PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Notes for Lecture 16:

Finish reading Chapter 6 (Sec. 6.6-6.10 in JDJ) (some sections covered in less detail)

1. Some details of Liénard-Wiechert results
2. Energy density and flux associated with electromagnetic fields
3. Time harmonic fields

Many events this week –

TODAY ➔

Many events this week –

TOMORROW →

The ADVANCED Photon Source
at Argonne National Laboratory

APS is one of several synchrotron light source facilities throughout the world.

PHYSICS COLLOQUIUM

THURSDAY

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FEBRUARY 22TH, 2024

Light-induced Structural Responses in Matter

Understanding light-matter interactions and controlling matter with light are of fundamental interest in physics and engineering. This knowledge is crucial for advancing solar energy materials and developing high-performance next-generation data storage and computing systems.
In this talk, I will fist introduce an emerging experimental approach: ultrafast X-ray diffraction, which enables the probing of dynamic behavior of matter induced by light excitation. I will then present recent investigations that push the boundaries of ultrafast X-ray diffraction techniques.

The first advancement involves the use of diffuse X-ray scattering, allowing for resolving light-induced local lattice deformations in matter associated with polaron formation and evolution. I will showcase the results from experiments conducted on a lead halide perovskite system, revealing evolving local strain fields over tens of picoseconds as photogenerated carriers localize.

Second, I will introduce the X-ray diffraction microscopy approach to resolve light-induced changes in matter across nano- and meso-scale heterogeneity. Specifically, I will present results on multiferroic bismuth ferrite, exhibiting stripe-ordered ferroelectric domains. I will illustrate how light can induce sub-nanosecond timescale manipulation of these domains through creation and annihilation of domain walls,

Burak Guzelturk, Ph.D.
Advanced Photon Source
Argonne National Laboratory

4 pm - Olin 101
Refreshments will be served in the Olin

Many events this week –

FRIDAY ➔

PH.D. DEFENSE

FRIDAY

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FEBRUARY 23RD, 2024

SYNERGISTIC SOLAR ENERGY HARVESTING SOLUTIONS

Nonrenewable energy resources, such as coal and fossil fuels, wreak havoc on the environment and public health. Further, their price volatility, associated geopolitical tensions, and finite supply restrict these sources from being a long-term solution. Renewable sources, including solar, geothermal, and wind power, present a sustainable approach to meet the world's energy demands. Solar energy is the most abundant among these resources and is the fastest growing renewable energy worldwide. The solar research community focuses on the challenges of enhancing efficiency, reducing costs, improving accessibility, and developing innovative technologies to maximize solar radiation harvesting.

In this work, we design two projects to address these challenges. The first project is a photovoltaic/thermal (PV/T) system. Materials are specifically chosen to be low cost and widely produced in order to promote accessibility. Our PV/T system combines thermal and infrared collection with visible light PVs. We show our 3D system can collect more daylight over a 12-hour period than a traditional planar PV. A prototype PV/T unit, with ethylene glycol as the thermal fluid, generated 1074 J of heat energy over a 30-minute period. This heat energy resulted from the synergistic relationship of the thermal collector pulling the heat away from the PV, keeping it cool and able to maintain performance.

Through a process of spectral splitting, the demonstrator above is expanded to include UV capture more effectively. From this an overall system efficiency of 73.1% was achieved. Importantly, this performance improvement was realized using a set of organic dyes which are widely available and inexpensive.

The second solar energy harvesting project incorporates thermoelectrics as a thermal capture mechanism. Using existing thin film thermoelectric platforms, this work focused on optimizing robust appliques of cost-effective thermoelectrics. This was done by developing new doping routes for the thermoelectric thin films. Ultimately the thermoelectric can be adhered to PVs to provide a path for heat to escape the PV, while generating additional electrical energy in the process.

Lindsey Gray
WFU Graduate Student
Physics Department
Wake Forest University

11 am - ZSR 404
Reception to follow a successful defense - Olin Lobby

14	Fri: 02/16/2024	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/19/2024
15	Mon: 02/19/2024	Chap. 6	Maxwell's Equations	#14	02/26/2024
16	Wed: 02/21/2024	Chap. 6	Electromagnetic energy and forces	#15	02/26/2024
17	Fri: 02/23/2024	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/26/2024	Chap. 7	Electromagnetic plane waves		
19	Wed: 02/28/2024	Chap. 7	Optical effects of refractive indices		
20	Fri: 03/01/2024	Chap. 1-7	Review		
21	Mon: 03/04/2024	Chap. 8	Short lectures on waveguides	Exam	
22	Wed: 03/06/2024	Chap. 8	Short lectures on waveguides	Exam	
23	Fri: 03/08/2024	Chap. 8	Short lectures on waveguides	Exam	
	Mon: 03/11/2024	No class	*Spring Break*		
	Wed: 03/13/2024	No class	*Spring Break*		
	Fri: 03/15/2024	No class	*Spring Break*		
24	Mon: 03/18/2024	Chap. 9	Radiation from localized oscillating sources		

PHY 712 – Problem Set #15

Assigned: 02/21/2024   Due: 02/26/2024

This problem is related to ideas presented in Chapter 6 of **Jackson**.

