PHY 712 Electrodynamics
10-10:50 AM in Olin 103

Discussion for Lecture 20:

Review of Chapters 1-7

1. **Comment on what to expect with the take- home exam**
2. **Main topics covered**
3. **Some details of past HW problems**

16	Wed: 02/21/2024	Chap. 6	Electromagnetic energy and forces	#15	02/26/2024
17	Fri: 02/23/2024	Chap. 7	Electromagnetic plane waves	#16	02/26/2024
18	Mon: 02/26/2024	Chap. 7	Electromagnetic plane waves	#17	03/01/2024
19	Wed: 02/28/2024	Chap. 7	Optical effects of refractive indices	#18	03/01/2024
20	Fri: 03/01/2024	Chap. 1-7	Review		
21	Mon: 03/04/2024	Chap. 8	Short lectures on waveguides	Exam	
22	Wed: 03/06/2024	Chap. 8	Short lectures on waveguides	Exam	
23	Fri: 03/08/2024	Chap. 8	Short lectures on waveguides	Exam	
	Mon: 03/11/2024	No class	*Spring Break*		
	Wed: 03/13/2024	No class	*Spring Break*		
	Fri: 03/15/2024	No class	*Spring Break*		
24	Mon: 03/18/2024	Chap. 9	Radiation from localized oscillating sources		

For 3/04/2024-3/08/2024:

- Individual work on take home exam
- Shortened class lectures on Chapter 8 of **Jackson**

Motivation for giving/taking mid-term exam

1. Opportunity to review/solidify knowledge in the topic
2. Opportunity to practice problem solving techniques appropriate to the topic
3. Assessment of performance. Accordingly, the work you turn in must be your own (of course).
   - You are encouraged to consult with your instructor (but no one else!) if any questions arise about the exam questions
   - Extra credit awarded if you report errors/inconsistencies/ambiguities in the exam questions

Instructions on exam:

Note: This is a ``take-home'' exam which can be turned in any time before 4 PM Friday, March 8, 2024. In addition to each worked problem, please attach ALL Maple (or Mathematica, Matlab, Wolfram, etc.), work sheets as well as a full list of resources used to complete these problems. It is assumed that all work on the exam is performed under the guidelines of the honor code. In particular, if you have any questions about the material, you may consult with the instructor ***but no one else***. For grading purposes, each question in multi-part problems are worth equal weight. Credit will be assigned on the basis of both the logical steps of the solution and on the correct answer.

More advice about exam –

- It is important that the instructor is able to read your work and understand your reasoning.
- Since you will be using Maple or Mathematica or ?? to evaluate some of your results, please include the software work (or snips of it) into your exam materials.
- Your exam paper does not need to be a work of art, but it does need to be readable. If you prefer to submit your exam paper electronically, that will be fine. (I may print it myself.)

More advice – accumulate trusted equations/mathematical relationships and know how to use them

Jackson pg. 783

Table 4 Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many SI or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.997 924 58), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.997\,924\,58 \times 4\pi \times 10^5$) and "9" is actually $10^{-16} c^2 = 8.987\,55 \ldots$. Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or SI units.

