PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Class notes for Lecture 8:

Finish reading Chap. 2 and start Chap. 3

Solution of Poisson/Laplace equation for special geometries –
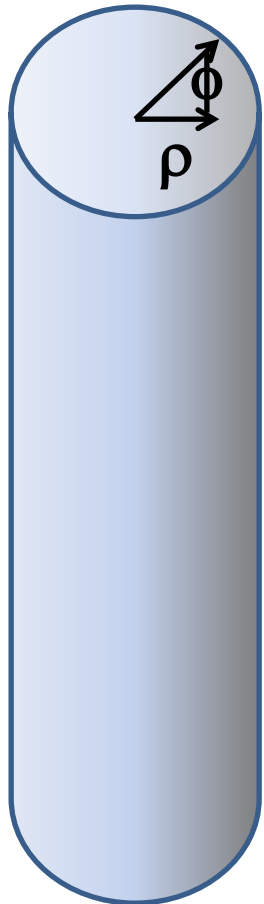
A. Cylindrical

B. Spherical

Course schedule for Spring 2023

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/9/2023	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/13/2023
2	Wed: 01/11/2023	Chap. 1	Electrostatic energy calculations	#2	01/18/2023
3	Fri: 01/13/2023	Chap. 1	Electrostatic energy calculations thanks to Ewald	#3	01/18/2023
	Mon: 01/16/2023		MLK Holiday -- no class		
4	Wed: 01/18/2023	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/20/2023
5	Fri: 01/20/2023	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/23/2023
6	Mon: 01/23/2023	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/25/2023
7	Wed: 01/25/2023	Chap. 2 & 3	Image charge constructions	#7	01/30/2023
8	Fri: 01/27/2023	Chap. 2 & 3	Cylindrical and spherical geometries		
9	Mon: 01/30/2023	Chap. 3 & 4	Spherical geometry and multipole moments		

Solution of the Poisson/Laplace equation in various geometries
➔cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

Laplace equation: $\qquad \nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0$$
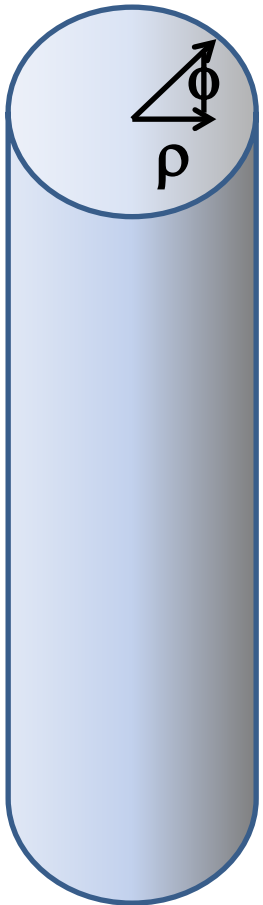
$$\Phi(\rho, \varphi) = \Phi(\rho, \varphi + m2\pi)$$

Assume: $\quad \Phi(\rho, \phi) = f(\rho) g(\phi)$

Suppose $\quad \dfrac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$

$$g(\phi) = \cos(m\phi + \alpha_m) \qquad \Rightarrow m = \text{integer}, \alpha_m = \text{phase}$$

Solution of the Poisson/Laplace equation in various geometries
➔cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

Laplace equation: $\qquad \nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$
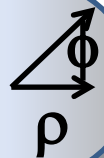
Assume: $\quad \Phi(\rho,\phi) = f(\rho)g(\phi)$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df_m(\rho)}{d\rho}\right) - \frac{m^2}{\rho^2}f_m(\rho) = 0$$

$$f_0(\rho) = \begin{cases} 1 \\ \ln\rho \end{cases} \qquad f_{m>0} = \rho^{\pm m}$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

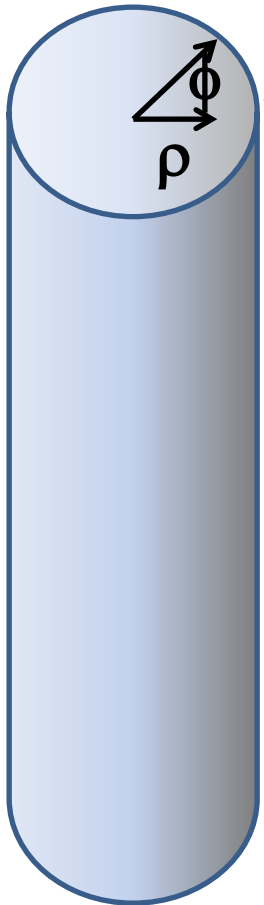Corresponding orthogonal functions from solution of Laplace equation:

$$\nabla^2 \Phi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

$$\Phi(\rho,\varphi) = \Phi(\rho,\varphi + m2\pi) \qquad \rightarrow m=\text{integer}$$

$$\Rightarrow \text{General solution of the Laplace equation}$$

$$\text{in these coordinates:}$$

$$\Phi(\rho,\varphi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty}\left(A_m\rho^m + B_m\rho^{-m}\right)\cos(m\varphi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) G(\rho, \rho', \varphi, \varphi') =$$

$$-4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\varphi - \varphi')$$

It can be shown that the following form can be used:

$$G(\rho, \rho', \varphi, \varphi') = -\ln\left(\rho_>{}^2\right) + 2\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>} \right)^m \cos\left(m(\varphi - \varphi')\right)$$

Note that the previous example is similar to the construction for the 2-d cartesian case --

For the 2-d cartesian case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y') \quad \text{where} \quad \frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

The y dependence of this equation will have the required

behavior, if we choose: $\quad \left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(y-y'),$

which in turn can be expressed in terms of the two independent

solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right]v_{n_i}(y) = 0,$$

and the Wronskian constant: $\quad K_n \equiv \frac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$

