PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 12:

Start reading Chap. 5 (Sec. 5.1-5.5 in JDJ)

A. Magnetostatics

B. Vector potential

C. Example: current loop

Course schedule for Spring 2024

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/17/2024	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/19/2024
2	Fri: 01/19/2024	Chap. 1	Electrostatic energy calculations	#2	01/29/2024
3	Mon: 01/22/2024	Chap. 1	Electrostatic energy calculations	#3	01/29/2024
4	Wed: 01/24/2024	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/29/2024
5	Fri: 01/26/2024	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/29/2024
6	Mon: 01/29/2024	Chap. 1 - 3	Brief introduction to numerical methods	#6	02/05/2024
7	Wed: 01/31/2024	Chap. 2 & 3	Image charge constructions	#7	02/05/2024
8	Fri: 02/2/2024	Chap. 2 & 3	Poisson equation in cylindrical geometries		
9	Mon: 02/05/2024	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/12/2024
10	Wed: 02/07/2024	Chap. 4	Dipoles and Dielectrics	#9	02/12/2024
11	Fri: 02/09/2024	Chap. 4	Dipoles and Dielectrics	#10	02/12/2024
12	Mon: 02/12/2024	Chap. 5	Magnetostatics	#11	02/19/2024
13	Wed: 02/14/2024	Chap. 5	Magnetic dipoles and hyperfine interaction		

PHY 712 -- Assignment #11

Assigned: 2/12/2024   Due: 2/19/2024

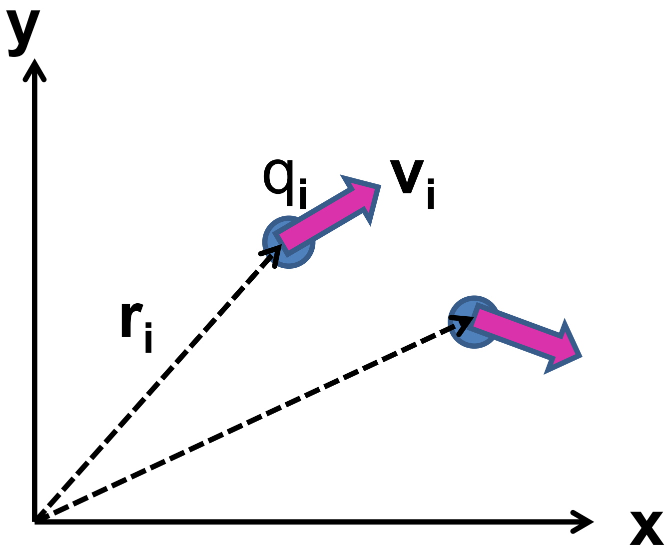
Start reading Chapter 5 (Sec. 5.1-5.5) in **Jackson** .

1. Consider an infinitely long cylindrical wire with radius *a*, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically, in terms of *r* the radial parameter of the cylindrical coordinates of the system the current density is $\mathbf{J}(r)=\mathbf{J_0}$, where $\mathbf{J_0}$ is a constant vector pointing along the z-axis, for $r \leq a$ and zero otherwise.
   a. Find the vector potential (**A**) for all *r*.
   b. Find the magnetic flux field (**B**) for all *r*.

Note that it is possible to solve this problem using ideas learned in beginning physics (such PHY 114).

Magnetostatics

Magnetic flux density or magnetic induction field **B**
Steady state (constant in time) current density **J**



$$\mathbf{J}(\mathbf{r}) = \sum_i q_i \mathbf{v}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Real life examples:

1. Steady macroscopic current
2. Quantum mechanical eigenstates with non-trivial current density.

Note that "statics" implies that $\nabla \cdot \mathbf{J} = 0$.

