$$

Note that the previous example is similar to the construction for the 2-d cartesian case --

For the 2-d cartesian case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y') \text{ where } \frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

The y dependence of this equation will have the required behavior, if we choose:

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(y-y'),$$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right]v_{n_i}(y) = 0,$$

and the Wronskian constant: $K_n \equiv \frac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$

Cartesian example continued --

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y, y') = -4\pi\delta(y - y'),$$

$$g_n(y, y') = \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>)$$

where: $$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0,$$

and $$K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n}\, y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n}(b - y))$

where $K_n = \sqrt{\alpha_n}\sinh(\sqrt{\alpha_n} b)$

using the identity: $\cosh(r)\sinh(s) + \sinh(r)\cosh(s) = \sinh(r + s)$

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>).$$

In the cylindrical geometry case,

$$u_n(x) \rightarrow \{\sin(m\varphi), \cos(m\varphi)\}$$

$$v_{n_{1,2}} \rightarrow \{1, \ln(\rho), \rho^m, \rho^{-m}\}$$

$$G(\rho, \rho', \varphi, \varphi') = -\ln\left(\rho_>{}^2\right) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos\left(m\left(\varphi - \varphi'\right)\right)$$

Note that, because we are using curvilinear coordinates, the Wronskian and the form of the delta function has to be modified.

Comments and details

Change notation

$\rho \Rightarrow r$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

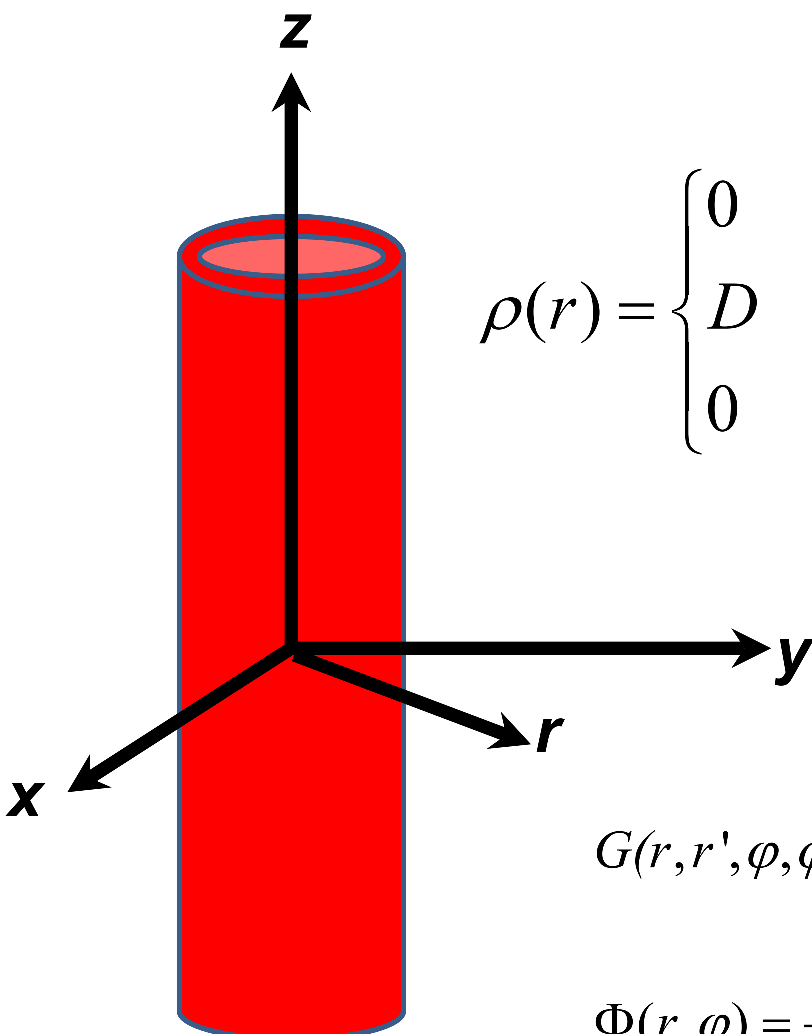
$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that: For this extended charge distribution, Coulomb's law in its original form diverges:

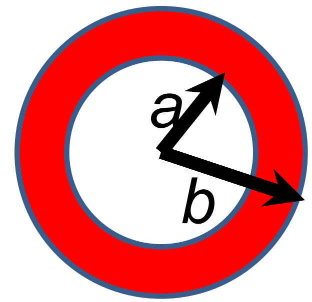
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that in this case, we have assumed that the surface integral contributions are trivial.

