PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes on Lecture 24:

Digression on some Mathematical Methods and

Sources of radiation Chap. 9 (Sec. 9.1-9.3)

A. Digression on tools for solving ordinary differential equations – Method of Frobenius
B. Electromagnetic waves due to specific sources
C. Dipole radiation patterns

Tentative schedule for the remaining semester --

	Fri: 03/15/2024	No class	*Spring Break*		
24	Mon: 03/18/2024	Chap. 9	Digression on Math methods and Radiation from localized oscillating sources		
25	Wed: 03/20/2024	Chap. 9	Radiation from localized oscillating sources		
26	Fri: 03/22/2024	Chap. 9 & 10	Radiation and scattering		
27	Mon: 03/25/2024	Chap. 11	Special Theory of Relativity		
28	Wed: 03/27/2024	Chap. 11	Special Theory of Relativity		
29	Fri: 03/29/2024	Chap. 11	Special Theory of Relativity		
30	Mon: 04/01/2024	Chap. 14	Radiation from moving charges		
31	Wed: 04/03/2024	Chap. 14	Radiation from accelerating charged particles		
32	Fri: 04/05/2024	Chap. 14	Synchrotron radiation and Compton scattering		
33	Mon: 04/08/2024	Chap. 15	Radiation from collisions of charged particles		
34	Wed: 04/10/2024	Chap. 13	Cherenkov radiation		
35	Fri: 04/12/2024		Special topic: E & M aspects of superconductivity		
36	Mon: 04/15/2024		Special topic: Quantum Effects in E & M		
37	Wed: 04/17/2024		Special topic: Quantum Effects in E & M		
38	Fri: 04/19/2024		Special topic: Quantum Effects in E & M		
	Mon: 04/22/2024		Presentations I		
	Wed: 04/24/2024		Presentations II		
	Fri: 04/26/2024		Presentations III		
39	Mon: 04/29/2024		Review		
40	Wed: 05/01/2024		Review		

PHY 712 – Problem Set #19

Assigned: 03/18/2024 Due: 03/25/2024

This problem (thanks to F. B. Hildebrand) reviews the Frobenius method of solving differential equations.

1. Use the Frobenius method to obtain two analytic solutions, valid near $r = 0$, to the following differential equation.

$$\left(r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} + \left(r^2 - \frac{1}{4} \right) \right) f(r) = 0.$$

Digression on tools for solving ordinary differential equations – Method of Frobenius

https://mathshistory.st-andrews.ac.uk/Biographies/Frobenius/

Ferdinand Georg Frobenius

Born: 26 October 1849
Berlin-Charlottenburg, Prussia (now Germany)

Died: 3 August 1917
Berlin, Germany

Summary: Georg Frobenius combined results from the theory of algebraic equations, geometry, and number theory, which led him to the study of abstract groups, the representation theory of groups and the character theory of groups. He also developed method for linear differential equations.

Why? Example seen recently -

Solutions of the differential equation: $\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)f(r)=0$

Frobenius method for finding solutions near $r=0$:

Guess series solution form: $f(r)=\sum_{n=0} A_n r^{s+n}$

Evaluate: $\mathcal{O}f(r)=\sum_{n=0} A_n \mathcal{O}r^{s+n}=0$ for each power of r^{s+m} to find relationships between coefficients A_m and the condition for non-trivial A_0.

Example (thanks to F. B. Hildebrand):

$$\mathcal{O}=2r\frac{d^2}{dr^2}+(1-2r)\frac{d}{dr}-1$$

$$\sum_{n=0} A_n \mathcal{O}r^{s+n}=0=\sum_{n=0} A_n\left((s+n)(2s+2n-1)r^{s+n-1}-(2s+2n+1)r^{s+n}\right)$$

Condition for non-trivial A_0: $s(2s-1)=0$

Continued --

Example (thanks to F. B. Hildebrand):

$$O = 2r\frac{d^2}{dr^2} + (1-2r)\frac{d}{dr} - 1$$

$$\sum_{n=0} A_n O r^{s+n} = 0 = \sum_{n=0} A_n \left(\left(s+n\right)\left(2s+2n-1\right) r^{s+n-1} - \left(2s+2n+1\right) r^{s+n} \right)$$