This follows from the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Comparison of electrostatics and magnetostatics

Electrostatic field due to charge density $\rho(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Alternative forms magnetostatic equations

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mu_0}{4\pi}\int d^3r' \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \times \mathbf{J}(\mathbf{r}')$$

Note that: $\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$

Also note that: $\nabla \times \left(s(\mathbf{r})\mathbf{V}(\mathbf{r})\right) = \nabla s(\mathbf{r}) \times \mathbf{V}(\mathbf{r}) + s(\mathbf{r})\nabla \times \mathbf{V}(\mathbf{r})$

let $s(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}$ and $\mathbf{V}(\mathbf{r}) = \mathbf{J}(\mathbf{r}')$, where $\mathbf{r}'$ is fixed

$$\left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \times \mathbf{J}(\mathbf{r}') = \nabla s(\mathbf{r}) \times \mathbf{V}(\mathbf{r}) = \nabla \times \left(s(\mathbf{r})\mathbf{V}(\mathbf{r})\right) - s(\mathbf{r})\nabla \times \mathbf{V}(\mathbf{r})$$

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times \mathbf{J}(\mathbf{r}')$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \left(\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla \times \mathbf{J}(\mathbf{r}')\right)$$

=0

$$\Rightarrow \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla \times \int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Alternative forms magnetostatic equations -- continued

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$\Rightarrow \nabla\cdot\mathbf{B}(\mathbf{r}) = 0$ No magnetic monopoles

$\Rightarrow \nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}(\mathbf{r})$ Ampere's law

"Proof" of Ampere's law for magnetostatic system:

$$\nabla\times\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\nabla\times\int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that: $\nabla\times\nabla\times\mathbf{V} = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}$

Recall that: $\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$ and $\nabla\cdot\mathbf{J}(\mathbf{r}) = 0$

Differential forms of magnetostatic equations:

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \text{No magnetic monopoles}$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \qquad \text{Ampere's law}$$

Magnetostatic vector potential

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \nabla s(\mathbf{r})$$

Non uniqueness of the magnetostatic vector potential

$$\text{Note that:} \quad \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \mathbf{A}'(\mathbf{r})$$

$$\text{if} \quad \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla s(\mathbf{r})$$

$$\text{Example:} \quad \text{for } \mathbf{B}(\mathbf{r}) = B_0 \hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2} B_0 \left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}} \right)$$

$$\text{or} \quad \mathbf{A}(\mathbf{r}) = B_0 x \hat{\mathbf{y}}$$

$$\text{or} \quad \mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$$

Differential form of Ampere's law in terms of vector potential:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}(\mathbf{r})) - \nabla^2 \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

$$\text{If} \quad \nabla \cdot \mathbf{A}(\mathbf{r}) = 0 \quad \text{(Coulomb gauge)} \quad \Rightarrow \nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Comment on determining $\mathbf{A}(\mathbf{r})$ in the Coulomb gauge:

Differential form: $\nabla^2\mathbf{A}(\mathbf{r}) = -\mu_0\mathbf{J}(\mathbf{r})$

Integral form: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

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1. Consider an infinitely long cylindrical wire with radius *a*, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically, in terms of *r* the radial parameter of the cylindrical coordinates of the system the current density is $\mathbf{J}(r)=\mathbf{J}_0$, where $\mathbf{J}_0$ is a constant vector pointing along the z-axis, for $r \leq a$ and zero otherwise.
   a. Find the vector potential (**A**) for all *r*.
   b. Find the magnetic flux field (**B**) for all *r*.

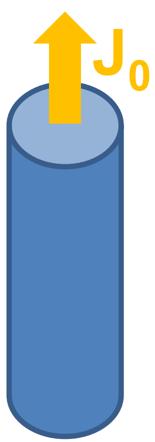
Integral form may not be a good idea in this case ---

Further comments about HW #11

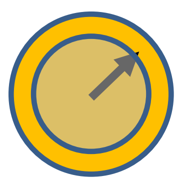
Using PHY 114 approach

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \text{No magnetic monopoles}$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \qquad \text{Ampere's law}$$



Top view



Integral form:

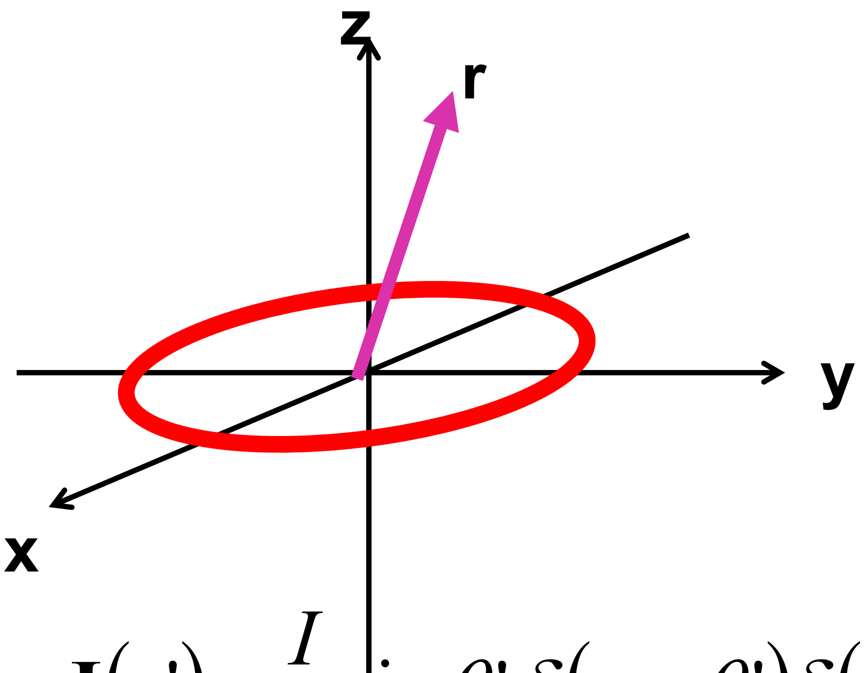
$$\int d\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mu_0 \int d\mathbf{A} \cdot \mathbf{J}$$

$$\int d\ell \cdot \mathbf{B} = \mu_0 \int d\mathbf{A} \cdot \mathbf{J}$$

For this case:

$$2\ \pi r B(r) = \mu_0 \pi r^2 J_0 \quad \text{for } r < a$$

Magnetostatics example: current loop



Integral form may be OK for this case…

$$\mathbf{J}(\mathbf{r}') = \frac{I}{a} \sin\theta' \, \delta(\cos\theta') \delta(r' - a)\left(-\sin\phi' \, \hat{\mathbf{x}} + \cos\phi' \, \hat{\mathbf{y}}\right)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Magnetostatics example: current loop -- continued

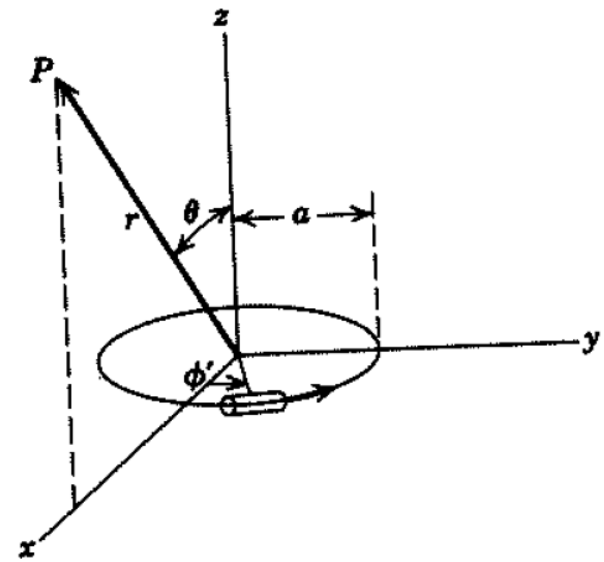
$$\mathbf{J}(\mathbf{r}') = \frac{I}{a}\sin\theta'\delta(\cos\theta')\delta(r'-a)(-\sin\varphi'\hat{\mathbf{x}}+\cos\varphi'\hat{\mathbf{y}})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \; \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi a}\int r'^2\,dr'\; d\cos\theta'\,d\varphi'\frac{\sin\theta'\delta(\cos\theta')\delta(r'-a)(-\sin\varphi'\hat{\mathbf{x}}+\cos\varphi'\hat{\mathbf{y}})}{\left(r^2+r'^2-2rr'\left(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')\right)\right)^{1/2}}$$

Completing integration over r' and θ':

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I a^2}{4\pi a}\int_0^{2\pi} d\varphi'\frac{(-\sin\varphi'\hat{\mathbf{x}}+\cos\varphi'\hat{\mathbf{y}})}{\left(r^2+a^2-2ra\left(\sin\theta\cos(\varphi-\varphi')\right)\right)^{1/2}}$$