Example – uniform cylindrical shell:



Top view:



$$\rho(r) = \begin{cases} 0 & r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \Phi(r,\phi) = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi(r,\phi)}{\partial^2 \phi}$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>{}^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Boundary condition: $\lim_{r\to\infty}\left(\frac{\partial \Phi(r,\varphi)}{\partial r}\right) = 0$

Question – Why only m=0 for this case?

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that $\int_0^{2\pi} d\varphi' \cos(m(\varphi-\varphi')) = 0$ for $m > 0$

Some details

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

In our case: $\Phi(r,\varphi) = \frac{2\pi D}{4\pi\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>^2)\right) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>)\right)$

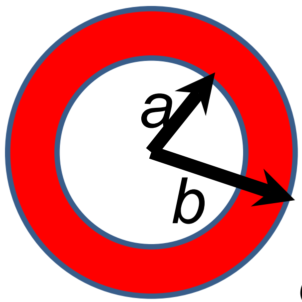
For $0 \le r < a$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r')\right)$

For $a \le r < b$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\left(\int_a^r r'dr'\left(-\ln(r)\right) + \int_r^b r'dr'\left(-\ln(r')\right)\right)$

For $r > b$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r)\right)$

Example continued -- *m=0* only --

Top view:



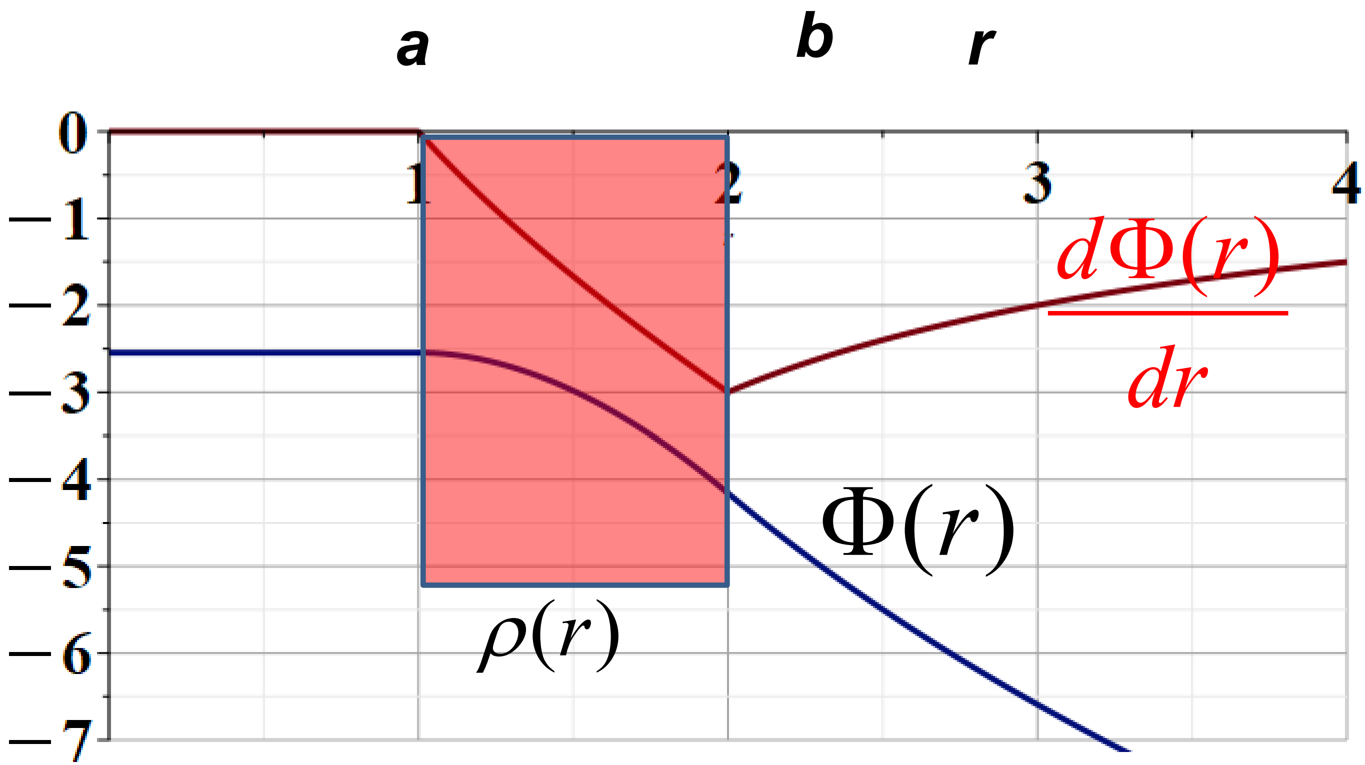
$$\rho(r) = \begin{cases} 0 & 0 < r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi - \varphi'\right)\right)$$