Condition for non-trivial A_0 : $\quad s\left(2s-1\right) = 0$

First solution: $\quad s = 0$

Coefficient of r^m : $\quad A_{m+1}(2m+1)(m+1) - A_m(2m+1) = 0$

$$f_1(r) = A_0\left(1 + r + \frac{r^2}{2} + \frac{r^3}{3!} + \ldots\right) = A_0 e^r$$

Second solution: $\quad s = \frac{1}{2}$

Coefficient of r^m : $\quad A_{m+1}(2m+3)(m+1) - A_m 2\left(m+1\right) = 0$

$$f_2(r) = A_0 r^{1/2}\left(1 + \frac{2}{3}r + \frac{2^2}{3\cdot 5}r^2 + \frac{2^3}{3\cdot 5\cdot 7}r^3 \ldots\right)$$

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial (\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Complicated coupled mess!

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require: $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

This choice decouples the equations for the scalar and vector potentials.

General equation form:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = -4\pi f$$

$$\Psi(\mathbf{r},t) = \begin{cases} \Phi(\mathbf{r},t) \\ A_x(\mathbf{r},t) \\ A_y(\mathbf{r},t) \\ A_z(\mathbf{r},t) \end{cases} \qquad f(\mathbf{r},t) = \begin{cases} \rho(\mathbf{r},t)/(4\pi\varepsilon_0) \\ \mu_0 J_x(\mathbf{r},t)/(4\pi) \\ \mu_0 J_y(\mathbf{r},t)/(4\pi) \\ \mu_0 J_z(\mathbf{r},t)/(4\pi) \end{cases}$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-|\mathbf{r}-\mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Electromagnetic waves from time harmonic sources

Charge density: $\rho(\mathbf{r},t)=\Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density: $\mathbf{J}(\mathbf{r},t)=\Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Note that the continuity condition applies:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t}+\nabla\cdot\mathbf{J}(\mathbf{r},t)=0 \quad \Rightarrow -i\omega\tilde{\rho}(\mathbf{r},\omega)+\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)=0$$

General source : $f(\mathbf{r},t)=\Re\left(\tilde{f}(\mathbf{r},\omega)e^{-i\omega t}\right)$

For $\tilde{f}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0}\tilde{\rho}(\mathbf{r},\omega)$

or $\tilde{f}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\tilde{J}_i(\mathbf{r},\omega)$

Electromagnetic waves from time harmonic sources – continued:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

$$\widetilde{\Psi}(\mathbf{r},\omega)e^{-i\omega t} = \widetilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t} + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) \widetilde{f}(\mathbf{r}',\omega)e^{-i\omega t'}$$

$$= \widetilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t} + \int d^3r' \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \widetilde{f}(\mathbf{r}',\omega)e^{-i\omega t}$$

Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}}_0(\mathbf{r},\omega) = 0$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1(x) = -\left(1 + \frac{i}{x}\right)\frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3\cos(x)}{x^2} \qquad h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right)\frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x << 1 \qquad \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x >> 1 \qquad \Rightarrow h_l(x) \approx (-i)^{l+1}\frac{e^{ix}}{x}$$

Digression on spherical Bessel functions --

Consider the homogeneous wave equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$$

Suppose $\tilde{\Phi}_0(\mathbf{r},\omega) = \psi_{lm}(r) Y_{lm}(\hat{\mathbf{r}})$

$\Rightarrow \psi_{lm}(r)$ must satisfy the following for $k = \omega / c$:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right)\psi_{lm}(r) = 0$$

General spherical Bessel function equation:

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{l(l+1)}{x^2} + 1\right)w_l(x) = 0 \qquad \Rightarrow \psi_{lm}(r) = w_l(kr)$$