Let $\varphi-\varphi'\equiv\phi$

$$\sin\varphi' = \sin(\varphi-\phi) = \sin\varphi\cos\phi - \cos\varphi\sin\phi$$

$$\cos\varphi' = \cos(\varphi-\phi) = \cos\varphi\cos\phi + \sin\varphi\sin\phi$$

Remaining non-trivial terms

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4\pi}(\sin\varphi\hat{\mathbf{x}}-\cos\varphi\hat{\mathbf{y}})\int_0^{2\pi} d\phi\frac{\cos\phi}{\left(r^2+a^2-2ra\left(\sin\theta\cos\phi\right)\right)^{1/2}}$$

Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4\pi}\left(\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}}\right)\int_0^{2\pi} d\phi \frac{\cos\phi}{\left(r^2 + a^2 - 2ra\left(\sin\theta\cos\phi\right)\right)^{1/2}}$$

Elliptic integrals:

$$K(m) = \int_0^{\pi/2} \frac{du}{\left(1 - m\sin^2 u\right)^{1/2}}$$

$$E(m) = \int_0^{\pi/2} \left(1 - m\sin^2 u\right)^{1/2} du$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} 4Ia \frac{\left(\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}}\right)}{\left(r^2 + a^2 + 2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2 - k^2\right)K(k) - 2E(k)}{k^2}\right]$$

where: $k^2 \equiv \dfrac{4ar\sin\theta}{r^2 + a^2 + 2ra\sin\theta}$

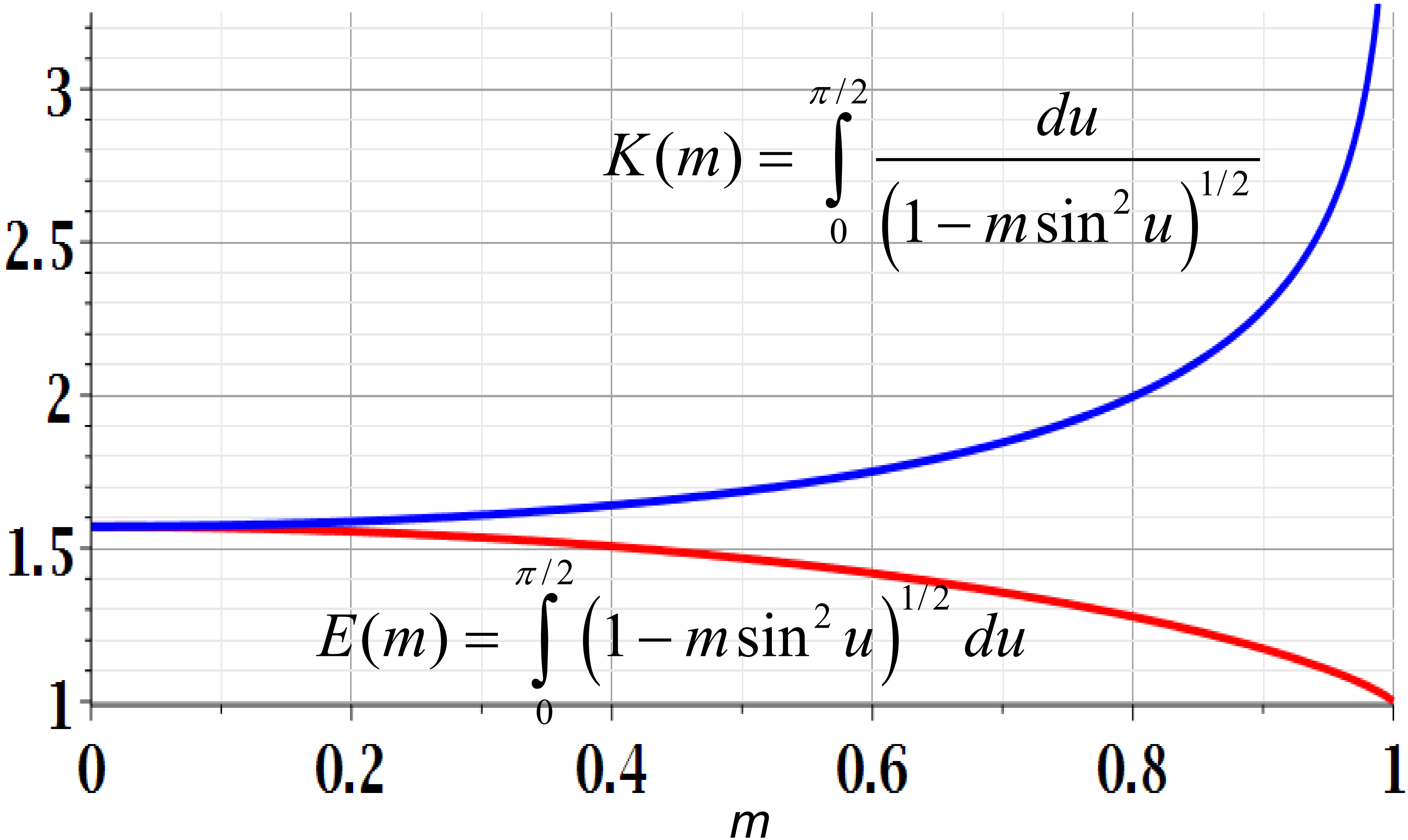
$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