Cartesian example continued --

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2} \right] g_n(y, y') = -4\pi \delta(y - y'),$$

$$g_n(y, y') = \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>)$$

where: $\quad \left[\dfrac{d^2}{dy^2} - \alpha_n \right] v_{n_i}(y) = 0,$

and $\quad K_n \equiv \dfrac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \dfrac{dv_{n_2}}{dy}$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n}\, y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n}(b - y))$

where $\quad K_n = \sqrt{\alpha_n} \sinh(\sqrt{\alpha_n} b)$

using the identity: $\cosh(r)\sinh(s) + \sinh(r)\cosh(s) = \sinh(r + s)$

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>).$$

In the cylindrical geometry case,

$$u_n(x) \rightarrow \left\{ \sin(m\varphi), \cos(m\varphi) \right\}$$

$$v_{n_{1,2}} \rightarrow \left\{ 1, \ln(\rho), \rho^m, \rho^{-m} \right\}$$

$$G(\rho, \rho', \varphi, \varphi') = -\ln\left(\rho_>^2\right) + 2\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>} \right)^m \cos\left(m(\varphi - \varphi') \right)$$
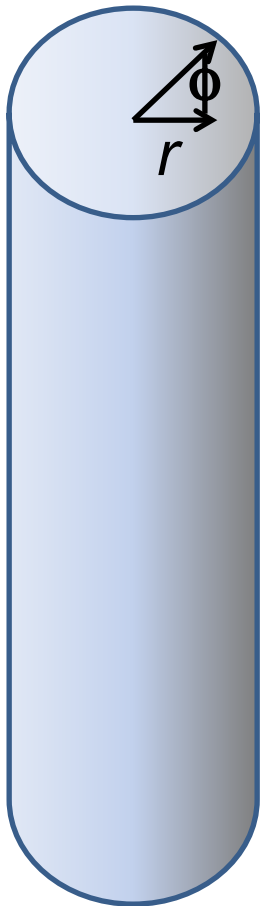
Note that, because we are using curvilinear coordinates, the Wronskian and the form of the delta function is modified.

More details given in **Jackson** Sec. 3.7 - 3.11.

Comments and details

Change notation

$$\rho \Rightarrow r$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi - \varphi')\right)$$
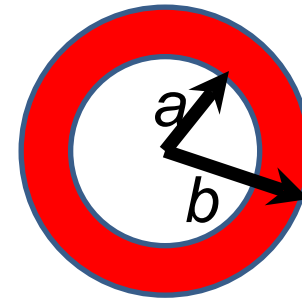
$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that in this case, we have assumed that the surface integral contributions are trivial.

Example – uniform cylindrical shell:
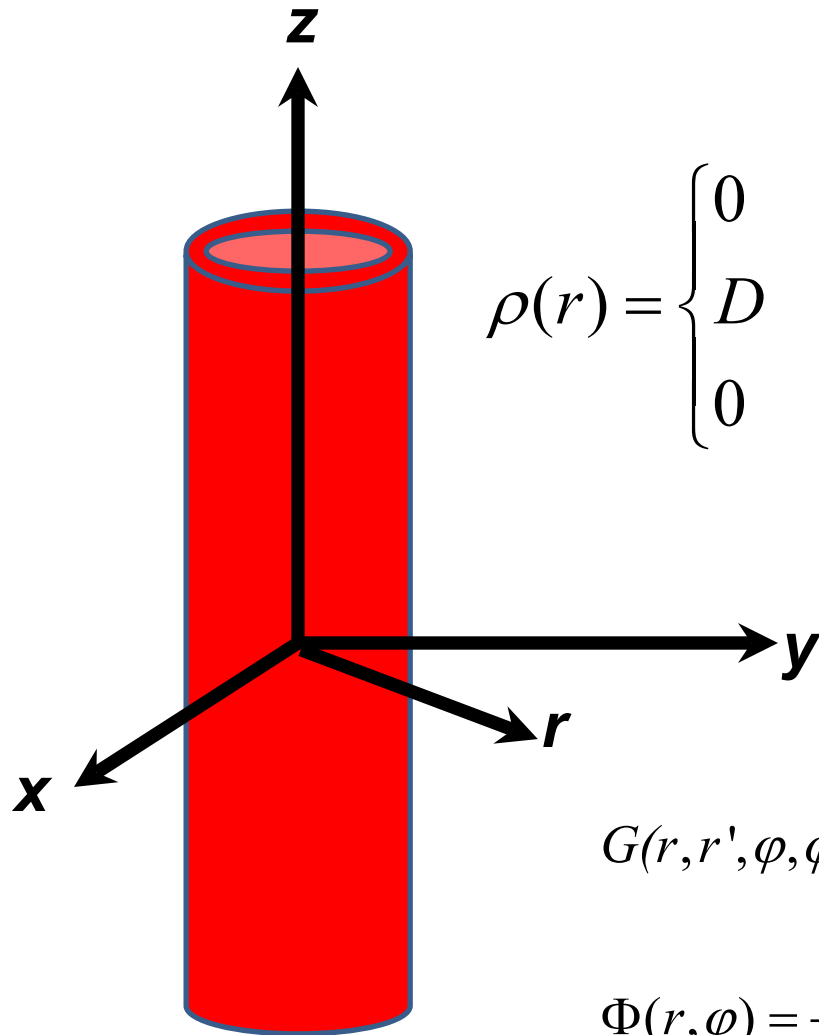
Top view:

$$\rho(r) = \begin{cases} 0 & r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \Phi(r,\phi) = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi(r,\phi)}{\partial^2 \phi}$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi - \varphi')\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi'\int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Boundary condition: $\quad \lim_{r\to\infty}\left(\frac{\partial \Phi(r,\varphi)}{\partial r}\right) = 0$

Question – Why only m=0 for this case?

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi-\varphi')\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}d\varphi'\int_0^{\infty}r'dr'\,G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that $\displaystyle\int_0^{2\pi}d\varphi'\cos(m(\varphi-\varphi')) = 0$ for $m > 0$

So that $\displaystyle\Phi(r,\varphi) = \frac{2\pi}{4\pi\epsilon_0}\int_0^{\infty}r'dr'\left(-\ln\left(r_>^{\,2}\right)\right)\rho(r')$

Some details

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi - \varphi')\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}d\varphi'\int_0^{\infty}r'dr'G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

In our case: $\quad \Phi(r,\varphi) = \dfrac{2\pi D}{4\pi\epsilon_0}\displaystyle\int_a^b r'dr'\left(-\ln(r_>^2)\right) = \dfrac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>)\right)$
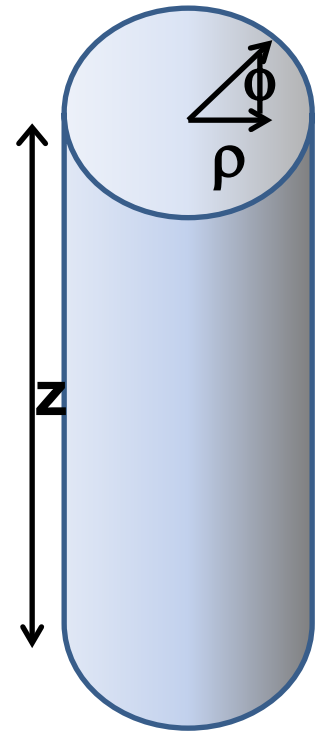
For $0 \le r < a:\quad \Phi(r,\varphi) = \dfrac{D}{\epsilon_0}\displaystyle\int_a^b r'dr'\left(-\ln(r')\right)$

