PHY 712 Electrodynamics
10-10:50 AM in Olin 103

Notes for Lecture 18:

Continue reading Chapter 7 (Sec. 7.5,7.10 in JDJ)

1. **Real and imaginary contributions to electromagnetic response**
2. **Frequency dependence of dielectric materials; Drude model**
3. **Kramers-Kronig relationships**

9	Mon: 02/05/2024	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/12/2024
10	Wed: 02/07/2024	Chap. 4	Dipoles and Dielectrics	#9	02/12/2024
11	Fri: 02/09/2024	Chap. 4	Dipoles and Dielectrics	#10	02/12/2024
12	Mon: 02/12/2024	Chap. 5	Magnetostatics	#11	02/19/2024
13	Wed: 02/14/2024	Chap. 5	Magnetic dipoles and hyperfine interaction	#12	02/19/2024
14	Fri: 02/16/2024	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/19/2024
15	Mon: 02/19/2024	Chap. 6	Maxwell's Equations	#14	02/26/2024
16	Wed: 02/21/2024	Chap. 6	Electromagnetic energy and forces	#15	02/26/2024
17	Fri: 02/23/2024	Chap. 7	Electromagnetic plane waves	#16	02/26/2024
18	Mon: 02/26/2024	Chap. 7	Electromagnetic plane waves	#17	03/01/2024
19	Wed: 02/28/2024	Chap. 7	Optical effects of refractive indices	#18	03/01/2024
20	Fri: 03/01/2024	Chap. 1-7	Review		
21	Mon: 03/04/2024	Chap. 8	Short lectures on waveguides	Exam	
22	Wed: 03/06/2024	Chap. 8	Short lectures on waveguides	Exam	
23	Fri: 03/08/2024	Chap. 8	Short lectures on waveguides	Exam	
	Mon: 03/11/2024	No class	*Spring Break*		
	Wed: 03/13/2024	No class	*Spring Break*		
	Fri: 03/15/2024	No class	*Spring Break*		
24	Mon: 03/18/2024	Chap. 9	Radiation from localized oscillating sources		

PHY 712 -- Assignment #17

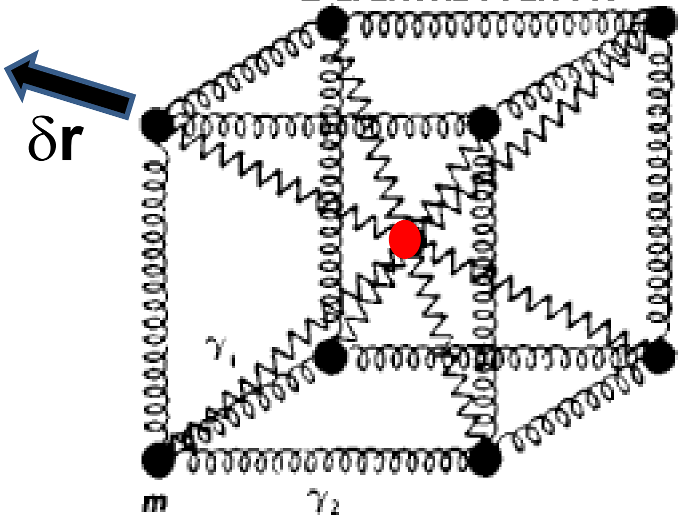
Assigned: 2/26/2024 Due: 3/01/2024

Continue reading Chapter 7, particularly Sec. 7.10 in **Jackson** .

1. Work problem 7.22 (a) in **Jackson**. In addition to the analytic results, plot the real and imaginary functions as a function of ω for your favorite values of the constants.

Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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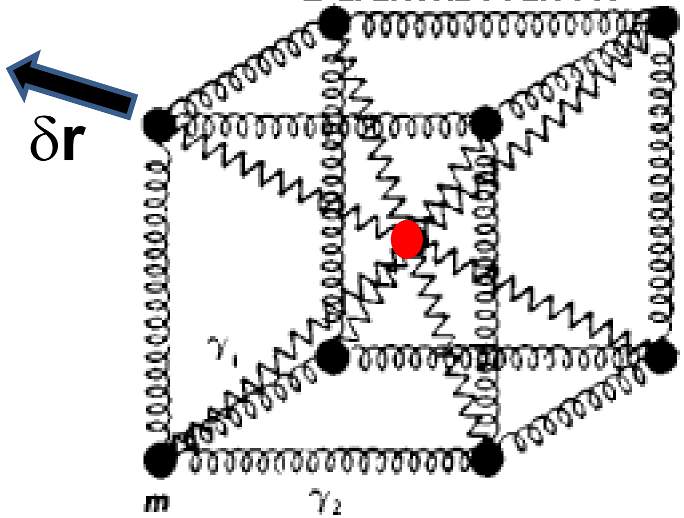
$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

Note that:

- $\gamma > 0$ represents dissipation of energy.
- ω_0 represents the natural frequency of the vibration; $\omega_0=0$ would represent a free (unbound) particle

Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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$$m\delta\,\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\,\mathbf{r} - m\gamma\delta\,\dot{\mathbf{r}}$$

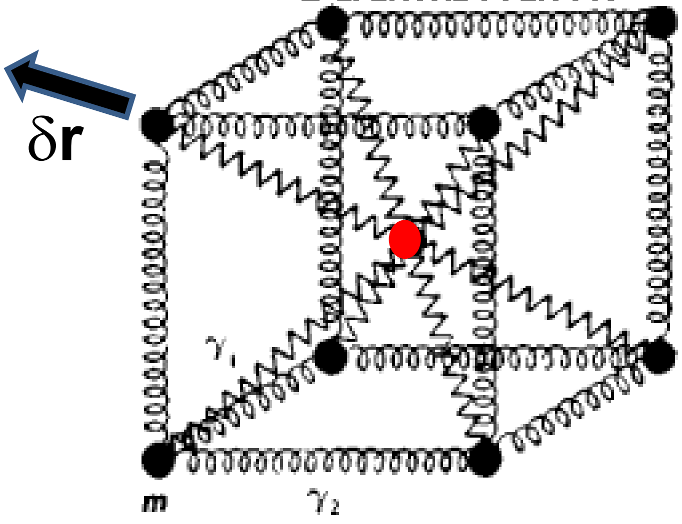
For $\delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}$, $\qquad \delta\mathbf{r}_0 = \dfrac{q\mathbf{E}_0}{m}\dfrac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Induced dipole:

$$\mathbf{p} = q\ \delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}e^{-i\omega t}$$

Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

Displacement field:

$$\mathbf{D} = \varepsilon\mathbf{E} \quad = \varepsilon_0\mathbf{E} + \mathbf{P}$$

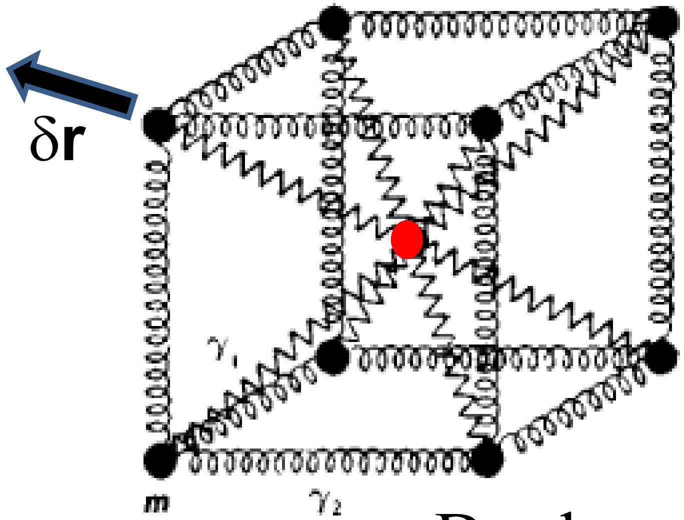
$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i) \approx N\sum_i f_i \; \mathbf{p}_i$$