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\phi}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0} \int d^3r' \,\tilde{\rho}(\mathbf{r}',\omega) j_l(kr_<) h_l(kr_>) Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0 \int d^3r' \,\tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr_<) h_l(kr_>) Y^*_{lm}(\hat{\mathbf{r}}')$$

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \,\tilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \,\tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

Some details:

$$\tilde{\Phi}(\mathbf{r},\omega)=\tilde{\Phi}_0(\mathbf{r},\omega)+\sum_{lm}\tilde{\varphi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\varphi}_{lm}(r,\omega)=\frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)\,h_l(kr_>)\,Y^*_{lm}(\hat{\mathbf{r}}')$$

$$=\frac{ik}{\varepsilon_0}\int d\Omega' Y^*_{lm}(\hat{\mathbf{r}}')\left(h_l(kr)\int_0^r r'^2\,dr'\,j_l(kr')\,\tilde{\rho}(\mathbf{r}',\omega)+j_l(kr)\int_r^\infty r'^2\,dr'\,h_l(kr')\,\tilde{\rho}(\mathbf{r}',\omega)\right)$$

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega)\approx\frac{ik}{\varepsilon_0}h_l(kr)\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr')Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega)\approx ik\mu_0 h_l(kr)\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr')Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources – continued -- some details:

$$\tilde{\varphi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\, j_l(kr_<)\, h_l(kr_>)\, Y^*_{lm}(\hat{\mathbf{r}}')$$

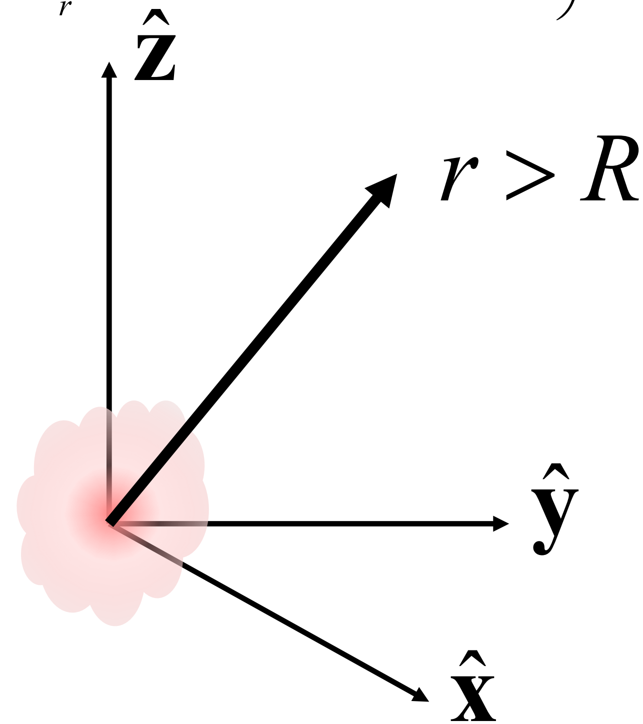
$$= \frac{ik}{\varepsilon_0}\left(h_l(kr)\int_0^r r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega)\, j_l(kr') + j_l(kr)\int_r^\infty r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega)\, h_l(kr') \right)$$

where $\rho_{lm}(\mathbf{r}',\omega) \equiv \int d\Omega'\,\tilde{\rho}(\mathbf{r}',\omega)\, Y^*_{lm}(\hat{\mathbf{r}}')$

note that for $r > R$, where $\tilde{\rho}(\mathbf{r},\omega) \approx 0$,

$$\tilde{\varphi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr)\int_0^\infty r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega)\, j_l(kr')$$

Similar relationships can be written for $\tilde{\mathbf{a}}_{lm}(r,\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}',\omega)$.



Electromagnetic waves from time harmonic sources – continued:

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

Note that these results are "exact" when *r* is outside the extent of the charge and current density.