For $a \le r < b:\quad \Phi(r,\varphi) = \dfrac{D}{\epsilon_0}\left(\displaystyle\int_a^r r'dr'\left(-\ln(r)\right) + \int_r^b r'dr'\left(-\ln(r')\right)\right)$

For $r > b:\quad \Phi(r,\varphi) = \dfrac{D}{\epsilon_0}\displaystyle\int_a^b r'dr'\left(-\ln(r)\right)$

Example continued -- *m=0* only --

Top view:
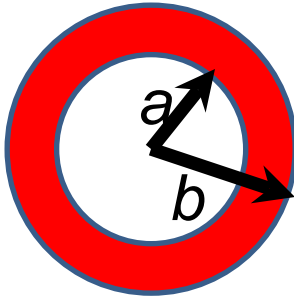
$$\rho(r) = \begin{cases} 0 & 0 < r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m(\varphi-\varphi')\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi'\int_0^{\infty} r'dr'G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

$$\Phi(r) = \begin{cases} \dfrac{D}{4\epsilon_0}\left(b^2 - a^2 - b^2\ln(b^2) + a^2\ln(a^2)\right) & 0 < r < a \\[2em] \dfrac{D}{4\epsilon_0}\left(b^2 - r^2 - b^2\ln(b^2) + a^2\ln(r^2)\right) & a \le r \le b \\[2em] \dfrac{D}{4\epsilon_0}\left(a^2 - b^2\right)\ln(r^2) & r > b \end{cases}$$

Example continued --
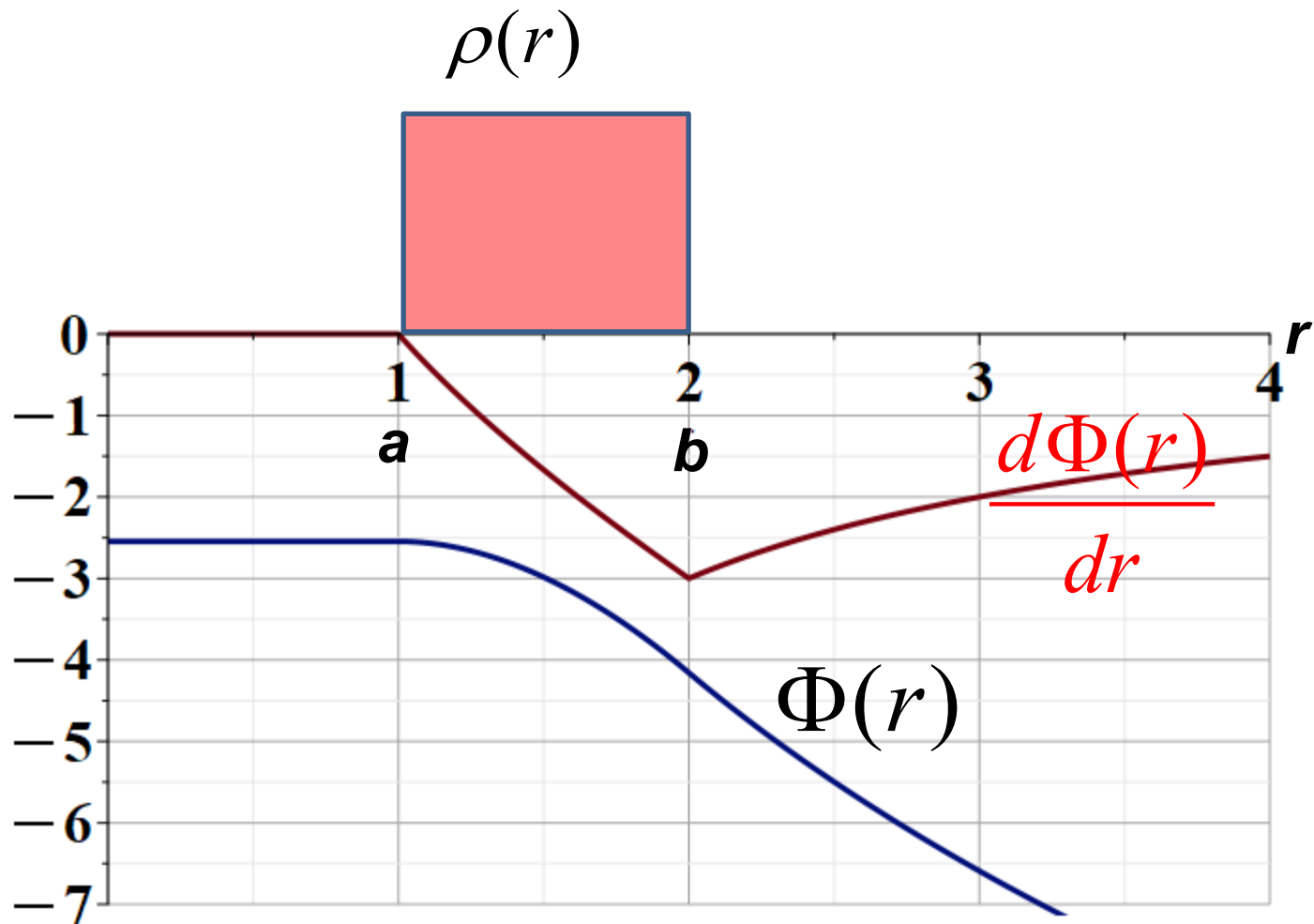
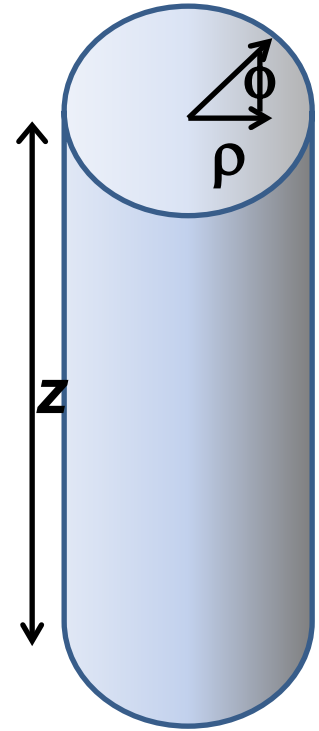# Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence

