PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Class notes for Lecture 13:

Continue reading Chap. 5 – Sec. 5.6-5.7 in JDJ

A. Examples of magnetostatic fields

B. Magnetic dipoles

C. Hyperfine interaction

8	Fri: 02/2/2024	Chap. 2 & 3	Poisson equation in cylindrical geometries		
9	Mon: 02/05/2024	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/12/2024
10	Wed: 02/07/2024	Chap. 4	Dipoles and Dielectrics	#9	02/12/2024
11	Fri: 02/09/2024	Chap. 4	Dipoles and Dielectrics	#10	02/12/2024
12	Mon: 02/12/2024	Chap. 5	Magnetostatics	#11	02/19/2024
13	Wed: 02/14/2024	Chap. 5	Magnetic dipoles and hyperfine interaction	#12	02/19/2024
14	Fri: 02/16/2024	Chap. 5	Magnetic dipoles and dipolar fields		
15	Mon: 02/19/2024	Chap. 6	Maxwell's Equations		
16	Wed: 02/21/2024	Chap. 6	Electromagnetic energy and forces		
17	Fri: 02/23/2024	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/26/2024	Chap. 7	Electromagnetic plane waves		
19	Wed: 02/28/2024	Chap. 7	Optical effects of refractive indices		
20	Fri: 03/01/2024	Chap. 1-7	Review		
21	Mon: 03/04/2024	Chap. 8	Short lectures on waveguides	Exam	
22	Wed: 03/06/2024	Chap. 8	Short lectures on waveguides	Exam	
23	Fri: 03/08/2024	Chap. 8	Short lectures on waveguides	Exam	
	Mon: 03/11/2024	No class	*Spring Break*		
	Wed: 03/13/2024	No class	*Spring Break*		

PHY 712 -- Assignment #12

Assigned: 2/14/2024 Due: 2/19/2024

Continue reading Chapter 5 (Sec. 5.6-5.7) in **Jackson** .

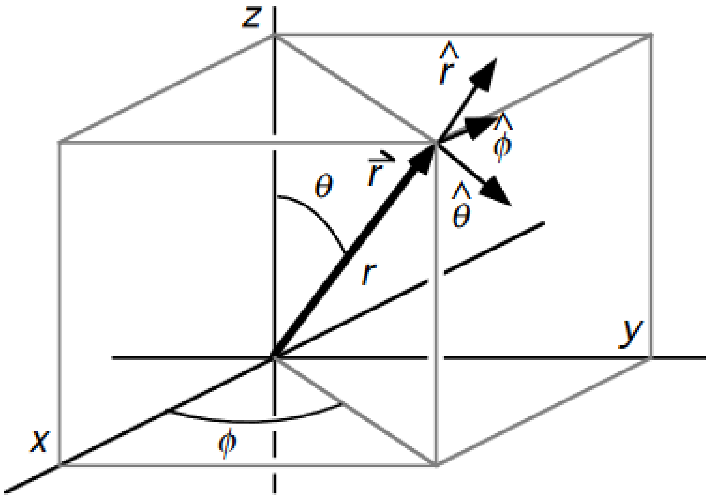
1. Consider the following equation and verify its validity (or otherwise).

$$\int d\Omega' \sum_{lm} Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}') \, \mathbf{r}' = \frac{r'}{r}\mathbf{r} \equiv r' \, \hat{\mathbf{r}} \, .$$

Comment about spherical polar coordinates

Ref: https://www.cpp.edu/~ajm/materials/delsph.pdf

Spherical Coordinates



Transforms

The forward and reverse coordinate transformations are

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r\sin\theta\cos\phi$$

$$\theta = \arctan\left(\sqrt{x^2 + y^2}, z\right) \qquad y = r\sin\theta\sin\phi$$

$$\phi = \arctan(y, x) \qquad z = r\cos\theta$$

where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.

Unit Vectors

The unit vectors in the spherical coordinate system are functions of position. It is convenient to express them in terms of the *spherical* coordinates and the unit vectors of the *rectangular* coordinate system which are *not* themselves functions of position.