Note that $\tilde{\rho}(\mathbf{r}',\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}',\omega)$ are connected via the continuity condition: $\quad -i\omega\,\tilde{\rho}(\mathbf{r},\omega)+\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)=0$

$$\tilde{\phi}_{lm}(r,\omega)\approx\frac{ik}{\varepsilon_0}h_l(kr)\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr')Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$=-\frac{k}{\omega\varepsilon_0}h_l(kr)\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\cdot\nabla'\left(j_l(kr')Y^*{}_{lm}(\hat{\mathbf{r}}')\right)$$

Electromagnetic waves from time harmonic sources – continued -- now considering the dipole approximation

Various approximations:

$$kr >> 1 \qquad \Rightarrow h_l(kr) \approx (-i)^{l+1} \frac{e^{ikr}}{kr}$$

$$kr' << 1 \qquad \Rightarrow j_l(kr') \approx \frac{(kr')^l}{(2l+1)!!}$$

Lowest (non-trivial) contributions in l expansions:

$$\tilde{\varphi}_{1m}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_1(kr) \int d^3r' \tilde{\rho}(\mathbf{r}',\omega) \frac{kr'}{3} Y^*_{1m}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{00}(r,\omega) \approx ik\mu_0 h_0(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}',\omega) Y^*_{00}(\hat{\mathbf{r}}')$$

Some details -- continued: (assuming confined source)

Recall continuity condition: $-i\omega\,\tilde{\rho}(\mathbf{r},\omega)+\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)=0$

$$-i\omega\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)+\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\int d^3r\;\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)=\frac{1}{i\omega}\int d^3r\;\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$=-\frac{1}{i\omega}\int d^3r\;\tilde{\mathbf{J}}(\mathbf{r},\omega)=\mathbf{p}(\omega)$$

Here we have used the identity:

$$\nabla\cdot(\psi\mathbf{V})=\nabla\psi\cdot\mathbf{V}+\psi(\nabla\cdot\mathbf{V})$$

We have also assumed that

$$\lim_{r\to\infty}\left(x\tilde{\mathbf{J}}(\mathbf{r},\omega)\right)=0$$

Electromagnetic waves from time harmonic sources – in the dipole approximation continued:

Lowest order contribution; dipole radiation:

Define dipole moment at frequency ω:

$$\mathbf{p}(\omega) \equiv \int d^3r \; \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r \; \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Note: in this case we have assumed a restricted extent of the source such that $kr' \ll 1$ for all r' with significant charge/current density.

Electromagnetic waves from time harmonic sources – in dipole approximation -- continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)\right)-\mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

Example of radiation source -- exact treatment

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega) = \frac{J_0}{-i\omega R} \cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \left(ik\mu_0\right) \int_0^\infty r'^2\, dr'\, e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{J_0 k}{\varepsilon_0 \omega R} \cos\theta \int_0^\infty r'^2\, dr'\, e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{\left(1 + k^2 R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{\left(1 + k^2 R^2\right)^2}$$

Example of radiation source – exact treatment continued

Evaluation for $r >> R$:

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}\;\frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\widetilde{\Phi}(\mathbf{r},\omega)=\frac{J_0 k}{\varepsilon_0\omega}\cos\theta\,\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\;\frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Relationship to dipole approximation (exact when *kR*➔0)

$$\mathbf{p}(\omega)\equiv\int d^3r\;\mathbf{r}\tilde{\rho}(\mathbf{r},\omega)=-\frac{1}{i\omega}\int d^3r\;\tilde{\mathbf{J}}(\mathbf{r},\omega)=-\frac{8\pi R^3 J_0}{i\omega}\hat{\mathbf{z}}$$

Corresponding dipole fields: $\tilde{\mathbf{A}}(\mathbf{r},\omega)=-\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Summary of results

Exact -- Evaluation for $r >> R$:

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \, \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right) \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Dipole approximation --

$$\mathbf{p}(\omega) \equiv \int d^3r \; \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r \; \tilde{\mathbf{J}}(\mathbf{r},\omega) = -\frac{8\pi R^3 J_0}{i\omega}\hat{\mathbf{z}}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r} = 2R^3 J_0 \mu_0 \hat{\mathbf{z}}\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r} = \frac{2R^3 J_0 k}{\varepsilon_0\omega}\cos\theta\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$