1. In deriving the Liénard Wiechert potentials, the retarded time

$$t_r = t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

was introduced. Note that this expression for the retarded time t_r may be difficult to evaluate in practice since the right hand side of the equation also depends on t_r. Here $\mathbf{r}, t$ represent the position and time at which the field is measured, and $\mathbf{R}_q(t')$ represents the trajectory of the charged particle as a function of its time denoted by t'. The velocity of the particle is given by

$$\mathbf{v} \equiv \mathbf{v}(t') \equiv \frac{d\mathbf{R}_q(t')}{dt'}.$$

Demonstrate the following identities:

(a)

$$\frac{\partial t_r}{\partial t} = \frac{1}{1 - \frac{\mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}}$$

(b)

$$-c\nabla t_r = \frac{\frac{\mathbf{r}-\mathbf{R}_q(t_r)}{|\mathbf{r}-\mathbf{R}_q(t_r)|}}{1 - \frac{\mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}}$$

Your questions:

From Gabby: If possible, for review on Wednesday, could we go over Green's functions and the process of how to derive a function we would need for a particular problem? I know we mainly covered it in electrostatics, but we didn't get to practice it too much in the HWs so I am still a bit confused.

Comment: This is a very good suggestion. For today, perhaps we can discuss the properties of the Green's function and how it works. The general construction of Green's functions for a particular differential operator is a course in itself that we have discussed in PHY 711 and 712. We should definitely include some of those points in the mid term review.

Slide from Lecture 15

Solution of Maxwell's equations in the Lorentz gauge

Let Ψ represent Φ, A_x, A_y, A_z Let f represent ρ, J_x, J_y, J_z

$$\nabla^2\Psi(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{r},t)}{\partial t^2}=-4\pi f(\mathbf{r},t)$$

Green's function:

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t')=-4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t)=\Psi_{f=0}(\mathbf{r},t)+\int d^3r'\int dt' G(\mathbf{r},t;\mathbf{r}',t')f(\mathbf{r}',t')$$

Operationally, $G(\mathbf{r},t;\mathbf{r}',t')$ is the inverse of the differential $\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$

Checking:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\mathbf{r},t) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

$$= \quad 0 \quad + \int d^3r' \int dt' \left(-4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')\right) f(\mathbf{r}',t')$$

$$= \quad -4\pi f(\mathbf{r},t)$$

For the case of isotropic boundary values at infinity:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)$$

"Proof" involved several steps which we can review at a later time.

Solution of Maxwell's equations in the Lorentz gauge – Review from previous lecture --

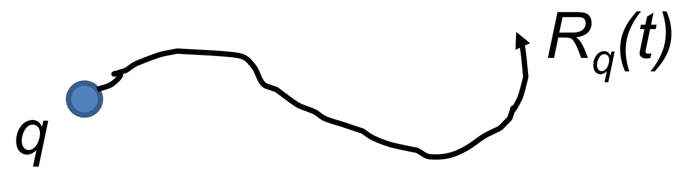
Liénard-Wiechert potentials and fields --

Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $R_q(t)$.

$$\text{Charge density: } \rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$$

$$\text{Current density: } \mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)), \quad \text{where} \quad \dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$$



Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\Big(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\Big)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',\mathrm{t'})}{|\mathbf{r}-\mathbf{r}'|}\delta\Big(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\Big).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