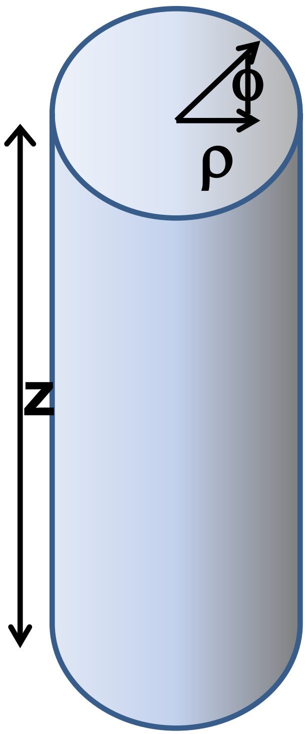
$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

$$\Phi(r) = \begin{cases} \frac{D}{4\epsilon_0}\left(b^2 - a^2 - b^2\ln(b^2) + a^2\ln(a^2)\right) & 0 < r < a \\ \frac{D}{4\epsilon_0}\left(b^2 - r^2 - b^2\ln(b^2) + a^2\ln(r^2)\right) & a \le r \le b \\ \frac{D}{4\epsilon_0}\left(a^2 - b^2\right)\ln(r^2) & r > b \end{cases}$$

Example continued --



Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence

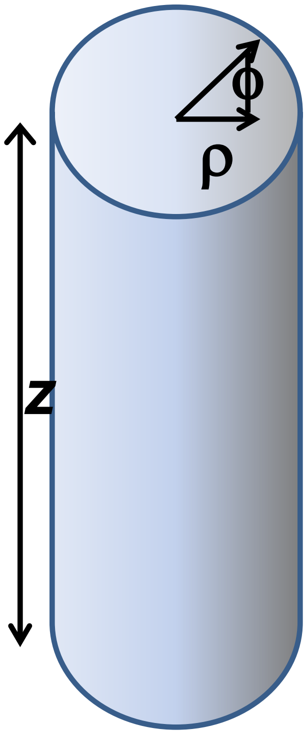


$$\text{Laplace equation:} \qquad \nabla^2 \Phi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho) Q(\phi) Z(z)$$

Cylindrical geometry continued:



$$\text{Laplace equation:} \qquad \nabla^2\Phi = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

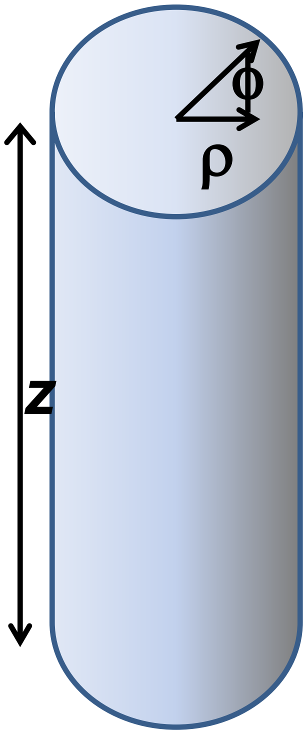
One possibility:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry continued:



$$\text{Laplace equation}: \qquad \nabla^2\Phi = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

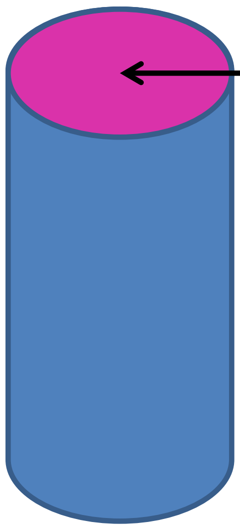
Another possibility:

$$\frac{d^2Z}{dz^2} + k^2 Z = 0 \qquad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow I_m(k\rho), K_m(k\rho)$$

Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*

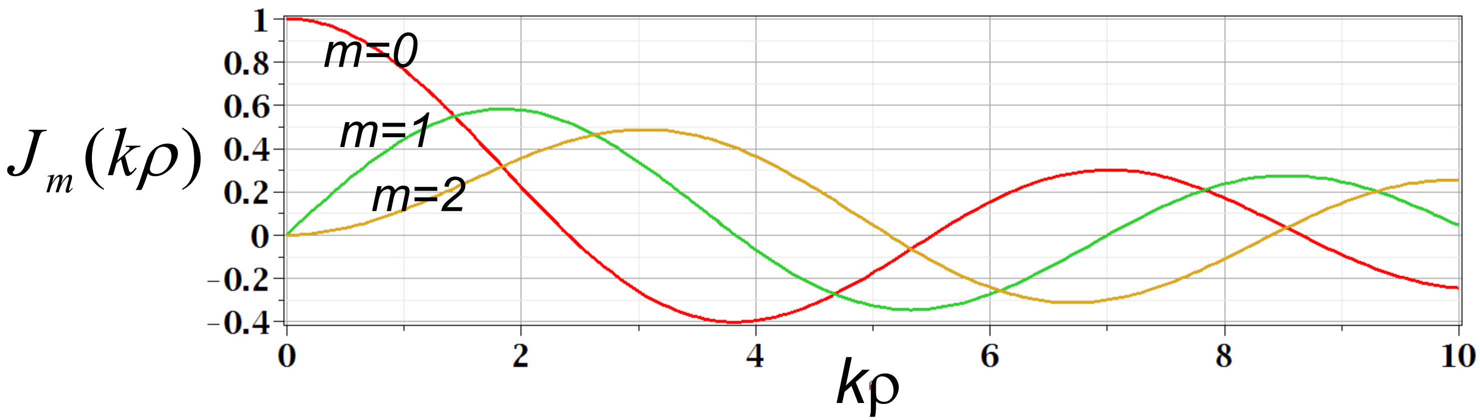


$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

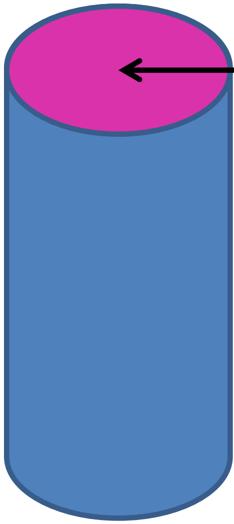
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn}J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\phi+\alpha_{mn}\right)$$

$$\text{where} \quad J_m(k_{mn}a)=0$$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*



$$\Phi(\rho,\varphi,z=L)=V(\rho,\varphi)$$

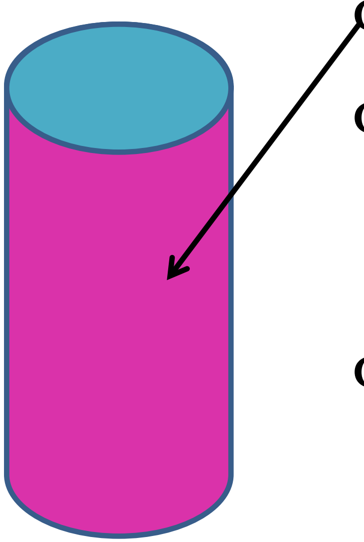
$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\varphi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(\rho,\varphi)$ is an even function of φ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\varphi \cos\left(m\varphi\right)\int_0^a \rho d\rho J_m(k_{mn}\rho)V(\rho,\varphi)}{\sinh(k_{mn}L)\int_0^{2\pi} d\varphi \cos^2\left(m\varphi\right)\int_0^a \rho d\rho J_m{}^2(k_{mn}\rho)}$$