Laplace equation : $\nabla^2 \Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

Cylindrical geometry continued:

Laplace equation : $\qquad \nabla^2 \Phi = 0$

$$\Phi(\rho, \phi, z) = R(\rho) Q(\phi) Z(z)$$

One possibility :

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2} \right) R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

https://dlmf.nist.gov/
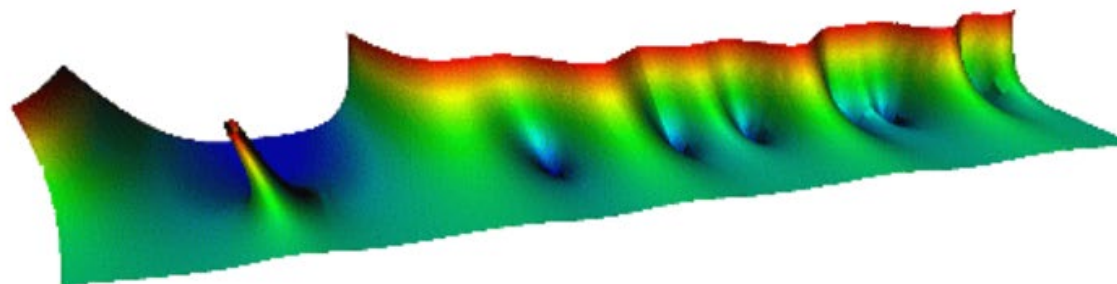
§10.3 Graphics

Contents

§10.3(i) Real Order and Variable

For the modulus and phase functions $M_\nu(x)$, $\theta_\nu(x)$, $N_\nu(x)$, and $\phi_\nu(x)$ see §10.18.
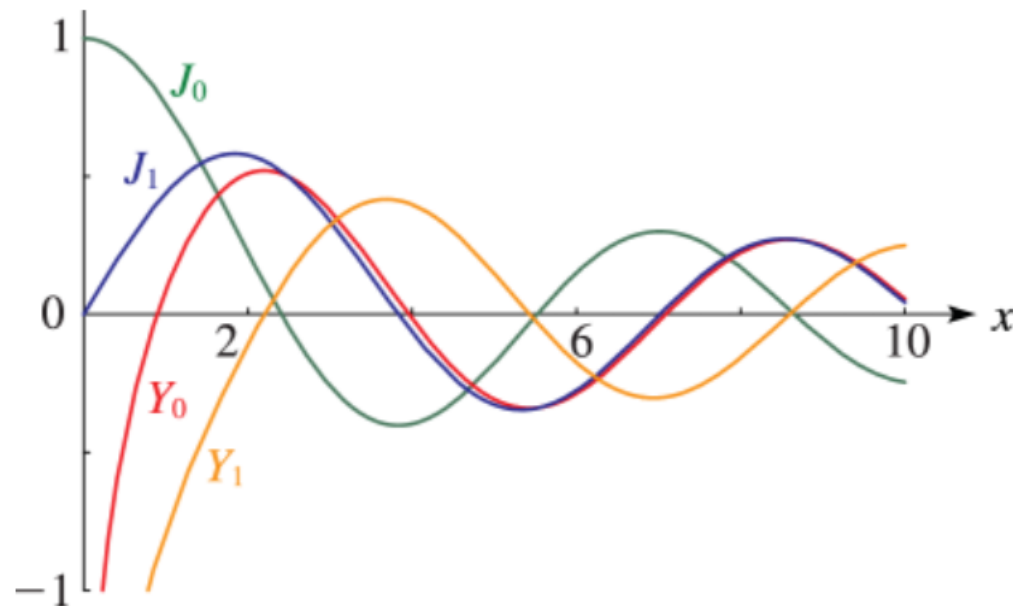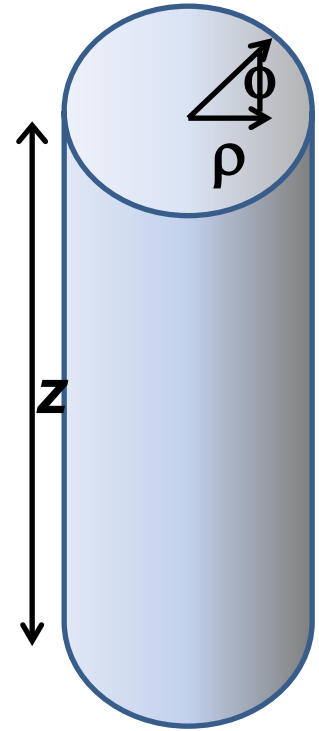

Figure 10.3.1: $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$,

Cylindrical geometry continued:

Laplace equation: $\nabla^2 \Phi = 0$

$$\Phi(\rho, \varphi, z) = R(\rho) Q(\varphi) Z(z)$$
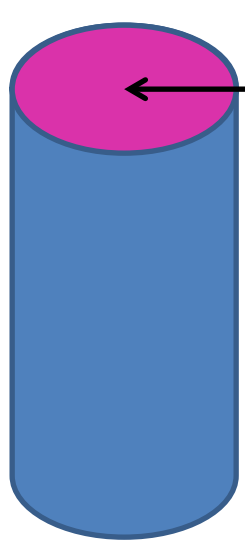
Another possibility:

$$\frac{d^2 Z}{dz^2} + k^2 Z = 0 \qquad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right) R = 0 \qquad \Rightarrow I_m(k\rho), K_m(k\rho)$$