$N \equiv$ number of dipoles/volume

$f_i \equiv$ fraction of type i dipoles

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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Drude model expression for permittivity:

$$\mathbf{D} = \varepsilon \mathbf{E} \quad = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + N \sum_i f_i \ \mathbf{p}_i$$

$$\mathbf{p}_i = q_i \ \delta \mathbf{r} = \frac{q_i^{\ 2} \mathbf{E}_0}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} e^{-i\omega t}$$

$$\varepsilon \mathbf{E} \ = \ \varepsilon_0 \mathbf{E}_0 e^{-i\omega t} \left(1 + N \sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} \right)$$

Drude model dielectric function:

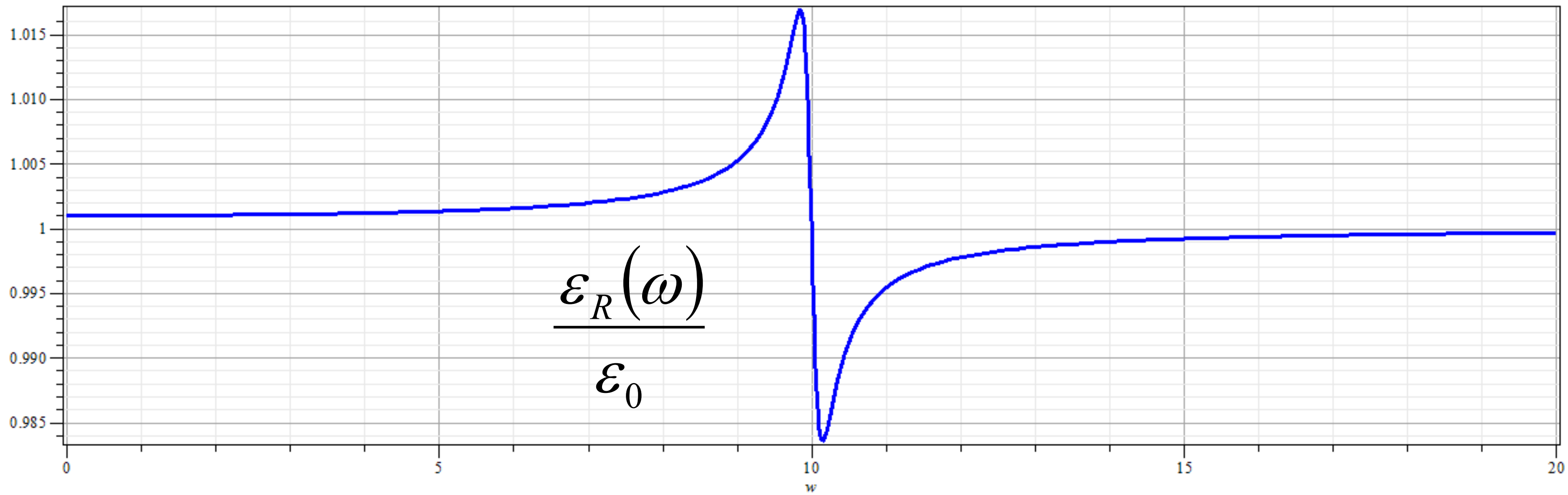
$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