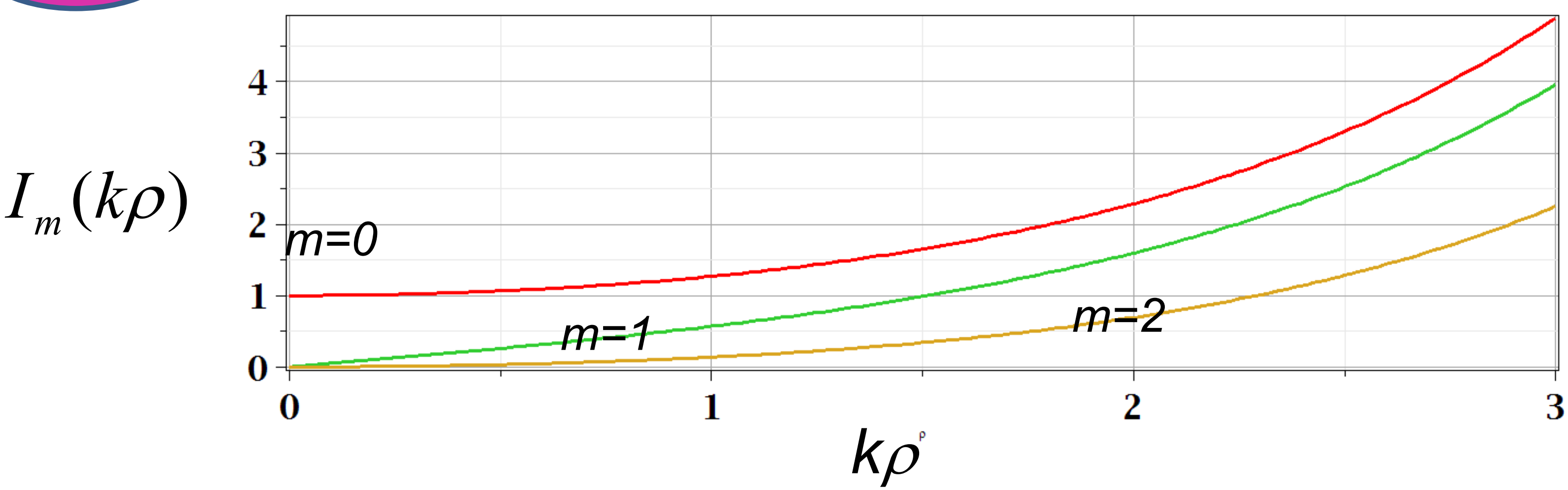
Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at $\rho=a$



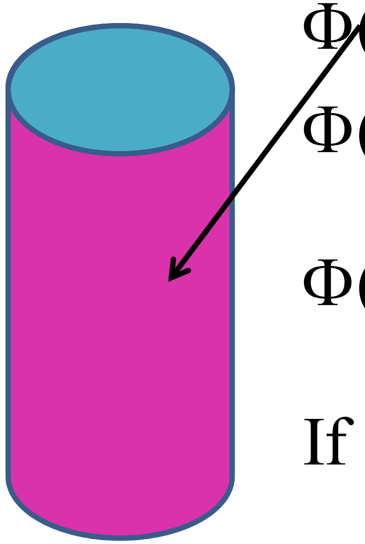
$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi+\alpha_{mn}\right)$$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at ρ=*a*



$$\Phi(\rho = a, \varphi, z) = V(\varphi, z)$$

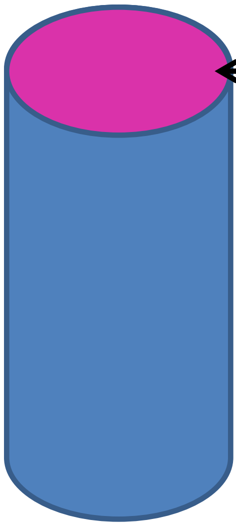
$$\Phi(\rho, \varphi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\varphi + \alpha_{mn}\right)$$

If $V(z,\varphi)$ is an even function of φ so that $\alpha_{mn} = \pi/2$:

$$A_{mn} = \frac{\int_0^{2\pi} d\varphi \cos(m\varphi) \int_0^L dz \sin\left(\frac{n\pi z}{L}\right) V(z,\varphi)}{I_m\left(\frac{n\pi a}{L}\right) \int_0^{2\pi} d\varphi \cos^2(m\varphi) \int_0^L dz \sin^2\left(\frac{n\pi z}{L}\right)}$$

Green's function for Dirchelet boundary value inside cylindar:



$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho=a,\phi,z)=0\,,\;\;\Phi(\rho,\phi,z=0)=0$$

Expansion in terms of Bessel function zeros : $J_m(k_{mn}a)=0$

$$G(\rho,\rho',\phi,\phi',z,z')=$$

$$\frac{8\pi}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}J_m(k_{mn}\rho)J_m(k_{mn}\rho')\sinh(k_{mn}z_<)\sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2\sinh(k_{mn}L)}$$

$$\Phi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_0}\int_V d\phi'\,\rho'\,d\rho'\,dz'\,G(\rho,\rho',\phi,\phi',z,z')\rho(\rho',\phi',z')$$