Solutions of Laplace equation inside cylindrical shape
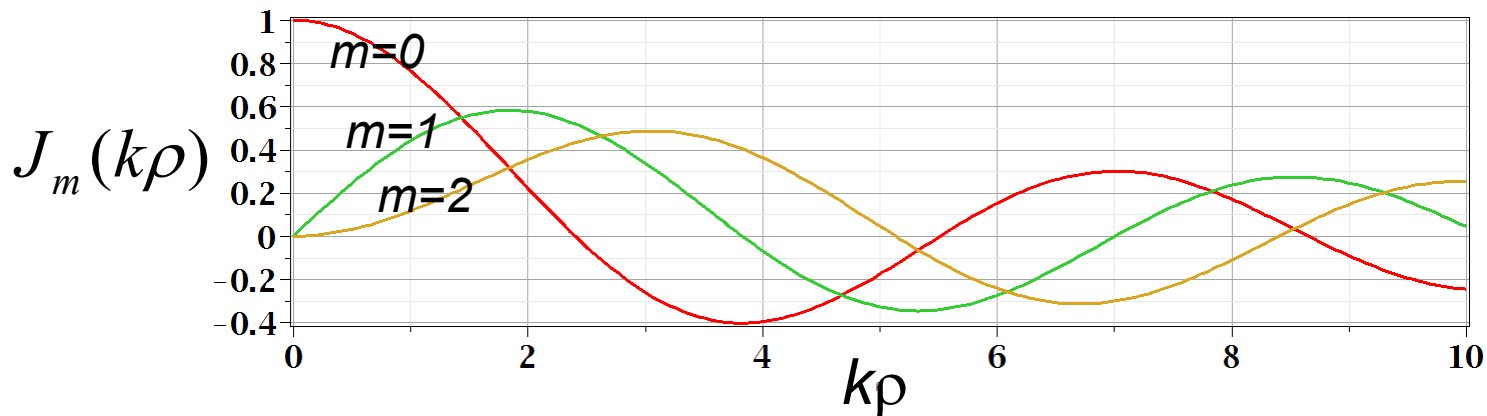Example with non-trivial boundary value at *z=L*

$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

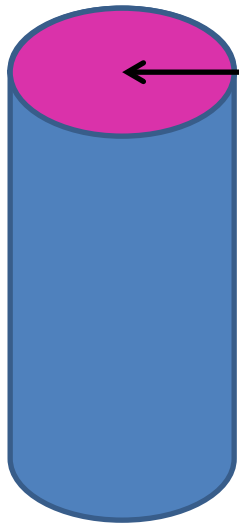$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

Well behaved at *ρ=0*

$$\Phi(\rho,\phi,z)=\sum_{n,m}A_{mn}J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin(m\phi+\alpha_{mn})$$

where $J_m(k_{mn}a)=0$

$J_m(k\rho)$

m=0

m=1

m=2

$k\rho$

Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*

$$\Phi(\rho,\varphi,z=L) = V(\rho,\varphi)$$

$$\Phi(\rho,\varphi,z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\varphi,z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\varphi + \alpha_{mn}\right)$$

If $V(\rho,\varphi)$ is an even function of φ so that $\alpha_{mn} = \pi/2$:

$$A_{mn} = \frac{\displaystyle\int_0^{2\pi} d\varphi\cos\left(m\varphi\right)\int_0^a \rho\, d\rho\, J_m(k_{mn}\rho)V(\rho,\varphi)}{\sinh(k_{mn}L)\displaystyle\int_0^{2\pi} d\varphi\cos^2\left(m\varphi\right)\int_0^a \rho\, d\rho\, J_m{}^2(k_{mn}\rho)}$$

§10.26 Graphics

Contents

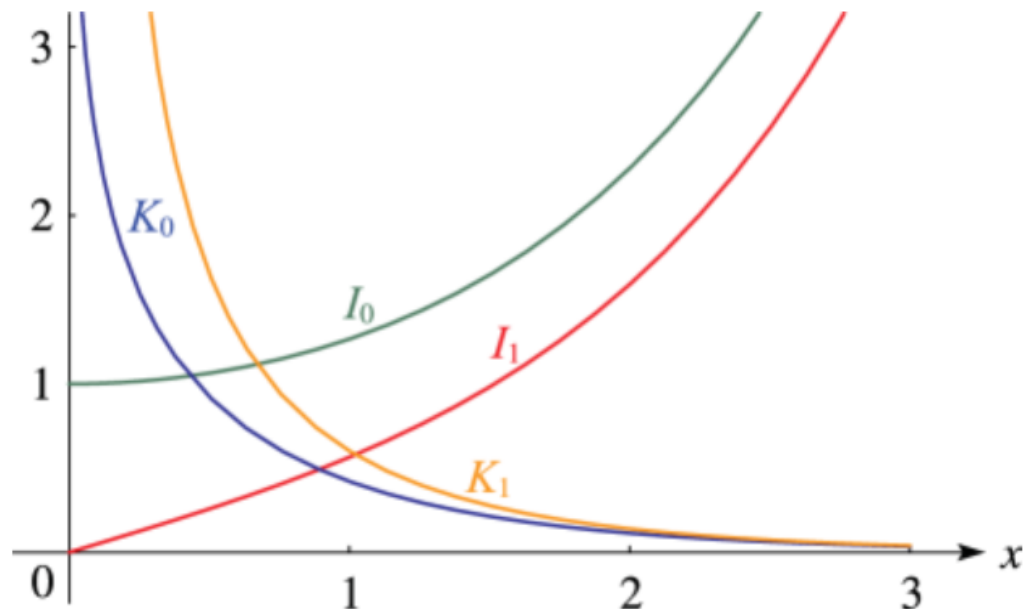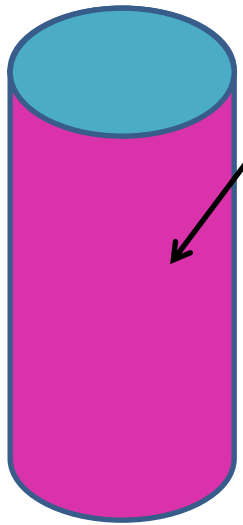## §10.26(i) Real Order and Variable

Figure   10.26.1: $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$,

Solutions of Laplace equation inside cylindrical shape
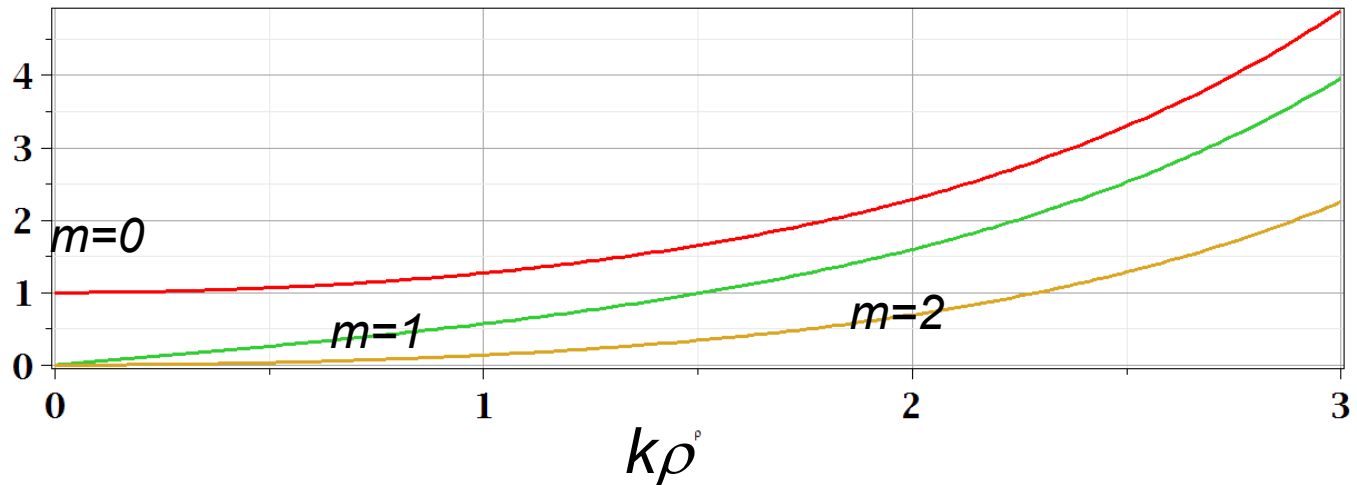Example with non-trivial boundary value at $\rho=a$