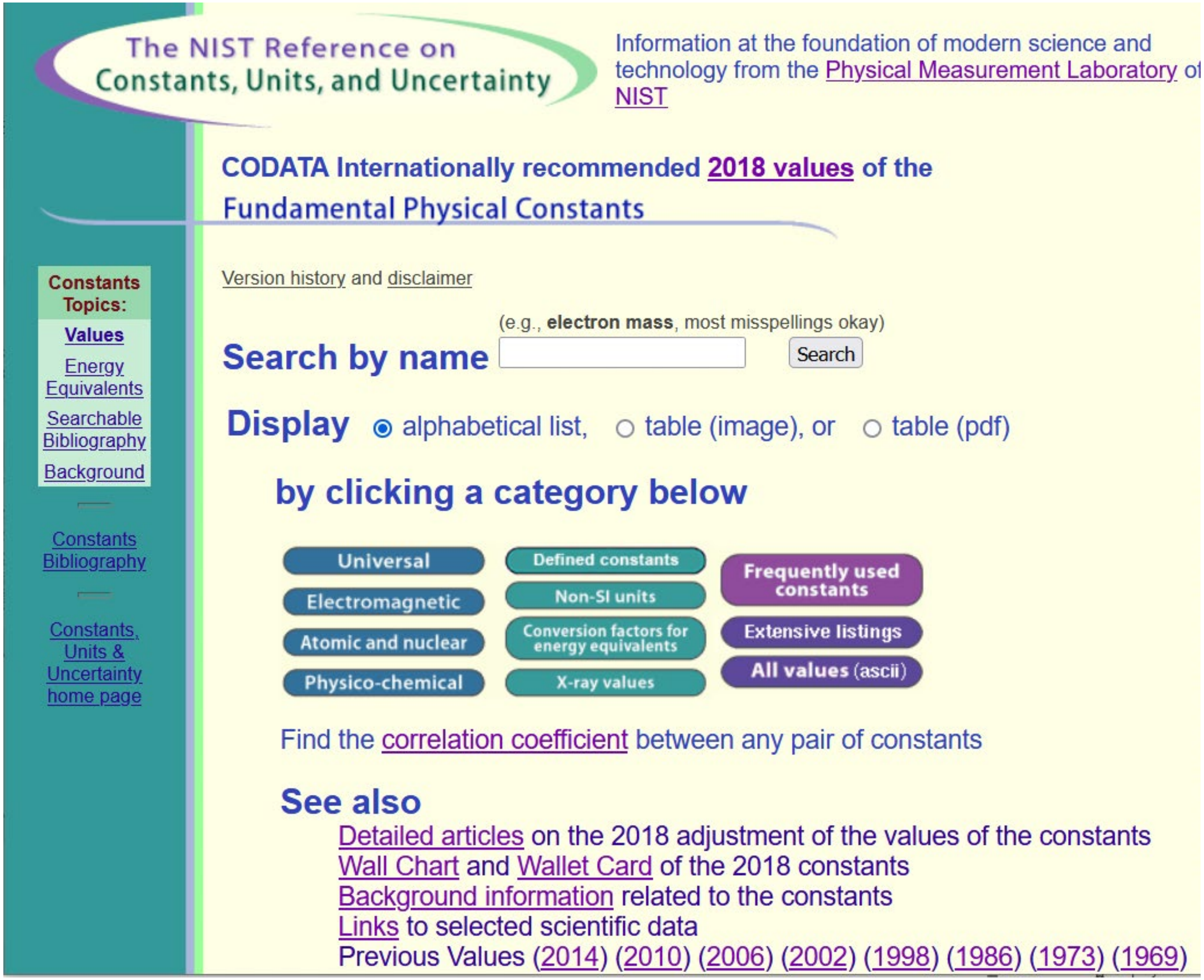
Jackson pg. 783

Physical Quantity	Symbol	SI		Gaussian
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (g)
Time	t	1 second (s)	1	second (s)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton (N)	10^5	dynes
Work / Energy	W / U	1 joule (J)	10^7	ergs
Power	P	1 watt (W)	10^7	ergs s^{-1}
Charge	q	1 coulomb (C)	3×10^9	statcoulombs
Charge density	ρ	1 C m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (A)	3×10^9	statamperes
Current density	J	1 A m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1} (Vm^{-1})	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt (V)	$\frac{1}{300}$	statvolt
Polarization	P	1 C m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 C m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 mho m^{-1}	9×10^9	s^{-1}
Resistance	R	1 ohm (Ω)	$\frac{1}{9} \times 10^{-11}$	s cm^{-1}
Capacitance	C	1 farad (F)	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber (Wb)	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla (T)	10^4	gauss (G)
Magnetic field	H	1 A m^{-1}	$4\pi \times 10^{-3}$	oersted (Oe)
Magnetization	M	1 A m^{-1}	10^{-3}	magnetic moment cm^{-3}
Inductance*	L	1 henry (H)	$\frac{1}{9} \times 10^{-11}$	