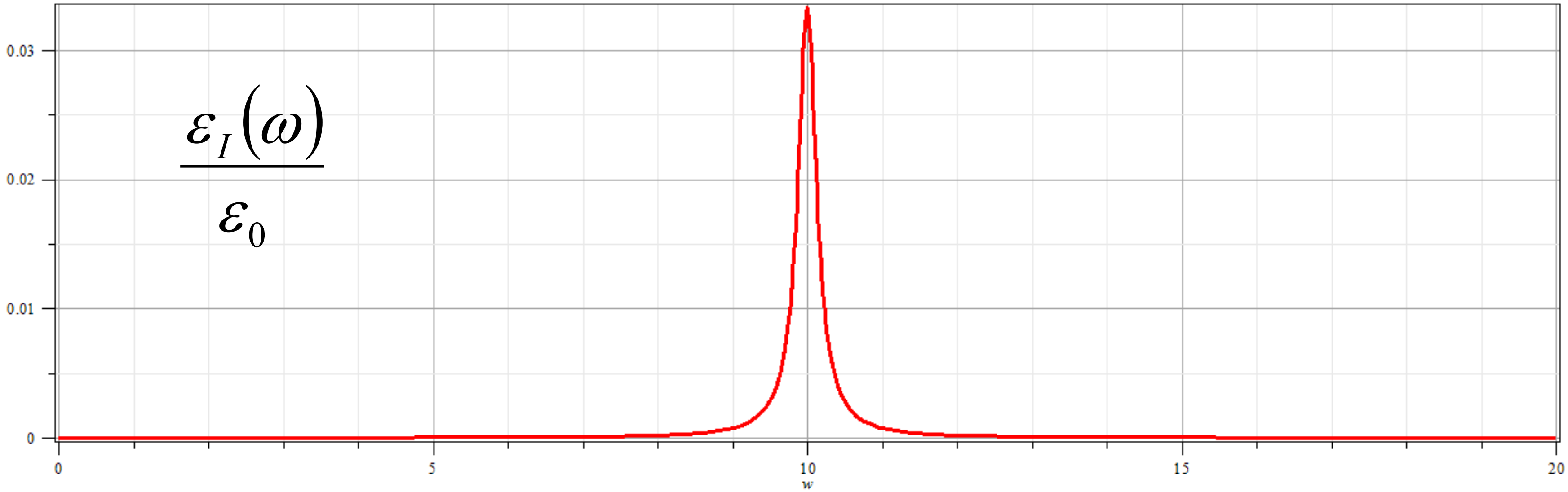
$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{\omega_i^2 - \omega^2}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^{\,2}}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{\omega\gamma_i}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^{\,2}}$$

Drude model dielectric function:





Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega >> \omega_i$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\underbrace{\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \right)}_{\equiv\, \omega_P^2}$$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega_0 = 0$ (representing a free particle of charge q_0, mass m_0

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_{i>0} f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} + iNf_0 \frac{q_0^{\,2}}{\varepsilon_0 m_0} \frac{1}{\omega(\gamma_0 - i\omega)}$$

$$\equiv \frac{\varepsilon_b(\omega)}{\varepsilon_0} + i\frac{\sigma(\omega)}{\varepsilon_0\omega}$$

Some details:

$$\mathbf{D} = \varepsilon_b \mathbf{E} \qquad\qquad \mathbf{J} = \sigma\mathbf{E}$$