$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

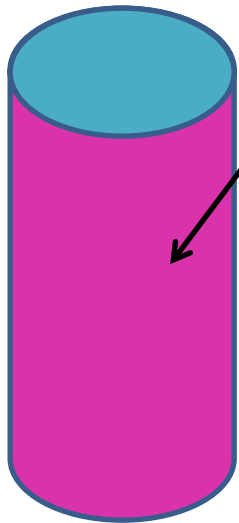$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m \left(\frac{n\pi\rho}{L} \right) \sin\left(\frac{n\pi z}{L} \right) \sin\left(m\phi + \alpha_{mn} \right)$$

$I_m(k\rho)$

m=0

m=1

m=2

$k\rho$

Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at $\rho = a$

$$\Phi(\rho = a, \varphi, z) = V(\varphi, z)$$

$$\Phi(\rho, \varphi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\varphi + \alpha_{mn}\right)$$

If $V(z, \varphi)$ is an even function of φ so that $\alpha_{mn} = \pi/2$:

$$A_{mn} = \frac{\int\limits_0^{2\pi} d\varphi \cos(m\varphi)\int\limits_0^L dz \sin\left(\frac{n\pi z}{L}\right)V(z,\varphi)}{I_m\left(\frac{n\pi a}{L}\right)\int\limits_0^{2\pi} d\varphi \cos^2(m\varphi)\int\limits_0^L dz \sin^2\left(\frac{n\pi z}{L}\right)}$$

Green's function for Dirchelet boundary value inside cylinder:
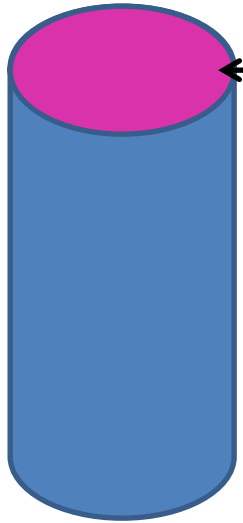
$$\Phi(\rho, \phi, z = L) = V(\rho, \phi)$$

$$\Phi(\rho = a, \phi, z) = 0, \quad \Phi(\rho, \phi, z = 0) = 0$$

Expansion in terms of Bessel function zeros : $J_m(k_{mn}a) = 0$

$$G(\rho, \rho', \phi, \phi', z, z') =$$

$$\frac{8\pi}{\pi a^2} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')} J_m(k_{mn}\rho) J_m(k_{mn}\rho') \sinh(k_{mn}z_<) \sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2 \sinh(k_{mn}L)}$$

$$\Phi(\rho, \phi, z) = \frac{1}{4\pi\varepsilon_0} \int_V d\phi' \, \rho' d\rho' dz' \, G(\rho, \rho', \phi, \phi', z, z') \rho(\rho', \phi', z')$$

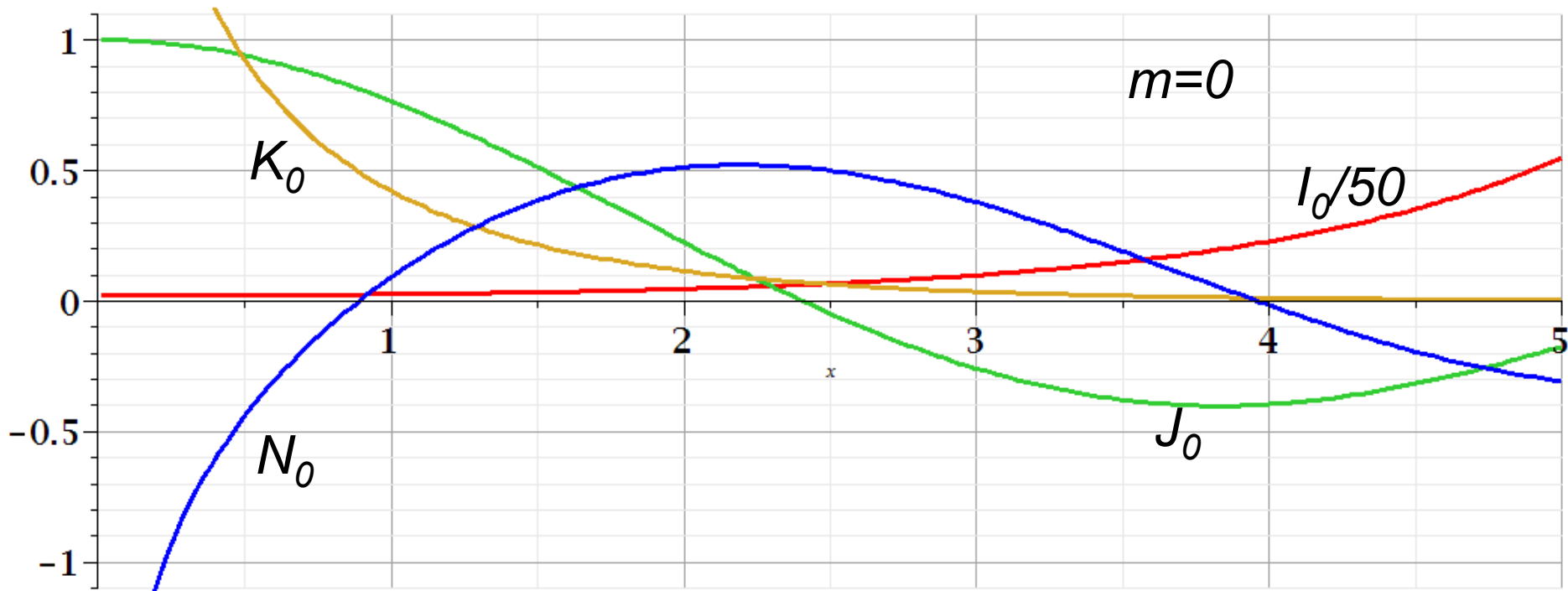$$+ \frac{1}{4\pi} \int_{S; z'=L} d\phi' \, \rho' d\rho' \frac{\partial G(\rho, \rho', \phi, \phi', z, z')}{\partial z'} \bigg|_{z'=L} V(\rho', \phi')$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2} \right) \right) F_m^{\pm}(u) = 0$$