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

$$\hat{\phi} = \frac{\hat{z}\times\hat{r}}{\sin\theta} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\hat{\theta} = \hat{\phi}\times\hat{r} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$

Note result given in “detailed” pdf file:

In order to evaluate the vector potential (1) for this problem, we can make use of the expansion:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}'}). \tag{3}$$

Noting that

$$\mathbf{r}' = r' \sqrt{\frac{4\pi}{3}} \left(Y_{1-1}(\hat{\mathbf{r}'}) \frac{\hat{\mathbf{x}} + \mathbf{i}\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}'}) \frac{-\hat{\mathbf{x}} + \mathbf{i}\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}'}) \hat{\mathbf{z}} \right), \tag{4}$$

we see that the angular integral in Eq. (1) can be simplified with the use of the identity:

$$\int d\Omega' \sum_{lm} Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}'})\ \mathbf{r}' = \frac{r'}{r}\mathbf{r}\ \equiv r'\ \hat{\mathbf{r}}\ . \tag{5}$$

Lecture Notes

- Lecture 1 -- Introduction and electrostatics PP slides PDF
- Lecture 2 -- Evaluation of electrostatic energy PP slides PDF Ewald notes PDF CsCl Maple input CsCl PDF
- Lecture 3 -- Ewald summation methods PP slides PDF Ewald notes PDF CsCl Maple input CsCl PDF
- Lecture 4 -- Electrostatic potentials and fields PP slides PDF
- Lecture 5 -- Poisson equation in 2 and 3 dimensions PP slides PDF
- Lecture 6 -- Short introduction to numerical methods (Sec. 1.12, 1.13, and 2.13 in JDJ) PP slides PDF Detailed notes (PDF)
- Lecture 7 -- Image charge tricks and also Poisson equation in cylindrical coordinates (Sec. 2.1-6,3.7-8 in JDJ) PP slides PDF
- Lecture 8 -- Laplace/Poisson equation in cylindrical coordinates (Sec. 2.11, 3.7, 3.8, 3.11 in JDJ) PP slides PDF
- Lecture 9 -- Laplace/Poisson equation in spherical coordinates (Sec. 3.1-3.6, also beginning Chap. 4 in JDJ) PP slides PDF
- Lecture 10 -- Dipole moments and dielectrics (Sec. 4.1-4.4 in JDJ) PP slides PDF
- Lecture 11 -- Microscopic and macroscopic dipolar effects (Sec. 4.5-4.7 in JDJ) PP slides PDF
- Lecture 12 -- Magnetostatics (Sec. 5.1-5.5 in JDJ) PP slides PDF
- Lecture 13 -- Magnetic dipoles (Sec. 5.6-5.7 in JDJ) PP slides PDF Detailed PDF

