PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Notes for Lecture 26:

Complete reading of Chap. 9 & 10

A. Antenna radiation

B. Superposition of radiation from multiple sources

C. Scattered radiation

	Fri: 03/15/2024	No class	Spring Break		
24	Mon: 03/18/2024	Chap. 9	Digression on Math methods and Radiation from localized oscillating sources	#19	03/25/2024
25	Wed: 03/20/2024	Chap. 9	Radiation from localized oscillating sources	#20	03/25/2024
26	Fri: 03/22/2024	Chap. 9 & 10	Radiation and scattering	#21	03/25/2024
27	Mon: 03/25/2024	Chap. 11	Special Theory of Relativity		
28	Wed: 03/27/2024	Chap. 11	Special Theory of Relativity		
29	Fri: 03/29/2024	Chap. 11	Special Theory of Relativity		
30	Mon: 04/01/2024	Chap. 14	Radiation from moving charges		
31	Wed: 04/03/2024	Chap. 14	Radiation from accelerating charged particles		
32	Fri: 04/05/2024	Chap. 14	Synchrotron radiation and Compton scattering		
33	Mon: 04/08/2024	Chap. 15	Radiation from collisions of charged particles		
34	Wed: 04/10/2024	Chap. 13	Cherenkov radiation		
35	Fri: 04/12/2024		Special topic: E & M aspects of superconductivity		
36	Mon: 04/15/2024		Special topic: Quantum Effects in E & M		
37	Wed: 04/17/2024		Special topic: Quantum Effects in E & M		
38	Fri: 04/19/2024		Special topic: Quantum Effects in E & M		
	Mon: 04/22/2024		Presentations I		
	Wed: 04/24/2024		Presentations II		
	Fri: 04/26/2024		Presentations III		
39	Mon: 04/29/2024		Review		
40	Wed: 05/01/2024		Review		

PHY 712 -- Assignment #20

Assigned: 3/20/2024   Due: 3/25/2024

Continue reading Chapter 9 (Sec. 9.1-9.2) in **Jackson** .

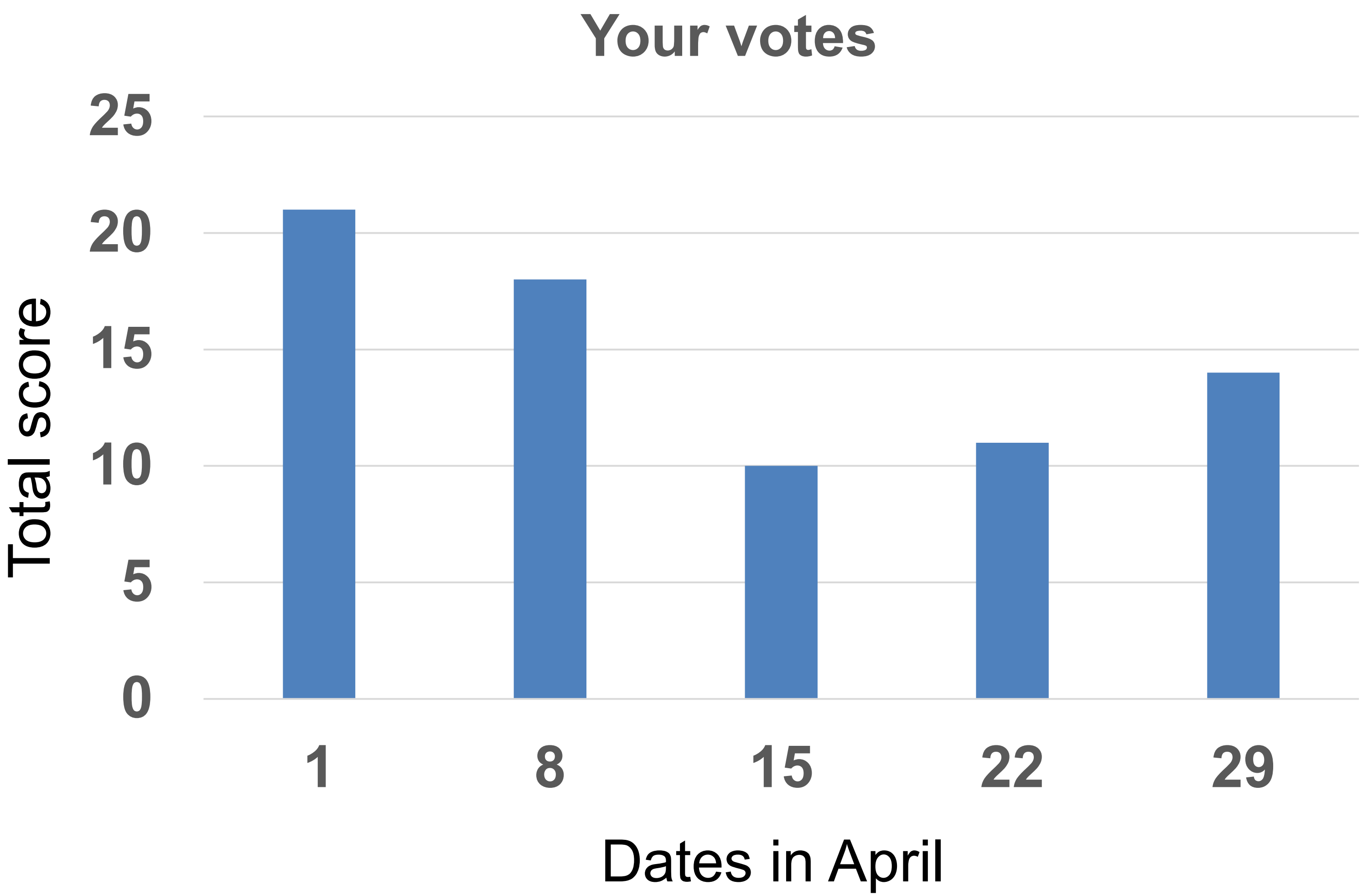
1. Problem 9.10 in **Jackson** lists the harmonic frequency dependent charge and current densities of a radiating H atom. Instead of answering **Jackson's** questions, calculate the exact scalar $\Phi(\mathbf{r},\omega_0)$ field for $r \gg a_0$ and compare your results with the scalar potential field calculated within the dipole approximation.