$$F_m^{+}(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm i N_m(u)$$
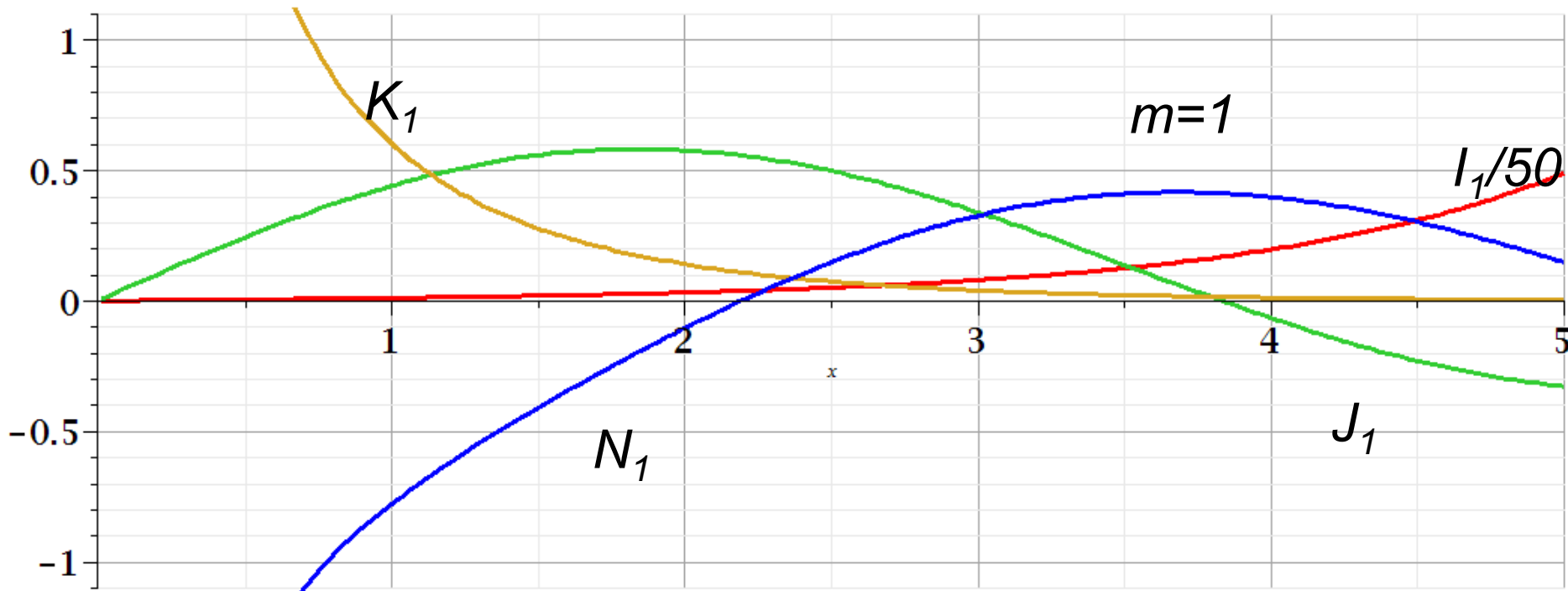
$$F_m^{-}(u) = I_m(u), K_m(u)$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2} \right) \right) F_m^{\pm}(u) = 0$$

$$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm i N_m(u)$$

$$F_m^-(u) = I_m(u), K_m(u)$$

Some useful identities involving cylindrical Bessel functions from **Jackson** Sec. 3.7

$$\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + \left(1 - \frac{m^2}{u^2}\right)\right)J_m(u) = 0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn}; $J_m(x_{mn}) = 0$

$$\int_0^a \rho\,d\rho\,J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn'}\rho}{a}\right) = \frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2 \delta_{nn'}$$

m=1
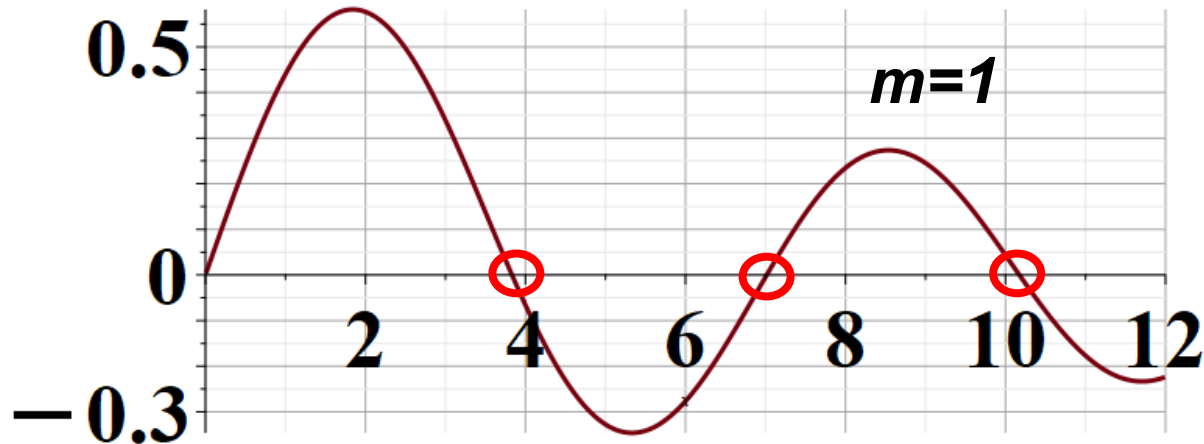
Poisson and Laplace equation in spherical polar coordinates