Examination of *k*

$$k^2 \equiv \frac{4ar\sin\theta}{r^2 + a^2 + 2ra\sin\theta}$$

Note that $k=0$ when $\theta=0$ and when $r \rightarrow infinity$
$k=1$ when $\theta=\pi/2$ and $r = a$

Standard eliptic functions --



Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}4Ia\frac{\left(\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}}\right)}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k)-2E(k)}{k^2}\right]$$

where: $$k^2 \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta} = \frac{4ax}{x^2+z^2+a^2+2ax}$$

For $\varphi = 0$: $x = r\sin\theta,\ \ y = 0,\ \ z = r\cos\theta$

$$\mathbf{A}(\mathbf{r}) = A_y(x,z)\hat{\mathbf{y}} = \frac{\mu_0}{4\pi}4Ia\hat{\mathbf{y}}\frac{1}{\left(x^2+z^2+a^2+2ax\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k)-2E(k)}{k^2}\right]$$

where: $$k^2 \equiv \frac{4ax}{x^2+z^2+a^2+2ax}$$

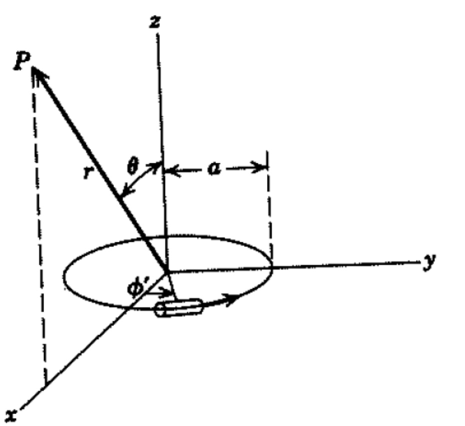
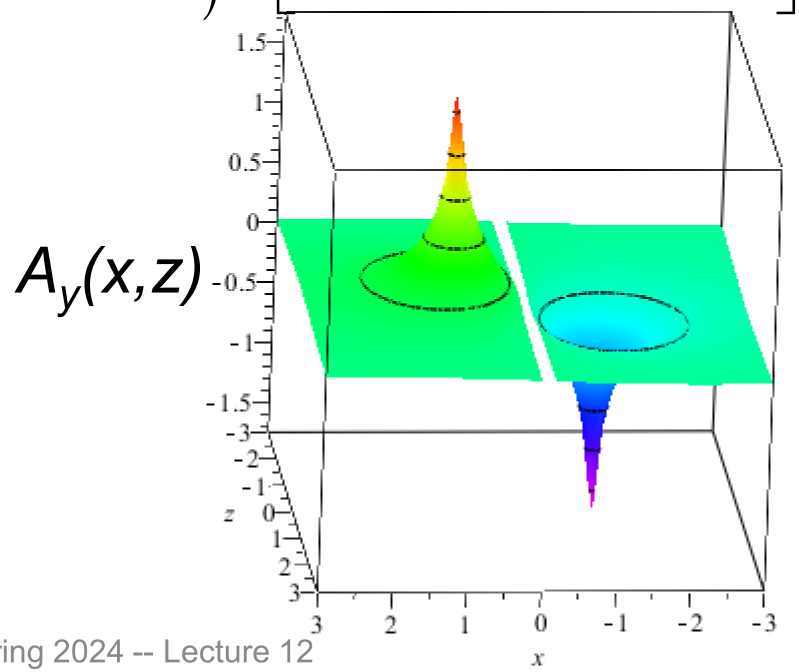