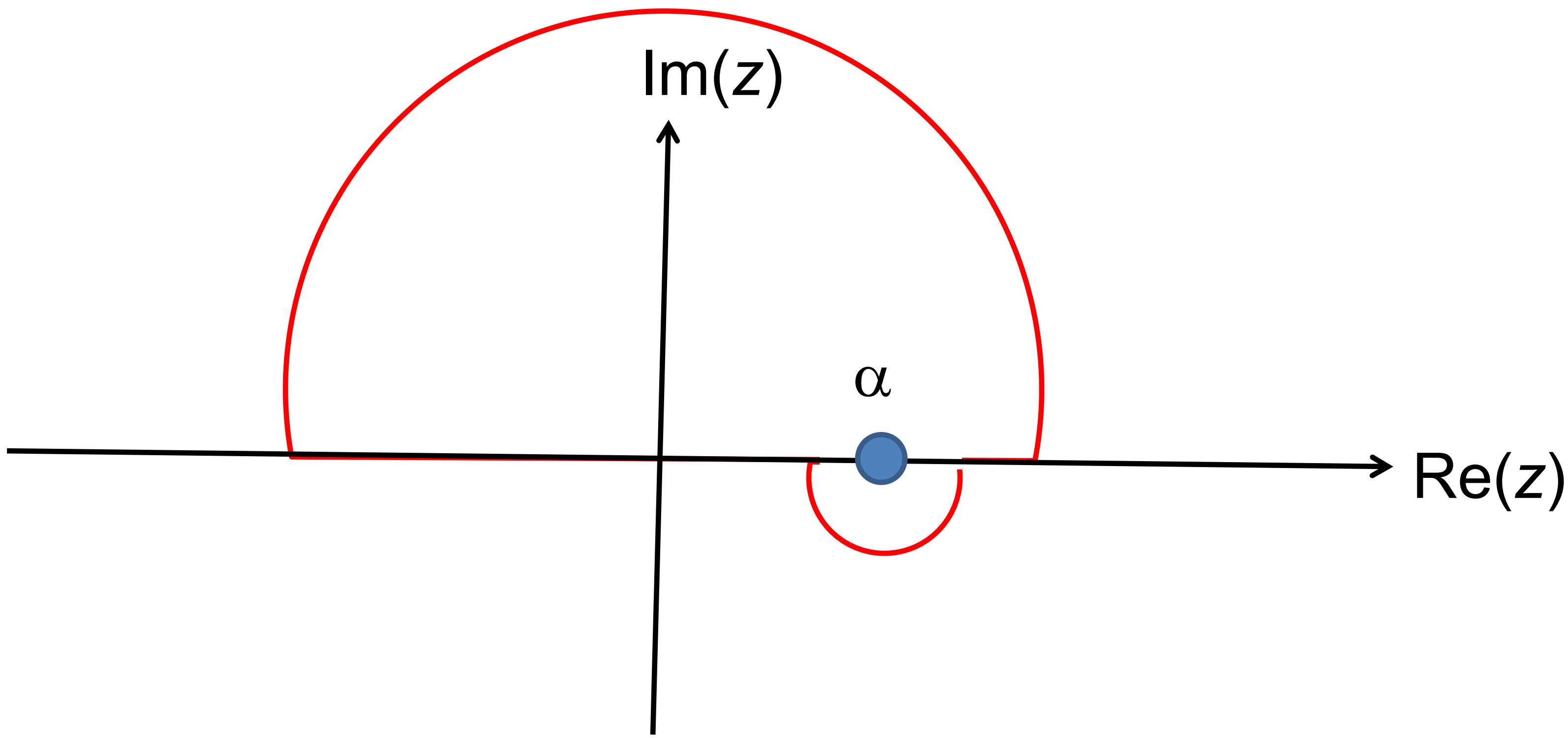
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = (\sigma - i\omega\varepsilon_b)\mathbf{E} = \varepsilon\frac{\partial \mathbf{E}}{\partial t} = -i\omega\left(\varepsilon_b + \frac{i\sigma}{\omega}\right)\mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0 \frac{q_0^{\,2}}{m_0} \frac{1}{(\gamma_0 - i\omega)}$$

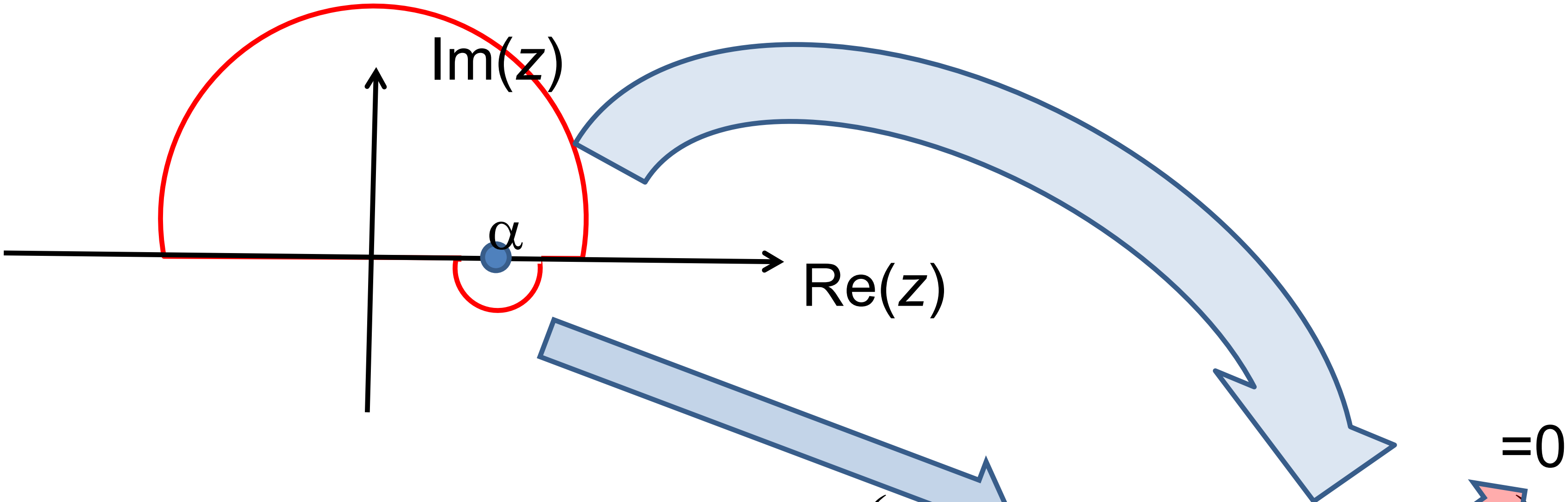
Analytic properties of the dielectric function (in the Drude model or from "first principles" -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz\, f(z) = 0 \qquad\qquad f(\alpha) = \frac{1}{2\pi i} \oint_{includes\ \alpha} dz\, \frac{f(z)}{z - \alpha}$$