*There is some confusion about the unit of inductance in Gaussian units. This stems from the use by some authors of a modified system of Gaussian units in which current is measured in electromagnetic units, so that the connection between charge and current is $I_m = (1/c)(dq/dt)$. Since inductance is defined through the induced voltage $V = L(dI/dt)$ or the energy $U = \frac{1}{2}LI^2$, the choice of current defined in Section 2 means that our Gaussian unit of inductance is equal in magnitude and dimensions (t^2l^{-1}) to the electrostatic unit of inductance. The electromagnetic current I_m is related to our Gaussian current I by the relation $I_m = (1/c)I$. From the energy definition of inductance, we see that the electromagnetic inductance L_m is related to our Gaussian inductance L through $L_m = c^2L$. Thus L_m has the dimensions of length. The modified Gaussian system generally uses the electromagnetic unit of inductance, as well as current. Then the voltage relation reads $V = (L_m/c)(dI_m/dt)$. The numerical connection between units of inductance is

$$1 \text{ henry} = \frac{1}{9} \times 10^{-11} \text{ Gaussian (es) unit} = 10^9 \text{ emu}$$

Source for standard measurements –
https://physics.nist.gov/cuu/Constants/index.html



Vector relations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$
$$\nabla\times\nabla\psi = 0$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$
$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$
$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$
$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$

Comment on cartesian unit vectors versus local (cylindrical or spherical) unit vectors

$$\hat{\mathbf{r}} = \sin\theta\ \cos\phi\ \hat{\mathbf{x}} + \sin\theta\ \sin\phi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\ \cos\phi\ \hat{\mathbf{x}} + \cos\theta\ \sin\phi\ \hat{\mathbf{y}} - \sin\theta\ \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\varphi}} = \ -\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}}$$

Note that $\nabla^2\mathbf{A} = \left(\nabla^2 A_x\right)\hat{\mathbf{x}} + \left(\nabla^2 A_y\right)\hat{\mathbf{y}} + \left(\nabla^2 A_z\right)\hat{\mathbf{z}}$

Also note that $\nabla^2 f(r) = \dfrac{\partial^2 f(r)}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial f(r)}{\partial r}$

Special functions -- many are described in Jackson
Additional source -- https://dlmf.nist.gov/

Digital Library of Mathematical Functions

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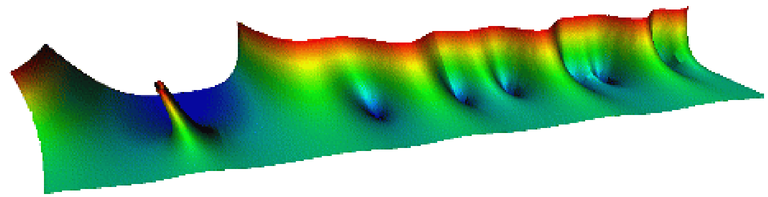


NIST Digital Library of Mathematical Functions

Project News

2022-03-15 *DLMF Update; Version 1.1.5*
2022-01-15 *DLMF Update; Version 1.1.4*
2021-09-15 *DLMF Update; Version 1.1.3*
2021-07-19 *Brian D. Sleeman, Associate Editor of the DLMF, dies at age 81*
More news

Basic equations of electrodynamics

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$	$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

More relationships

CGS (Gaussian)

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

MKS (SI)

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\epsilon \quad \Leftrightarrow \quad \epsilon / \epsilon_0$$

$$\mu \quad \Leftrightarrow \quad \mu / \mu_0$$

More SI relationships:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \qquad \mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}) \qquad \mathbf{B} = \mu\mathbf{H} \qquad \mathbf{B} = F(\mathbf{H})$$

for ferromagnet

More Gaussian relationships:

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \qquad \mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}) \qquad \mathbf{B} = \mu\mathbf{H} \qquad \mathbf{B} = F(\mathbf{H})$$

for ferromagnet

elementary charge: $e = 1.602176634 \times 10^{-19}$ C

(when using Gaussian units, charge is in “stat Coulombs”)

Energy and power (SI units)

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)$

Poynting vector: $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right)\equiv\frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}+\widetilde{\mathbf{E}}^{*}(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$\left\langle u\left(\mathbf{r},t\right)\right\rangle_{t\ \mathrm{avg}}=\frac{1}{4}\Re\left(\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\cdot\tilde{\mathbf{D}}^{*}(\mathbf{r},\omega)+\tilde{\mathbf{B}}(\mathbf{r},\omega)\cdot\tilde{\mathbf{H}}^{*}(\mathbf{r},\omega)\right)\right)$$

$$\left\langle \mathbf{S}\left(\mathbf{r},t\right)\right\rangle_{t\ \mathrm{avg}}=\frac{1}{2}\Re\left(\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^{*}(\mathbf{r},\omega)\right)\right)$$