Figure 5.5



Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} 4Ia \frac{\left(\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}}\right)}{\left(r^2 + a^2 + 2ra\sin\theta\right)^{1/2}} \left[\frac{\left(2-k^2\right)K(k) - 2E(k)}{k^2}\right]$$

$$= \hat{\boldsymbol{\varphi}} \frac{\mu_0}{4\pi} \frac{4Ia}{\left(r^2 + a^2 + 2ra\sin\theta\right)^{1/2}} \left[\frac{\left(2-k^2\right)K(k) - 2E(k)}{k^2}\right]$$

where: $\quad k^2 \equiv \dfrac{4ar\sin\theta}{r^2 + a^2 + 2ra\sin\theta}$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

Evaluation for special cases

For $k^2 \to 0$:

$$\frac{\left(2-k^2\right)K(k) - 2E(k)}{k^2} \approx \frac{\pi}{16}k^2$$

Evaluation for special case $k^2 \rightarrow 0$

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\varphi}} \frac{\mu_0 I a^2}{4} \frac{r\sin\theta}{\left(r^2 + a^2 + 2ra\sin\theta\right)^{3/2}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 I a^2}{4} \frac{\left(\hat{\mathbf{r}}\cos\theta\left(2a^2 + 2r^2 + ar\ \sin\theta\right) - \hat{\boldsymbol{\theta}}\sin\theta\left(2a^2 - r^2 + ar\ \sin\theta\right)\right)}{\left(r^2 + a^2 + 2ra\sin\theta\right)^{5/2}}$$

$$\text{where } k^2 \equiv \frac{4ar\sin\theta}{r^2 + a^2 + 2ra\sin\theta} \rightarrow 0$$

For $\theta = 0$: $r = z$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \hat{\mathbf{z}} \frac{\mu_0 I a^2}{2} \frac{1}{\left(z^2 + a^2\right)^{3/2}}$$

Evaluation for special case $k^2 \rightarrow 0$

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\varphi}} \frac{\mu_0 I a^2}{4} \frac{r\sin\theta}{\left(r^2 + a^2 + 2ra\sin\theta\right)^{3/2}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 I a^2}{4} \frac{\left(\hat{\mathbf{r}}\cos\theta\left(2a^2 + 2r^2 + ar\ \sin\theta\right) - \hat{\boldsymbol{\theta}}\sin\theta\left(2a^2 - r^2 + ar\ \sin\theta\right)\right)}{\left(r^2 + a^2 + 2ra\sin\theta\right)^{5/2}}$$

$$\text{where } k^2 \equiv \frac{4ar\sin\theta}{r^2 + a^2 + 2ra\sin\theta} \rightarrow 0$$

For $\quad r \rightarrow \infty$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 I a^2}{4r^3}\left(\hat{\mathbf{r}}\left(2\cos\theta\right) + \hat{\boldsymbol{\theta}}\sin\theta\right)$$

Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} 4Ia \frac{(\sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}})}{(r^2 + a^2 + 2ra\sin\theta)^{1/2}} \left[\frac{(2-k^2)K(k) - 2E(k)}{k^2} \right]$$

$$\text{where:} \quad k \equiv \frac{4ar\sin\theta}{r^2 + a^2 + 2ra\sin\theta}$$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

Note that for spherical polar coordinates: $\hat{\boldsymbol{\varphi}} = \sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}$

$$\mathbf{A}(\mathbf{r}) = A_\phi(\mathbf{r})\hat{\boldsymbol{\varphi}}$$

where $A_\phi(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{4Ia}{(r^2 + a^2 + 2ra\sin\theta)^{1/2}} \left[\frac{(2-k^2)K(k) - 2E(k)}{k^2} \right]$

$$\mathbf{B}(\mathbf{r}) = \frac{1}{r\sin\theta} \frac{\partial(\sin\theta A_\phi(\mathbf{r}))}{\partial\theta}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial(rA_\phi(\mathbf{r}))}{\partial r}\hat{\boldsymbol{\theta}}$$