Kramers-Kronig transform -- continued



$$f(\alpha)=\frac{1}{2\pi i}\oint_{includes\ \alpha} dz\ \frac{f(z)}{z-\alpha}=\frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} dz_R\ \frac{f(z_R)}{z_R-\alpha}+\int_{rest} dz\ \frac{f(z)}{z-\alpha}\right)$$

$$f(\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz_R\frac{f(z_R)}{z_R-\alpha}=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\frac{f(z_R)}{z_R-\alpha}+\frac{1}{2}f(\alpha)$$

Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \, \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

Suppose $f(z_R) = f_R(z_R) + i f_I(z_R)$:

$$\Rightarrow \frac{1}{2}\left(f_R(\alpha) + i f_I(\alpha)\right) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \, \frac{f_R(z_R) + i f_I(z_R)}{z_R - \alpha}$$

$$\Rightarrow f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \, \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \, \frac{f_R(z_R)}{z_R - \alpha}$$

Kramers-Kronig transform -- continued

$$f_R(\alpha)=\frac{1}{\pi}P\int_{-\infty}^{\infty}dz_R\;\frac{f_I(z_R)}{z_R\text{-}\alpha}$$

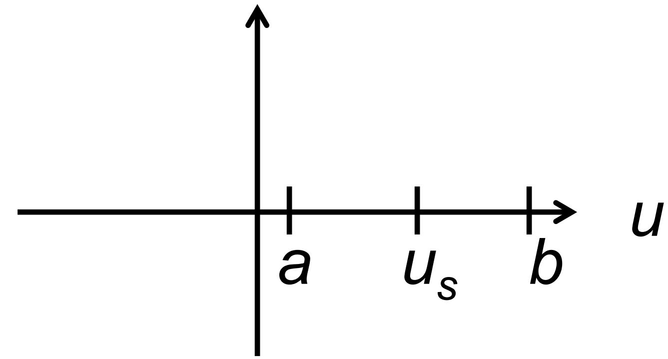
$$f_I(\alpha)=-\frac{1}{\pi}P\int_{-\infty}^{\infty}dz_R\;\frac{f_R(z_R)}{z_R\text{-}\alpha}$$

This Kramers-Kronig transform is useful for the dielectric function

when $\quad f(z_R)\Rightarrow\frac{\varepsilon(\omega)}{\varepsilon_0}-1$

Must show that: 1. $f(z)$ is analytic for $z_I>0$

2. $f(z)$ vanishes for $z\to\infty$



Some practical considerations

Principal parts integration :

$$P\int_a^b du\; g(u) = \lim_{\upsilon \to 0}\left(\int_a^{u_s-\upsilon} du\; g(u) + \int_{u_s+\upsilon}^{b} du\; g(u) \right)$$