Solution of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

Introduction of vector and scalar potentials:

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \qquad \text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Scalar and vector potentials continued:

$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$:

$$-\nabla^2 \Phi - \frac{\partial \left(\nabla \cdot \mathbf{A} \right)}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \left(\nabla \times \mathbf{A} \right) + \frac{1}{c^2} \left(\frac{\partial \left(\nabla \Phi \right)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Analysis of the scalar and vector potential equations:

$$-\nabla^2\Phi - \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times(\nabla\times\mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla\Phi)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Lorentz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Solution methods for scalar and vector potentials and their electrostatic and magnetostatic analogs:

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

In your "bag" of tricks:

- ❑ **Direct (analytic or numerical) solution of differential equations**
- ❑ **Solution by expanding in appropriate orthogonal functions**
- ❑ **Green's function techniques**
- ❑ **Solving Maxwell's equations**

How to choose most effective solution method --

- In general, Green's functions methods work well when source is contained in a finite region of space

Consider the electrostatic problem:

$$-\nabla^2\Phi = \rho / \varepsilon_0$$

Define: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Note that the Green's function is effectively the inverse of the differential operator and can be designed to effectly treat the boundary values as well.

How to construct and use Green's functions

- Starting with relevant physical equations (in this case, typically Maxwell's equations), reduce them so that you are working with a differential equation for single multivariable function.
- Construct a Green's function, ideally one which is both the inverse of the differential operator and also handles the boundary values. (Always check)
- Evaluate the integrals.
- Check that everything makes sense.

Methods for constructing Green's functions for second order differential equations

- Use two independent solutions of the homogeneous differential equation
- Orthogonal function expansion
- Combination of both

General procedure for constructing Green's function for one-dimensional system using 2 independent solutions of the homogeneous equations

Consider two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0$$

where $i = 1$ or 2. Let

$$G(x,x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>).$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

"Wronskian": $$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}.$$

Beautiful method; but only works in one dimension.

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}.$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x').$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x').$$

Orthogonal function expansion -- continued

Suppose the orthogonal functions satisfy an eigenvalue equation:

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

where the functions $u_n(x)$ also satisfy the appropriate boundary conditions, then we can construct the Green's function:

$$G(x,x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}.$$

Check:

$$\frac{d^2}{dx^2}G(x,x') = 4\pi \sum_n \frac{\left(-\alpha_n u_n(x)\right)u_n(x')}{\alpha_n} = -4\pi \sum_n u_n(x)u_n(x')$$

$$= -4\pi\delta(x-x')$$

Orthogonal function expansions in 2 and 3 dimensions – for cartesian coordinates:

$$\nabla^2\Phi(\mathbf{r}) \equiv \frac{\partial^2\Phi(\mathbf{r})}{\partial x^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial y^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial z^2} = -\rho(\mathbf{r}) / \epsilon_0.$$

Let $\{u_n(x)\}$, $\{v_n(y)\}$, $\{w_n(z)\}$ denote complete orthogonal function sets in the x, y, and z dimensions, respectively. The Green's function construction becomes:

$$G(x,x',y,y',z,z') = 4\pi\sum_{lmn}\frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l+\beta_m+\gamma_n},$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x),\ \frac{d^2}{dy^2}v_m(y) = -\beta_m v_m(y),\ \text{and}\ \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z).$$

(See Eq. 3.167 in Jackson for example.)

Combined orthogonal function expansion and homogeneous solution construction of Green's function in 2 and 3 dimensions.

An alternative method of finding Green's functions for a second order ordinary differential equations (in 1 dimension) is based on a product of two independent solutions of the homogeneous equation, $\phi_1(x)$ and $\phi_2(x)$:

$$G(x,x') = K\phi_1(x_<)\phi_2(x_>), \text{ where } K \equiv \frac{4\pi}{\frac{d\phi_1}{dx}\phi_2 - \phi_1\frac{d\phi_2}{dx}},$$

where $x_<$ denotes the smaller of x and x'.

For the two and three dimensional cases, we can use this technique in one of the dimensions in order to reduce the number of summation terms. These ideas are discussed in Section 3.11 of Jackson.