For $r \to \infty$:

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mu_0}{4\pi} \frac{I\pi a^2}{r^3} (2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

Other examples of current density sources:

Quantum mechanical expression for current density

for a particle of mass M and charge e and of probability amplitude $\Psi(\mathbf{r})$:

$$\mathbf{J}(\mathbf{r}) = -\frac{e\hbar}{2Mi}\left(\Psi^*(\mathbf{r})\nabla\Psi(\mathbf{r}) - \Psi(\mathbf{r})\nabla\Psi^*(\mathbf{r})\right)$$

For an electron in a spherical potential (such as in an atom):

$$\Psi(\mathbf{r}) \equiv \Psi_{nlm_l}(\mathbf{r}) = R_{nl}(r)Y_{lm_l}(\hat{\mathbf{r}})$$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{2Mi}\left|R_{nl}(r)\right|^2 \frac{1}{r\sin\theta}\left(Y_{lm_l}{}^*(\hat{\mathbf{r}})\frac{\partial Y_{lm_l}(\hat{\mathbf{r}})}{\partial\phi} - Y_{lm_l}(\hat{\mathbf{r}})\frac{\partial Y_{lm_l}{}^*(\hat{\mathbf{r}})}{\partial\phi}\right)\hat{\boldsymbol{\varphi}}$$

$$= \frac{e\hbar}{M}\frac{m_l}{r\sin\theta}\left|\Psi_{nlm_l}(\mathbf{r})^2\right|\hat{\boldsymbol{\varphi}}$$

Note that: $\hat{\boldsymbol{\varphi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} = \dfrac{\hat{\mathbf{z}}\times\mathbf{r}}{r\sin\theta}$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{M}\frac{m_l}{r^2\sin^2\theta}\left|\Psi_{nlm_l}(\mathbf{r})^2\right|(\hat{\mathbf{z}}\times\mathbf{r})$$

Magnetic vector potential for this case:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{J}(\mathbf{r}') = \frac{e\hbar}{M}\frac{m_l}{r'^2 \sin^2\theta'}\left|\Psi_{nlm_l}(\mathbf{r}')^2\right|(\hat{\mathbf{z}}\times\mathbf{r}')$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar m_l}{M}\int d^3r' \frac{(\hat{\mathbf{z}}\times\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\frac{\left|\Psi_{nlm_l}(\mathbf{r}')^2\right|}{r'^2 \sin^2\theta'}$$

For example: electron in the $nlm_l = 211$ state of H:

$$\left|\Psi_{211}(\mathbf{r}')^2\right| = \frac{1}{64\pi a^3}\left(\frac{r'}{a}\right)^2 e^{-r'/a}\sin^2\theta'$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{8\pi}\frac{e\hbar}{M}\frac{(\hat{\mathbf{z}}\times\mathbf{r})}{r^3}\left[1-e^{-r/a}\left(1+\frac{r}{a}+\frac{r^2}{2a^2}+\frac{r^3}{8a^3}\right)\right]$$

Some details -- (note – this case simplifies more quickly than most…)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar m_l}{M}\int d^3r' \; \frac{(\hat{\mathbf{z}}\times\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\frac{\left|\Psi_{nlm_l}(\mathbf{r}')^2\right|}{r'^2\sin^2\theta'}$$

$$\left|\Psi_{211}(\mathbf{r}')^2\right| = \frac{1}{64\pi a^3}\left(\frac{r'}{a}\right)^2 e^{-r'/a}\sin^2\theta'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{64\pi a^3}\frac{e\hbar m_l}{M}\hat{\mathbf{z}}\times\int r'dr'\,d\Omega'\frac{\hat{\mathbf{r}}'}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{r'}{a}\right)^2 e^{-r'/a}$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{8\pi}\frac{e\hbar}{M}\frac{(\hat{\mathbf{z}}\times\mathbf{r})}{r^3}\left[1-e^{-r/a}\left(1+\frac{r}{a}+\frac{r^2}{2a^2}+\frac{r^3}{8a^3}\right)\right]$$