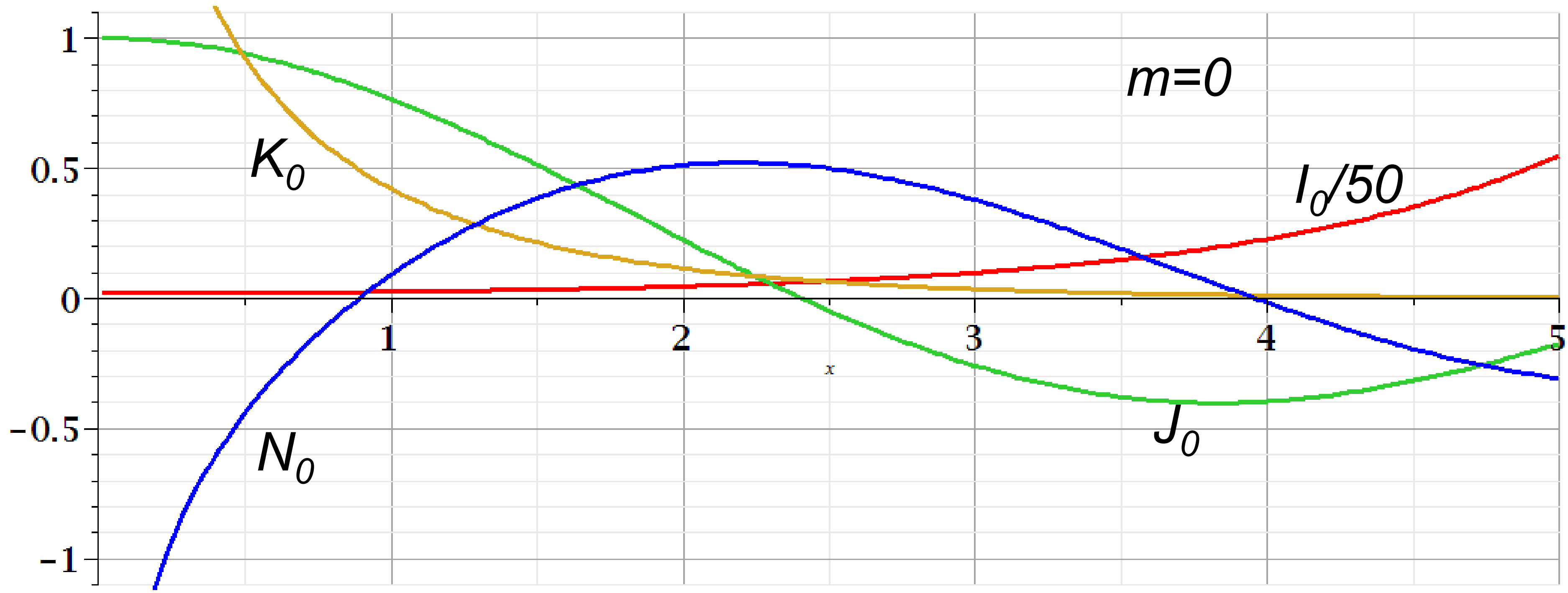
$$+\frac{1}{4\pi}\int_{S;z'=L}d\phi'\,\rho'\,d\rho'\left.\frac{\partial G(\rho,\rho',\phi,\phi',z,z')}{\partial z'}\right|_{z'=L}V(\rho',\phi')$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$