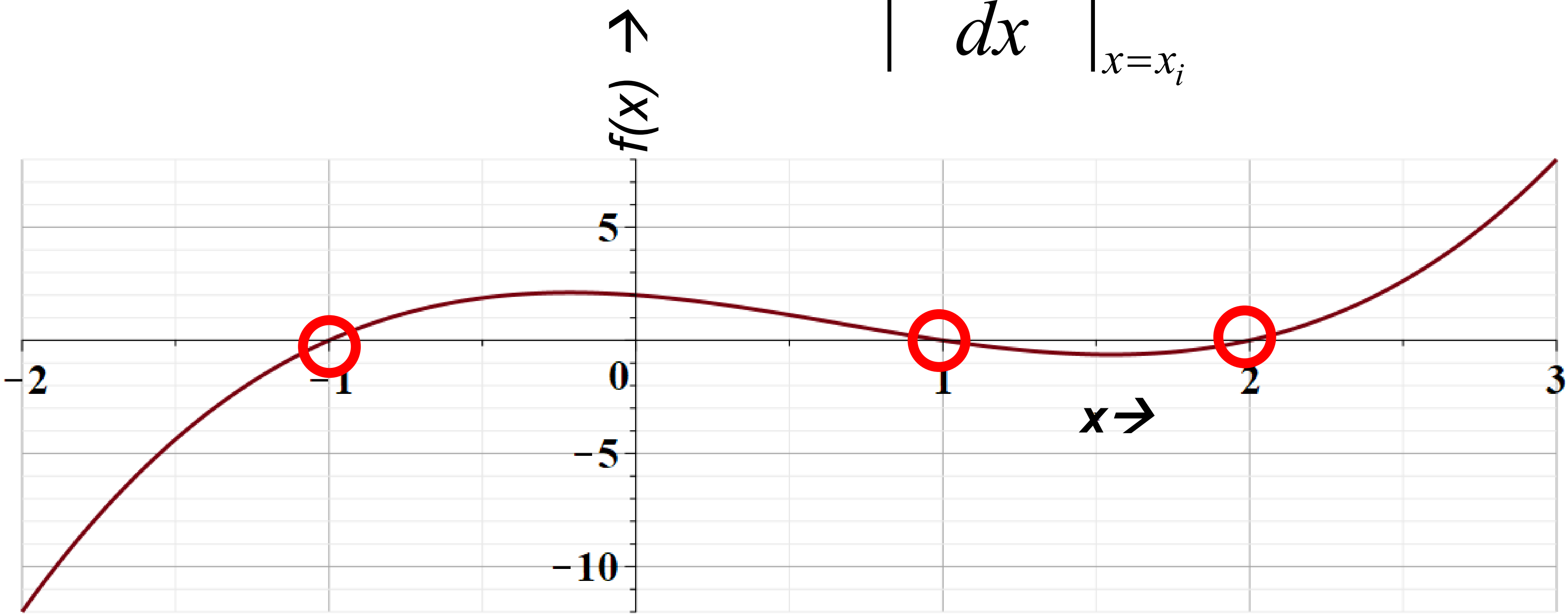
$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\Big(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\Big) = \frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}-\ \mathbf{R}_q(t_r)|}{c}.$$

Comment about delta functions -- See Pg. 26 in **Jackson**

$$\int_{-\infty}^{\infty} dx\ \Psi(x)\delta(f(x)) = \sum_i \frac{\Psi(x_i)}{\left|\frac{df(x)}{dx}\right|_{x=x_i}}$$



Some details --

$$\delta\Big(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\Big) = \delta(t' - t_r) \qquad t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

$$\frac{\partial t_r}{\partial t} = 1 + \frac{\Big(\mathbf{r} - \mathbf{R}_q(t_r)\Big) \cdot \dfrac{d\mathbf{R}_q(t_r)}{dt_r}}{c\,|\mathbf{r} - \mathbf{R}_q(t_r)|} \frac{\partial t_r}{\partial t}$$

Using notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r) \quad \mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$

$$\Rightarrow \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \dfrac{\mathbf{v} \cdot \mathbf{R}}{c}\right)}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R}\equiv\mathbf{r}-\mathbf{R}_q(t_r)$ $\quad t_r\equiv t-\dfrac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$

$$\mathbf{v}\equiv\dot{\mathbf{R}}_q(t_r),$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1-\frac{v^2}{c^2}\right)-\frac{v}{c}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)+\mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2}-\frac{\mathbf{v}\cdot\mathbf{R}}{Rc}-\frac{\dot{\mathbf{v}}\cdot R}{c^2}\right)-\frac{\dot{\mathbf{v}}R}{c^2}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\left(\mathbf{R}\times\left\{\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right)-\frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

Back to general case --

Maxwell's equations

$$\text{Coulomb's law:} \qquad \nabla \cdot \mathbf{D} = \rho_{free}$$

$$\text{Ampere-Maxwell's law:} \qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$$

$$\text{Faraday's law:} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles:} \qquad \nabla \cdot \mathbf{B} = 0$$

Energy analysis of electromagnetic fields and sources

Rate of work done on source $\mathbf{J}(\mathbf{r},t)$ by electromagnetic field:

$$\frac{dW_{mech}}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}_{free}$$

Expressing source current in terms of fields it produces:

$$\frac{dW_{mech}}{dt} = \int d^3r \;\; \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dW_{mech}}{dt} = \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}_{free} = \int d^3r \;\; \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

$$= -\int d^3r \;\; \left(\nabla \cdot \left(\mathbf{E} \times \mathbf{H} \right) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right)$$

Let $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$ "Poynting vector"

$$u \equiv \frac{1}{2} \left(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} \right)$$ energy density