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html
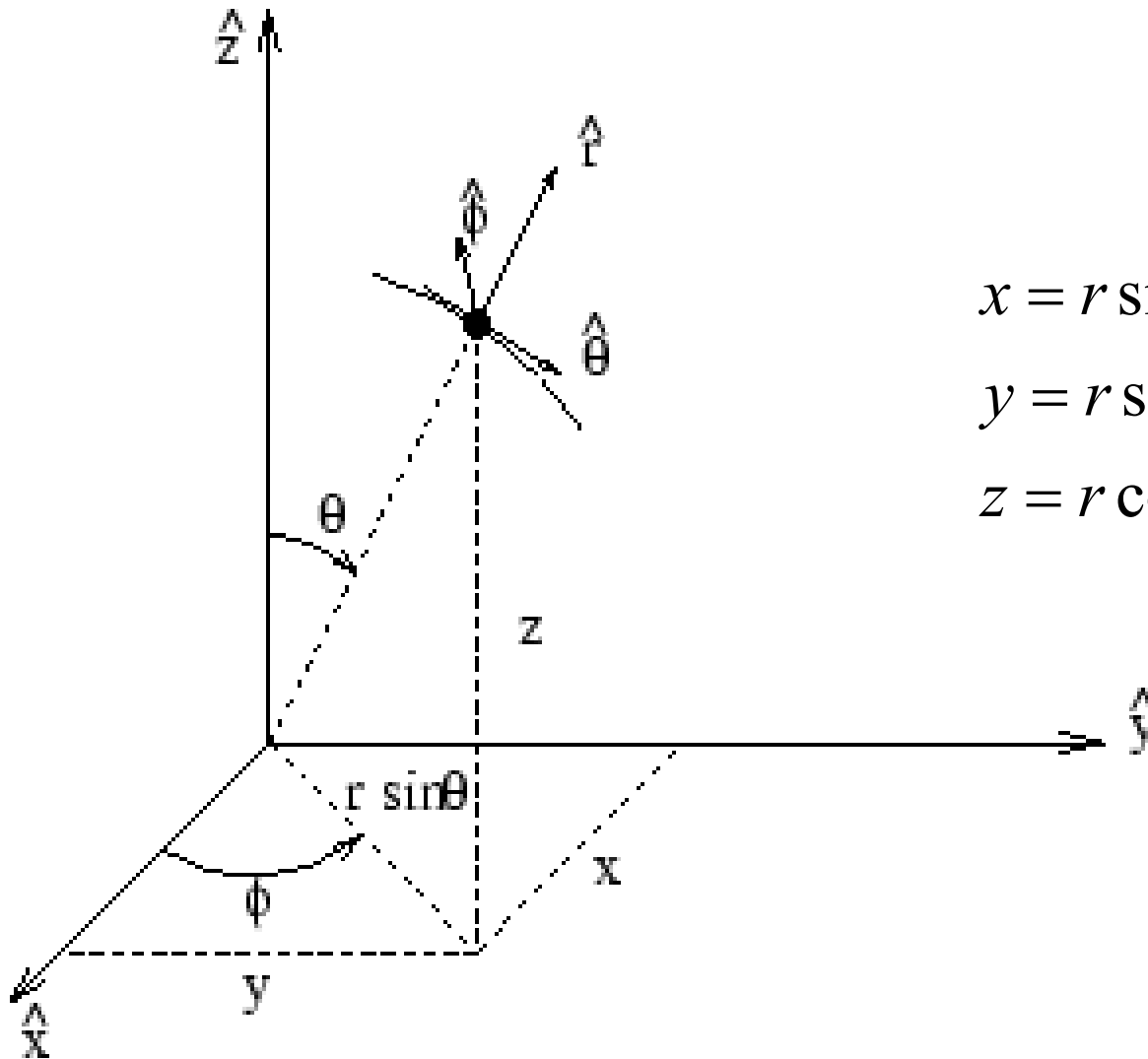
Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r,\theta,\varphi)$:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\Phi = 0$$

$$\Phi(r,\theta,\varphi) = \sum_{lm} R_{lm}(r)Y_{lm}(\theta,\varphi)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\varphi) = -l(l+1)Y_{lm}(\theta,\varphi)$$

Properties of spherical harmonic functions

$$Y_{lm}(\theta,\phi) = (-1)^m Y_{l(-m)}^*(\theta,\phi) \qquad \text{(standard Condon - Shortley convention)}$$

$$\int d\Omega\, Y_{lm}(\theta,\varphi) Y_{l'm'}^*(\theta,\varphi) \equiv \int \sin\theta\, d\theta\, d\varphi\, Y_{lm}(\theta,\varphi) Y_{l'm'}^*(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

Completeness :

$$\sum_{lm} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$$

Relationship to Legendre polynomials :

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r'}|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r \to \infty$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|}$$

$$= \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \rho(\mathbf{r'}) \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') \right)$$

Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3 r' \, \rho(\mathbf{r'}) \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^{\,*}(\theta',\varphi') \right)$$

$$\text{Suppose}: \quad \rho(\mathbf{r'}) = \frac{Q}{a^3 \pi^{3/2}} e^{-r'^2/a^2}$$

$$\int d\Omega' \, Y_{lm}^{\,*}(\theta',\varphi') = \sqrt{4\pi}\, \delta_{l0} \delta_{m0}$$

$$\Phi(\mathbf{r}) = \frac{4\pi}{4\pi\varepsilon_0} \int_0^\infty r'^2 \, dr' \frac{r_<^{\,0}}{r_>^{\,1}} \frac{Q}{a^3 \pi^{3/2}} e^{-r'^2/a^2}$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{\text{erf}(r/a)}{r}$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi')$$

Elements of "proof":

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r_> \left(1 + \left(\dfrac{r_<}{r_>}\right)^2 - 2\left(\dfrac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^{1/2}} =$$

$$= \frac{1}{r_>} \left(1 + \left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' + \left(\frac{r_<}{r_>}\right)^2 \left(\frac{3}{2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2 - \frac{1}{2}\right) + \cdots\right)$$

$$= \frac{1}{r_>} \left(\sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\right)$$

Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r'}|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^{*}(\theta', \varphi')$$

Elements of "proof" -- continued :

Sum rule for spherical harmonics :

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{r}}) \, Y_{lm}^{*}(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = 1$

$$\Rightarrow \frac{4\pi}{2l+1} \sum_{m=-l}^{l} |Y_{lm}(\hat{\mathbf{r}})|^2 = 1$$

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{2(\pm2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\phi}$$

$$Y_{2(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2\theta - \frac{1}{2} \right)$$