Green's function construction -- continued

For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y') \quad \text{where} \quad \frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

The y dependence of this equation will have the required behavior, if we choose: $\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y,y') = -4\pi\delta(y-y'),$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_1}(y) = 0, \quad \left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_2}(y) = 0$$

and the Wronskian constant: $K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y, y') = -4\pi\delta(y - y'),$$

$$g_n(y, y') = \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>)$$

where: $$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0,$$

and $$K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n}\, y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n}\,(b - y))$

where $K_n = \sqrt{\alpha_n} \sinh(\sqrt{\alpha_n}\, b)$

using the identity: $\cosh(r)\sinh(s) + \sinh(r)\cosh(s) = \sinh(r + s)$

General form for a 2-dimensional example

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')\frac{4\pi}{K_n}v_{n_1}(y_<)v_{n_2}(y_>).$$

where $$K_n \equiv \frac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$$

Note that the idea is very general, but the details are highly dependent on the form of the differential equation.

Example that is useful for spherical polar coordinates where the eigenfunction expansion is used for the angular variables and the homogeneous solution is used for the radial variable. This form is designed to produce solutions that vanish for $r \rightarrow \infty$.

For electrostatic problems where $\rho(\mathbf{r})$ is contained in a small region of space and $S \rightarrow \infty$, $\quad G(\mathbf{r},\mathbf{r}') = \dfrac{1}{|\mathbf{r}-\mathbf{r}'|}$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi')$$

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D} = \varepsilon \mathbf{E}; \quad \mathbf{B} = \mu \mathbf{H}$

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

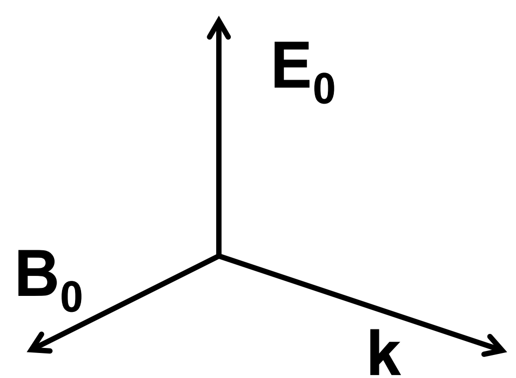
Summary of plane electromagnetic waves:

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{B}(\mathbf{r},t)=\Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$\mathbf{D}(\mathbf{r},t)=\varepsilon\mathbf{E}(\mathbf{r},t) \qquad \mathbf{H}(\mathbf{r},t)=\frac{1}{\mu}\mathbf{B}(\mathbf{r},t)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \quad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0=0$$

$$\mathbf{B}_0=\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} \quad \text{and} \quad \hat{\mathbf{k}}\cdot\mathbf{B}_0=0$$



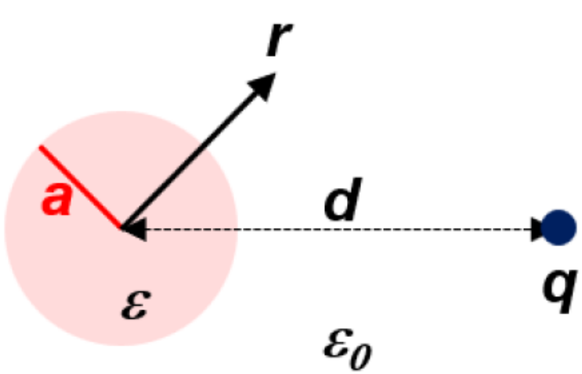
HW10

PHY 712 -- Assignment #10

Assigned: 2/9/2024 Due: 2/12/2024

Finish reading Chapter 4 in **Jackson** .

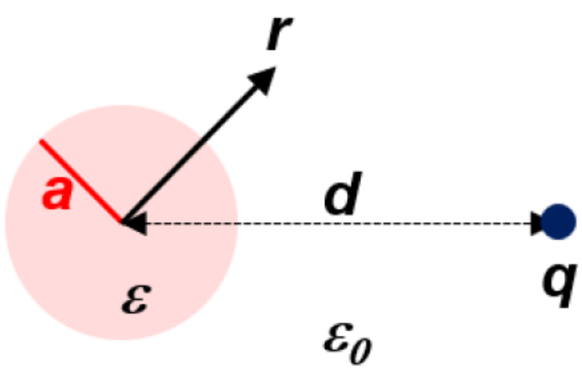
1. Work problem 4.9(a) in **Jackson**. Hint: It may be convenient to use a coordinate system with the origin at the center of the dielectric sphere. Also, you may benefit from considering the case where $\varepsilon/\varepsilon_0=1$ to check that your expression makes sense.