Example macroscopic localized current source --

For the model current density --

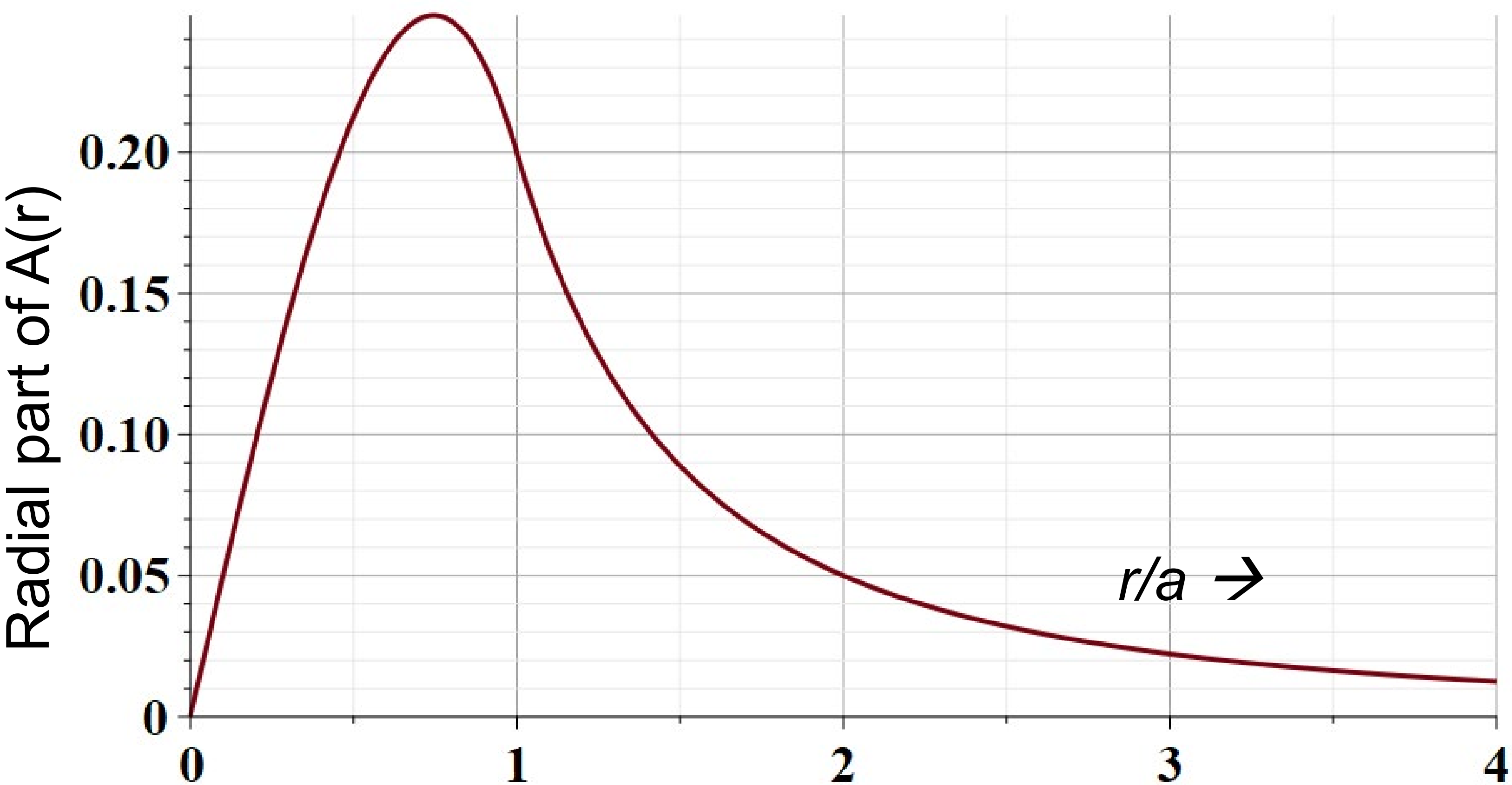
$$\mathbf{J}(\mathbf{r}) = \begin{cases} \rho_0 \boldsymbol{\omega} \times \mathbf{r} & \text{for r } \leq a \\ 0 & \text{otherwise} \end{cases}$$

Here ρ_0 is a charge density constant $\boldsymbol{\omega}$ is a constant angular velocity