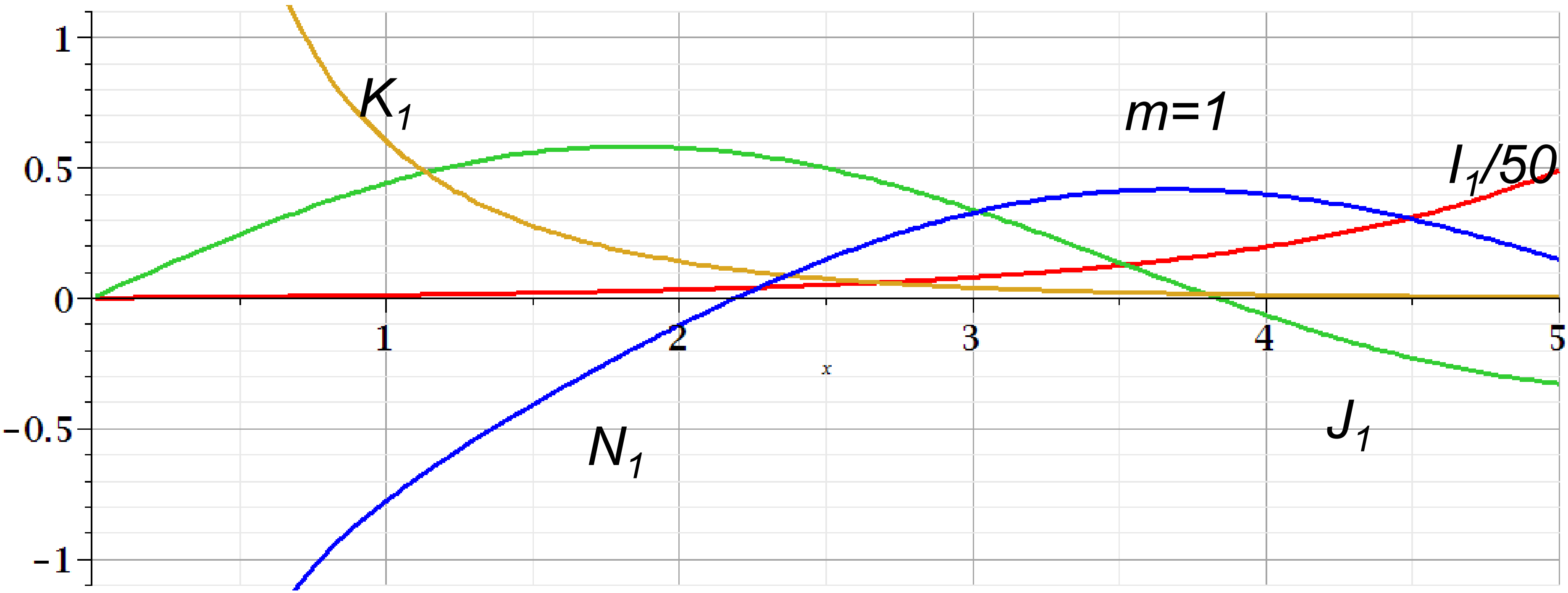


Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$



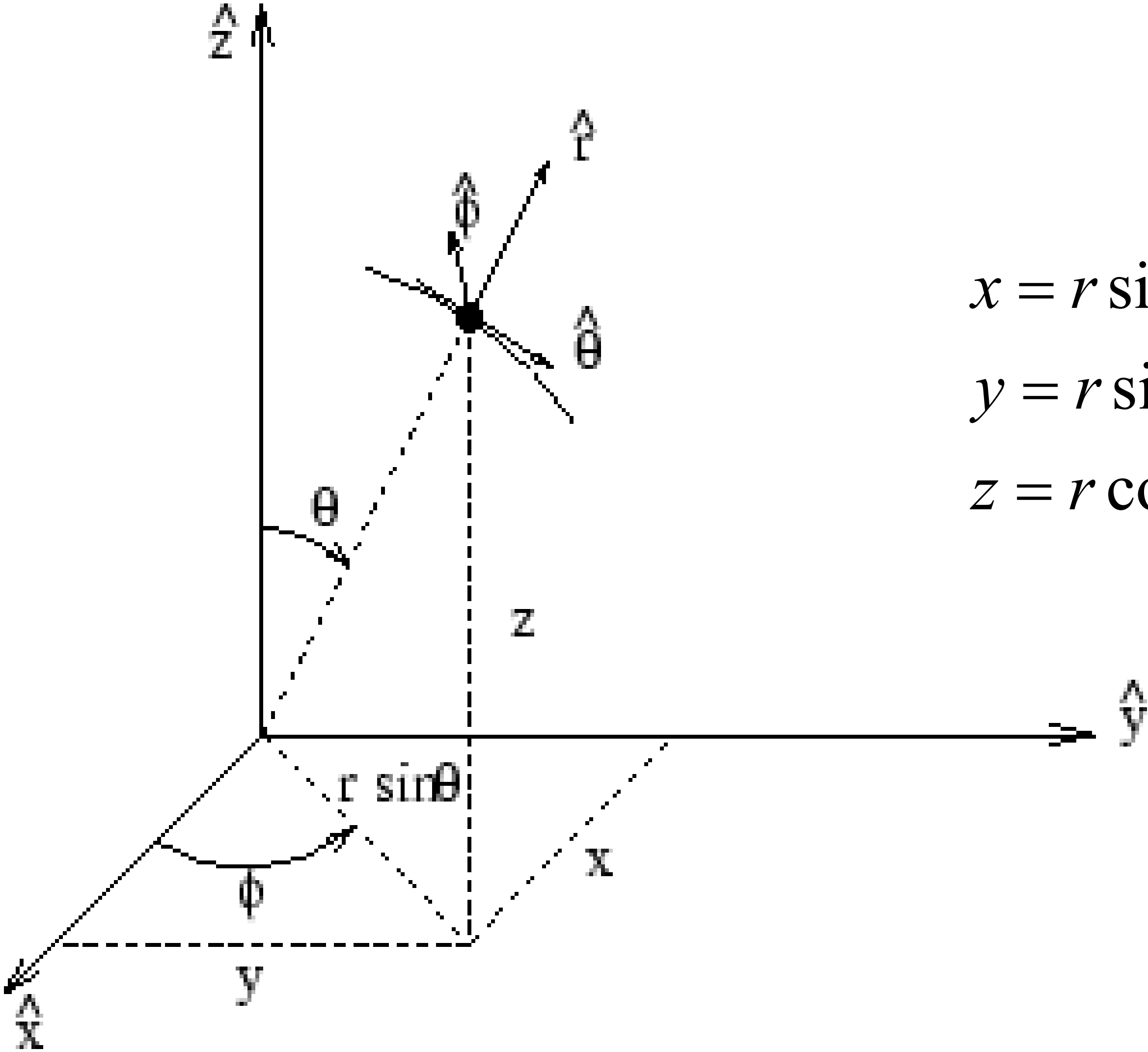
Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(1-\frac{m^2}{u^2}\right)\right)J_m(u)=0 \qquad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn}; $\qquad J_m(x_{mn})=0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho / a) J_m(x_{mn'}\rho / a)=\frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2 \delta_{nn'}$$