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$$

Assuming that $\mathbf{D} = \epsilon \mathbf{E}$ and that $\mathbf{B} = \mu \mathbf{H}$

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}_{free}$$

Electromagnetic energy density: $\quad u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$

$$E_{field} \equiv \int d^3r \;\; u\left(\mathbf{r},t\right)$$

Poynting vector: $\quad \mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \;\; \nabla\cdot\mathbf{S}\left(\mathbf{r},t\right) = -\oint d^2r \; \hat{\mathbf{r}}\cdot\mathbf{S}\left(\mathbf{r},t\right)$$

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \quad \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)$$

Follows by analogy with Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \quad \left(\mathbf{E}\times\mathbf{B}\right)$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r \, \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j + c^2 B_i B_j - \delta_{ij} \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

Summary -- By considering a complete system involving self-contained sources and fields, we examined the energy and force relationships and introduce energy and force equivalents of the electromagnetic fields

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$

Poynting vector: $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

Differential relationship: $\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$

Maxwell stress tensor (for vacuum case):

$$T_{ij} \equiv \varepsilon_0\left(E_iE_j + c^2B_iB_j - \delta_{ij}\frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

Integral relationships:

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}_{free}$$

$$E_{field} \equiv \int d^3r \;\; u(\mathbf{r},t)$$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \;\; \nabla \cdot \mathbf{S}(\mathbf{r},t) = -\oint d^2r \; \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r},t)$$

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt} \right)_i = \sum_j \int d^3r \, \frac{\partial T_{ij}}{\partial r_j}$$

Comment on treatment of time-harmonic fields

Fourier transformation in time domain :

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \widetilde{\mathbf{E}}(\mathbf{r},\omega) \, e^{-i\omega t}$$

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt \, \mathbf{E}(\mathbf{r},t) \, e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r},t)$ is real $\Rightarrow \widetilde{\mathbf{E}}(\mathbf{r},\omega) = \widetilde{\mathbf{E}}^*(\mathbf{r},-\omega)$

These relations and the notion of the superposition principle, lead to the common treatment:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) e^{-i\omega t} \right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega) e^{i\omega t} \right)$$

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Equations	in time domain	in frequency domain
Coulomb's law:	$\nabla \cdot \mathbf{D} = \rho_{free}$	$\nabla \cdot \widetilde{\mathbf{D}} = \widetilde{\rho}_{free}$
Ampere-Maxwell's law:	$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla \times \widetilde{\mathbf{H}} + i\omega\widetilde{\mathbf{D}} = \widetilde{\mathbf{J}}_{free}$
Faraday's law:	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \widetilde{\mathbf{E}} - i\omega\widetilde{\mathbf{B}} = 0$
No magnetic monopoles:	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \widetilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Poynting vector: $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)\times\left(\widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}^*(\mathbf{r},\omega) + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}(\mathbf{r},\omega)\right)$$

$$+\frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-2i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{2i\omega t}\right)$$

$$\left\langle \mathbf{S}(\mathbf{r},t)\right\rangle_{t\,\text{avg}} = \Re\left(\frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Maxwell's equations

For linear isotropic media -- $\mathbf{D} = \varepsilon\mathbf{E}; \quad \mathbf{B} = \mu\mathbf{H}$

and no sources:

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

$$\text{Coulomb's law:} \qquad \nabla \cdot \mathbf{E} = 0$$

$$\text{Ampere-Maxwell's law:} \qquad \nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\text{Faraday's law:} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles:} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Analysis of Maxwell's equations without sources -- continued:
Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where} \quad v^2 \equiv c^2 \frac{\mu_0 \varepsilon_0}{\mu \varepsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t)=\Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

from Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

$$\Rightarrow \mathbf{B}_0=\frac{\mathbf{k}\times\mathbf{E}_0}{\omega}=\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}$$

also note: $\hat{\mathbf{k}}\cdot\mathbf{E}_0=0$ and $\hat{\mathbf{k}}\cdot\mathbf{B}_0=0$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves:

$$\langle\mathbf{S}\rangle_{avg} = \frac{1}{2}\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\times\frac{1}{\mu}\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Note that:

$$\mathbf{E}_0\times\left(\hat{\mathbf{k}}\times\mathbf{E}_0\right) = \hat{\mathbf{k}}\left(\mathbf{E}_0\cdot\mathbf{E}_0\right) - \mathbf{E}_0\left(\hat{\mathbf{k}}\cdot\mathbf{E}_0\right.$$

$$= \hat{\mathbf{k}}|\mathbf{E}_0|^2$$

Analysis of Maxwell's equations without sources -- continued:

Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{4}\Re\left(\varepsilon\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right) +$$

$$\frac{1}{4}\Re\left(\frac{1}{\mu}\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$