Example :

$$P\int_a^b du\; \frac{1}{u\text{-}u_s} = \lim_{\upsilon \to 0}\left(\int_a^{u_s-\upsilon} du\, \frac{1}{u\text{-}u_s} + \int_{u_s+\upsilon}^{b} du\, \frac{1}{u\text{-}u_s} \right)$$

$$= \lim_{\upsilon \to 0}\left(\ln\left(\frac{\upsilon}{u_s - a}\right) + \ln\left(\frac{b-u_s}{\upsilon}\right)\right) = \ln\left(\frac{b-u_s}{u_s-a}\right)$$

More practical considerations

For dielectric function $\varepsilon(\omega)$:

$$\varepsilon(-\omega) = \varepsilon^*(\omega)$$

$$\Rightarrow \varepsilon_R(-\omega) = \varepsilon_R(\omega)$$

$$\Rightarrow \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Analytic properties the dielectric function which justify

the treatment of $f(z) \Rightarrow \dfrac{\varepsilon(z)}{\varepsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_I > 0$

2. $f(z)$ vanishes for $z \to \infty$ (for $z_I > 0$)

Analysis for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $|z| >> \omega_i$

$$f(z) \approx -\frac{1}{z^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right) \Rightarrow \text{vanishes at large } z$$

Analysis for Drude model dielectric function – continued -- Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^{\ 2} - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

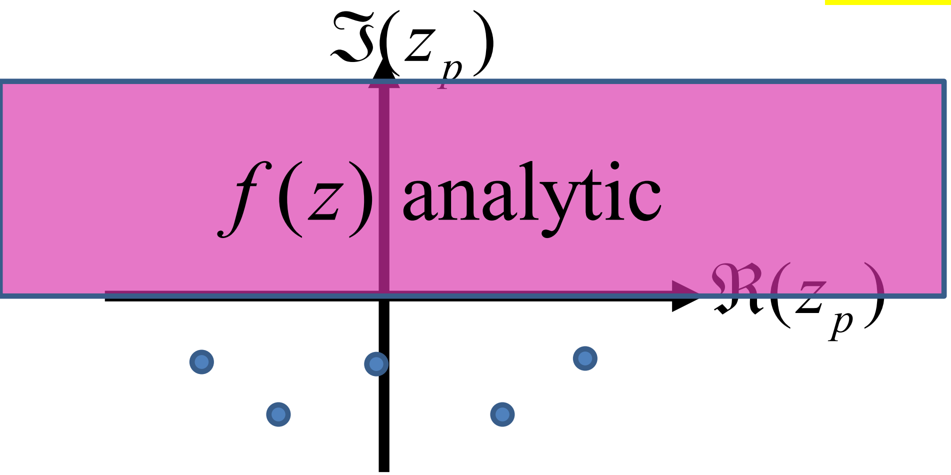
Note that $\Im(z_P) \leq 0 \quad \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \Rightarrow$ $f(z)$ is analytic for $\Im(z_P) > 0$



Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with} \quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^\infty d\tau \; G(\tau)\mathbf{E}(\mathbf{r},t-\tau) \right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^\infty d\tau \; G(\tau) e^{i\omega\tau}$$

Some details: Consider a convolution integral such as

$f(t) = \int_{-\infty}^{\infty} g(t')h(t-t')dt'$ where the functions $f(t), g(t),$ and $h(t)$

are all well-defined functions with Fourier transforms such as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t')e^{i\omega t'}dt' \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t}d\omega$$

It follows that: $\tilde{f}(\omega) = \tilde{g}(\omega)\tilde{h}(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^\infty d\tau \; G(\tau) \mathbf{E}(\mathbf{r},t-\tau) \right\}$$

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right) e^{-i\omega\tau} d\omega \qquad \tilde{G}(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^\infty d\tau \; G(\tau) e^{i\omega\tau}$$

For $\displaystyle \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^{\,2}}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$

$$G(\tau) = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^{\,2}}{m_i} e^{-\gamma_i \tau/2} \frac{\sin(\nu_i \tau)}{\nu_i} \Theta(\tau)$$

where $\nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2 / 4}$

Some details

$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}-1\right)e^{-i\omega\tau}d\omega=\frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz$$