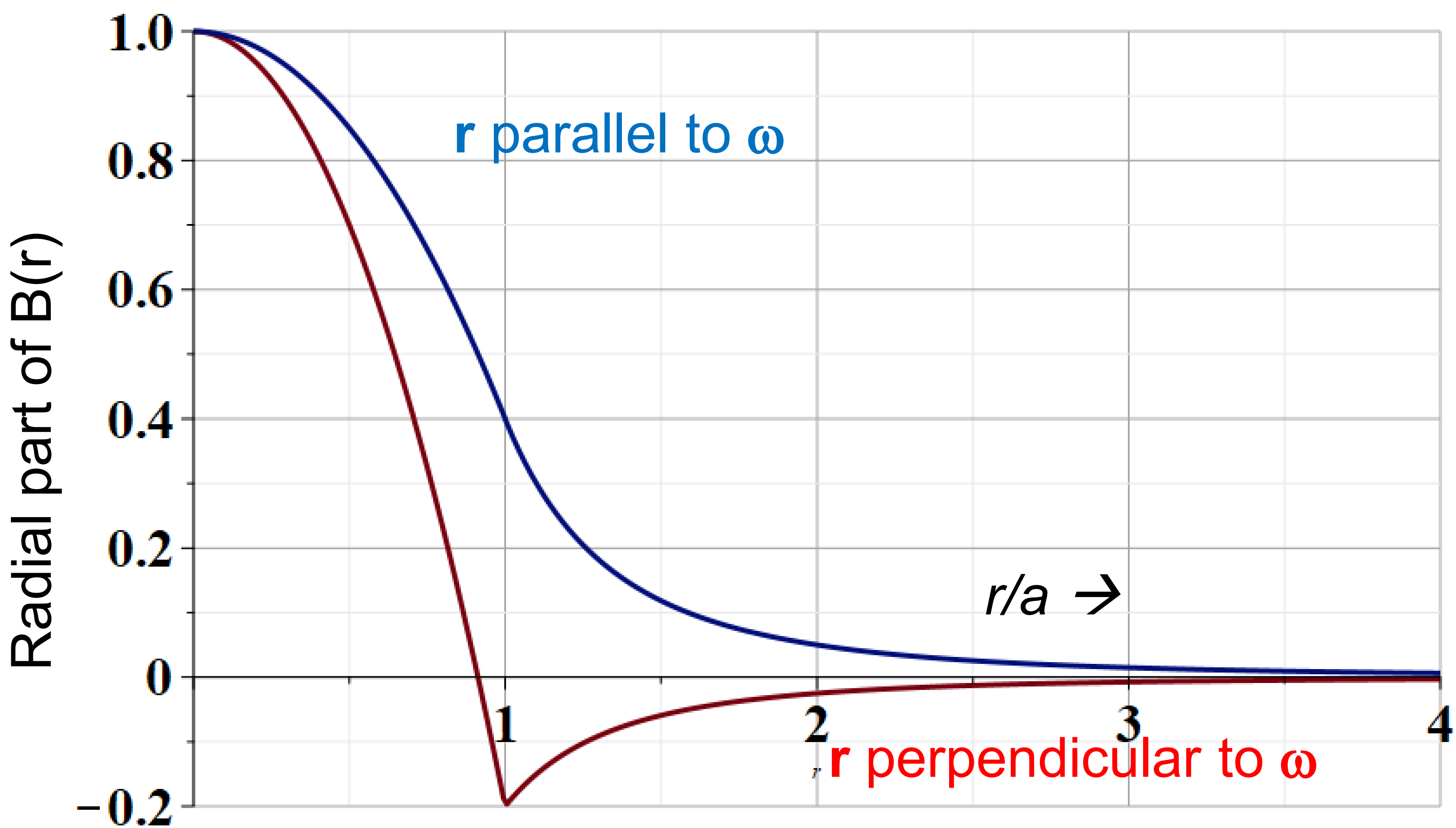
$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \left(\frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \\ \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \frac{a^5}{5r^3} & \text{for } r \geq a \end{cases} .$$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \left[\boldsymbol{\omega} \left(a^2 - \frac{6}{5} r^2 \right) + \frac{3}{5} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \leq a \\ \frac{\mu_0 \rho_0}{3} \left[-\boldsymbol{\omega} \frac{a^5}{5r^3} + \frac{3a^5}{5r^5} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \geq a \end{cases}$$

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \left(\frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \\ \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \frac{a^5}{5r^3} & \text{for } r \geq a \end{cases} .$$



$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 \rho_0}{3}\left[\boldsymbol{\omega}\left(a^2 - \dfrac{6}{5}r^2\right) + \dfrac{3}{5}\mathbf{r}(\boldsymbol{\omega}\cdot\mathbf{r})\right] & \text{for } r \le a \\ \dfrac{\mu_0 \rho_0}{3}\left[-\boldsymbol{\omega}\dfrac{a^5}{5r^3} + \dfrac{3a^5}{5r^5}\mathbf{r}(\boldsymbol{\omega}\cdot\mathbf{r})\right] & \text{for } r \ge a \end{cases}$$



Various forms of Ampere's law :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

Vector potential: $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$

For Coulomb gauge: $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$

$$\Rightarrow \nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$$

For confined current density :

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Other examples of current density sources:

Quantum mechanical expression for current density

for a particle of mass M and charge e and of probability amplitude $\Psi(\mathbf{r})$:

$$\mathbf{J}(\mathbf{r}) = -\frac{e\hbar}{2Mi}\left(\Psi^*(\mathbf{r})\nabla\Psi(\mathbf{r}) - \Psi(\mathbf{r})\nabla\Psi^*(\mathbf{r})\right)$$