Note that

$$\frac{1}{|\mathbf{r}-\mathbf{d}|}=\sum_{\ell=0}^{\infty}\frac{r^{\ell}}{d^{\ell+1}}P_{\ell}(\hat{\mathbf{r}}\cdot\hat{\mathbf{d}})$$

There are several ways of approaching this problem. One convenient way is to consider the effects of the dielectric sphere and point charge separately



$$\Phi(r,\theta) = \Phi_{\text{sphere}}(r,\theta) + \frac{q}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{d}|}$$

$$\Phi_{\text{sphere}}(r,\theta) = \begin{cases} \sum_{\ell=0} A_\ell r^\ell P_\ell(\cos\theta) & \text{for } r \le a \\ \sum_{\ell=0} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) & \text{for } r \ge a \end{cases}$$

Boundary values at $r = 0$:

$$\varepsilon \left.\frac{\partial\Phi(r,\theta)}{\partial r}\right|_{r=a_-} = \varepsilon_0 \left.\frac{\partial\Phi(r,\theta)}{\partial r}\right|_{r=a_+}$$

$$\left.\frac{\partial\Phi(r,\theta)}{\partial\theta}\right|_{r=a_-} = \left.\frac{\partial\Phi(r,\theta)}{\partial\theta}\right|_{r=a_+}$$

Also: $$\left.\frac{q}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{d}|}\right|_{r=a} = \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}\frac{a^\ell}{d^{\ell+1}}P_\ell(\cos\theta)$$

$$\ell a^{\ell-1}\left(\varepsilon\left(A_\ell + \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}\right)\right) = -(\ell+1)\varepsilon_0\frac{B_\ell}{a^{\ell+2}} + \ell a^{\ell-1}\frac{q}{4\pi d^{\ell+1}}$$

$$a^\ell\left(\left(A_\ell + \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}\right)\right) = \frac{B_\ell}{a^{\ell+1}} + a^\ell\frac{q}{4\pi\varepsilon_0 d^{\ell+1}} \quad \Rightarrow A_\ell = \frac{B_\ell}{a^{2\ell+1}}$$

$$\frac{\varepsilon}{\varepsilon_0}\left(A_\ell + \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}\right) = -\frac{(\ell+1)}{\ell}\frac{B_\ell}{a^{2\ell+1}} + \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}$$

$$A_\ell = \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}\left(\frac{1-\frac{\varepsilon}{\varepsilon_0}}{\frac{\varepsilon}{\varepsilon_0}+\frac{(\ell+1)}{\ell}}\right)$$

PHY 712 -- Assignment #11

Assigned: 2/12/2024   Due: 2/19/2024

Start reading Chapter 5 (Sec. 5.1-5.5) in **Jackson** .

1. Consider an infinitely long cylindrical wire with radius *a*, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically, in terms of *r* the radial parameter of the cylindrical coordinates of the system the current density is $\mathbf{J}(r)=\mathbf{J}_0$, where $\mathbf{J}_0$ is a constant vector pointing along the z-axis, for $r \leq a$ and zero otherwise.
   a. Find the vector potential (**A**) for all *r*.
   b. Find the magnetic flux field (**B**) for all *r*.

Simple solution using Ampere's law

Know that magnetic field is uniform and pointing in the ϕ direction

$$\text{For } r < a \qquad 2\pi r B = \mu_0 \pi r^2 J_0 \qquad \Rightarrow B = \frac{\mu_0 r J_0}{2}$$

$$\text{For } r > a \qquad 2\pi r B = \mu_0 \pi a^2 J_0 \qquad \Rightarrow B = \frac{\mu_0 a^2 J_0}{2r}$$

$$\mathbf{A} = A_z(r)\hat{\mathbf{z}}$$

$$\text{For } r < a \qquad A_z(r) = -\frac{\mu_0 r^2 J_0}{4}$$

$$\text{For } r > a \qquad A_z(r) = -\frac{\mu_0 a^2 J_0}{4}\left(1 + 2\ln\left(\frac{r}{a}\right)\right)$$

Is this answer unique?