Poisson and Laplace equation in spherical polar coordinates



$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html

Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r,\theta,\varphi)$:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\Phi=0$$

$$\Phi(r,\theta,\varphi)=\sum_{lm}R_{lm}(r)Y_{lm}(\theta,\varphi)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\varphi)=-l(l+1)Y_{lm}(\theta,\varphi)$$

Properties of spherical harmonic functions

$$Y_{lm}(\theta,\phi) = (-1)^m Y_{l(-m)}{}^*(\theta,\phi) \qquad \text{(standard Condon - Shortley convention)}$$

$$\int d\Omega Y_{lm}(\theta,\varphi) Y_{l'm'}{}^*(\theta,\varphi) \equiv \int \sin\theta \, d\theta \, d\varphi \, Y_{lm}(\theta,\varphi) Y_{l'm'}{}^*(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

Completeness :

$$\sum_{lm} Y_{lm}(\theta,\varphi) Y_{lm}{}^*(\theta',\varphi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$$

Relationship to Legendre polynomials :

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^{\ *}(\theta',\varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r\rightarrow\infty$:

$$\Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$=\frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^{\ *}(\theta',\varphi')\right)$$

Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^{\;*}(\theta',\varphi')\right)$$

$$\text{Suppose:}\quad \rho(\mathbf{r}') = \frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$$

$$\int d\Omega'\,Y_{lm}^{\;*}(\theta',\varphi') = \sqrt{4\pi}\,\delta_{l0}\delta_{m0}$$

$$\Phi(\mathbf{r}) = \frac{4\pi}{4\pi\varepsilon_0}\int_0^\infty r'^2\,dr'\frac{r_<^{\,0}}{r_>^{\,1}}\frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{\mathrm{erf}(r/a)}{r}$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi')$$

Elements of "proof":

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r_>\left(1+\left(\frac{r_<}{r_>}\right)^2 - 2\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^{1/2}} =$$

$$= \frac{1}{r_>}\left(1+\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' + \left(\frac{r_<}{r_>}\right)^2\left(\frac{3}{2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2 - \frac{1}{2}\right) + \cdots\right)$$

$$= \frac{1}{r_>}\left(\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\right)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{r}})\, Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') = 1$

$$\Rightarrow \frac{4\pi}{2l+1} \sum_{m=-l}^{l} |Y_{lm}(\hat{\mathbf{r}})|^2 = 1$$

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm 1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right)$$