For an electron in a spherical potential (such as in an atom):

$$\Psi(\mathbf{r}) \equiv \Psi_{nlm_l}(\mathbf{r}) = R_{nl}(r)Y_{lm_l}(\hat{\mathbf{r}})$$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{2Mi}\left|R_{nl}(r)\right|^2 \frac{1}{r\sin\theta}\left(Y_{lm_l}{}^*(\hat{\mathbf{r}})\frac{\partial Y_{lm_l}(\hat{\mathbf{r}})}{\partial\varphi} - Y_{lm_l}(\hat{\mathbf{r}})\frac{\partial Y_{lm_l}{}^*(\hat{\mathbf{r}})}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$$

$$= \frac{e\hbar}{M}\frac{m_l}{r\sin\theta}\left|\Psi_{nlm_l}(\mathbf{r})^2\right|\hat{\boldsymbol{\varphi}}$$

Note that: $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}} = \frac{\hat{\mathbf{z}}\times\mathbf{r}}{r\sin\theta}$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{M}\frac{m_l}{r^2\sin^2\theta}\left|\Psi_{nlm_l}(\mathbf{r})\right|^2(\hat{\mathbf{z}}\times\mathbf{r})$$

Details of the electron orbital magnetic dipole moment

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{m_e}\frac{m_l}{r\sin\theta}\left|\Psi_{nlm_l}(\mathbf{r})^2\right|\hat{\boldsymbol{\varphi}}$$

Note that: $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$

Magnetic dipole moment:

$$\mathbf{m} = \frac{1}{2}\int d^3r'\mathbf{r}'\times\mathbf{J}(\mathbf{r}') = -\frac{e\hbar m_l}{2m_e}\int d^3r'\frac{\mathbf{r}'\times\hat{\boldsymbol{\varphi}}'}{r'\sin\theta'}\left|\Psi_{nlm_l}(\mathbf{r})^2\right|$$

$$= -\frac{e\hbar m_l}{2m_e}\int d^3r'\frac{-r'\hat{\boldsymbol{\theta}}'}{r'\sin\theta'}\left|\Psi_{nlm_l}(\mathbf{r})^2\right|$$

Note that: $\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$

$$\mathbf{m} = -\frac{e\hbar m_l\hat{\mathbf{z}}}{2m_e}\int d^3r'\left|\Psi_{nlm_l}(\mathbf{r})^2\right|$$

$$= -\frac{e\hbar m_l}{2m_e}\hat{\mathbf{z}}$$

Significance of magnetic dipole – multipole approximation to vector potential and magnetic field

Magnetic dipolar field

The magnetic dipole moment is defined by

$$\mathbf{m} = \frac{1}{2}\int d^3r'\,\mathbf{r}' \times \mathbf{J}(\mathbf{r}'),$$

with the corresponding potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2},$$

and magnetostatic field

$$\mathbf{B}_{\mathbf{m}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}})-\mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta^3(\mathbf{r})\right\}.$$

Summary of magnetic field generated by point magnetic dipole moment discussed in the detailed notes:

$$\mathbf{B}_{\boldsymbol{\mu}_e}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\hat{\mathbf{r}}\left(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}\right) - \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_e \delta(\mathbf{r}) \right)$$

Magnetic field near nucleus due to orbiting electron:

$$\mathbf{B}_O(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e}{m_e} L_z \hat{\mathbf{z}} \left\langle \frac{1}{r^3} \right\rangle$$

"Hyperfine" interaction energy:

$$\mathcal{H}_{HF} = -\boldsymbol{\mu}_N \cdot \left(\mathbf{B}_{\boldsymbol{\mu}_e}(\mathbf{r}) + \mathbf{B}_O(\mathbf{r}) \right)$$

$$= \frac{\mu_0}{4\pi}\left(\frac{3\left(\boldsymbol{\mu}_N \cdot \hat{\mathbf{r}}\right)\left(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}\right) - \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e \delta(\mathbf{r}) + \frac{e}{m_e}\left\langle \frac{\mathbf{L} \cdot \boldsymbol{\mu}_N}{r^3} \right\rangle \right)$$

$$\mathcal{H}_{HF} = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{\mu}_N \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e \delta(\mathbf{r}) + \frac{e}{m_e}\left\langle \frac{\mathbf{L}\cdot\boldsymbol{\mu}_N}{r^3} \right\rangle \right)$$