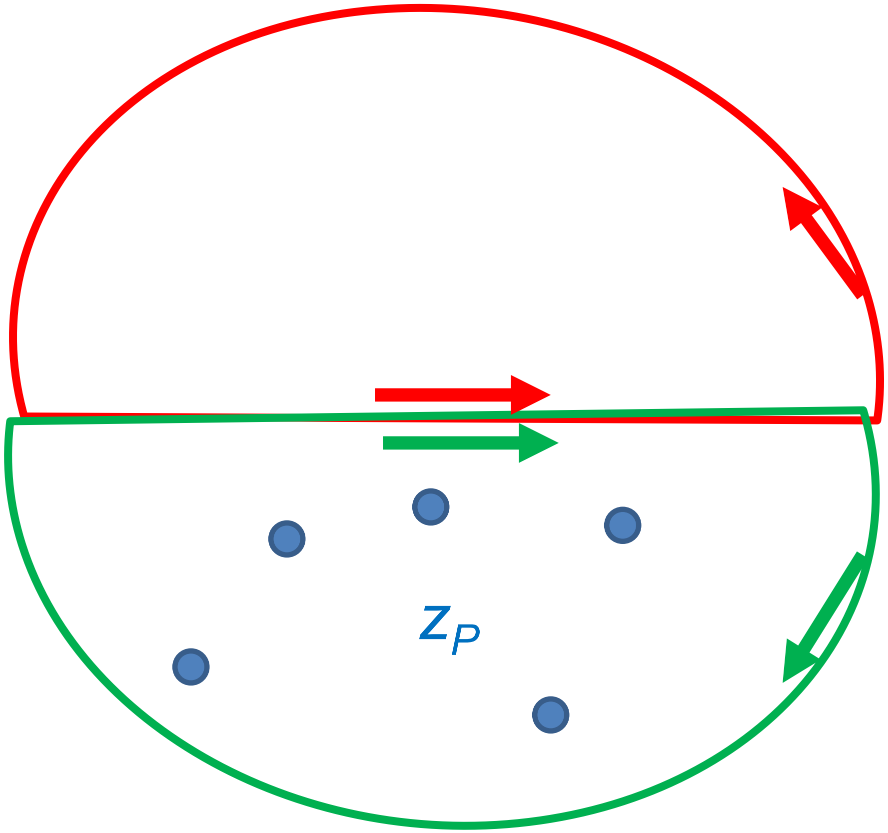
Let $f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-\ 1=N\sum_i f_i\frac{q_i^{\ 2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2-z_P^{\ 2}-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}\quad\text{or}\quad z_P=-i\left(\frac{\gamma_i}{2}\pm\sqrt{\left(\frac{\gamma_i}{2}\right)^2-\omega_i^2}\right)$$

$$G(\tau) = \frac{1}{2\pi} \oint f(z) e^{-iz\tau} dz = i \sum_P \text{Res}(z_P)$$

Note that: $e^{-iz\tau} = e^{-iz_R\tau} e^{z_I\tau}$



Valid contour for $\tau < 0$

$G(\tau) = 0$ for $\tau < 0$

Valid contour for $\tau > 0$

$$G(\tau) = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i \tau/2} \frac{\sin(\nu_i \tau)}{\nu_i}$$

$$G(\tau)=\frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz = i\sum_P \mathrm{Res}(z_P)$$

Let $f\left(z\right)=\frac{\varepsilon\left(z\right)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^{\;2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$

$f\left(z\right)$ has poles z_P at $\omega_i^2-z_P^{\;2}-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}\quad\text{or}\quad z_P=-i\left(\frac{\gamma_i}{2}\pm\sqrt{\left(\frac{\gamma_i}{2}\right)^2-\omega_i^2}\right)$$

$$G(\tau)=\frac{N}{\varepsilon_0}\sum_i f_i\frac{q_i^{\;2}}{m_i}e^{-\gamma_i\tau/2}\frac{\sin\left(\nu_i\tau\right)}{\nu_i}\Theta(\tau)$$

where $\nu_i\equiv\sqrt{\omega_i^2-\gamma_i^2/4}$ assuming $\omega_i^2-\gamma_i^2/4\geq 0$