PHY 712 -- Assignment #21

Assigned: 3/22/2024   Due: 3/25/2024

Continue reading Chapter 9 (Sec. 9.1-9.4) in **Jackson** .

1. Problem 9.16(a) in **Jackson** . In this case, "exactly" really means following the approach discussed in Sec. 9.4 using instead the current density given in the problem.


Your votes
25
20
15
10
5
0
Total score
1
8
15
22
29
Dates in April

	Fri: 03/15/2024	No class	Spring Break		
24	Mon: 03/18/2024	Chap. 9	Digression on Math methods and Radiation from localized oscillating sources	#19	03/25/2024
25	Wed: 03/20/2024	Chap. 9	Radiation from localized oscillating sources	#20	03/25/2024
26	Fri: 03/22/2024	Chap. 9 & 10	Radiation and scattering	#21	03/25/2024
27	Mon: 03/25/2024	Chap. 11	Special Theory of Relativity		
28	Wed: 03/27/2024	Chap. 11	Special Theory of Relativity		
29	Fri: 03/29/2024	Chap. 11	Special Theory of Relativity		
30	Mon: 04/01/2024	Chap. 14	Radiation from moving charges		
31	Wed: 04/03/2024	Chap. 14	Radiation from accelerating charged particles		
32	Fri: 04/05/2024	Chap. 14	Synchrotron radiation and Compton scattering		
33	Mon: 04/08/2024	Chap. 15	Radiation from collisions of charged particles		
34	Wed: 04/10/2024	Chap. 13	Cherenkov radiation		
35	Fri: 04/12/2024		Special topic: E & M aspects of superconductivity		
36	Mon: 04/15/2024		Special topic: Quantum Effects in E & M		
37	Wed: 04/17/2024		Special topic: Quantum Effects in E & M		
38	Fri: 04/19/2024		Special topic: Quantum Effects in E & M		
	Mon: 04/22/2024		Presentations I		
	Wed: 04/24/2024		Presentations II		
	Fri: 04/26/2024		Presentations III		
39	Mon: 04/29/2024		Review		
40	Wed: 05/01/2024		Review		

10 votes (Special topic: Quantum Effects in E & M)

11 votes (Presentations)

Electromagnetic waves from time harmonic sources – review:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

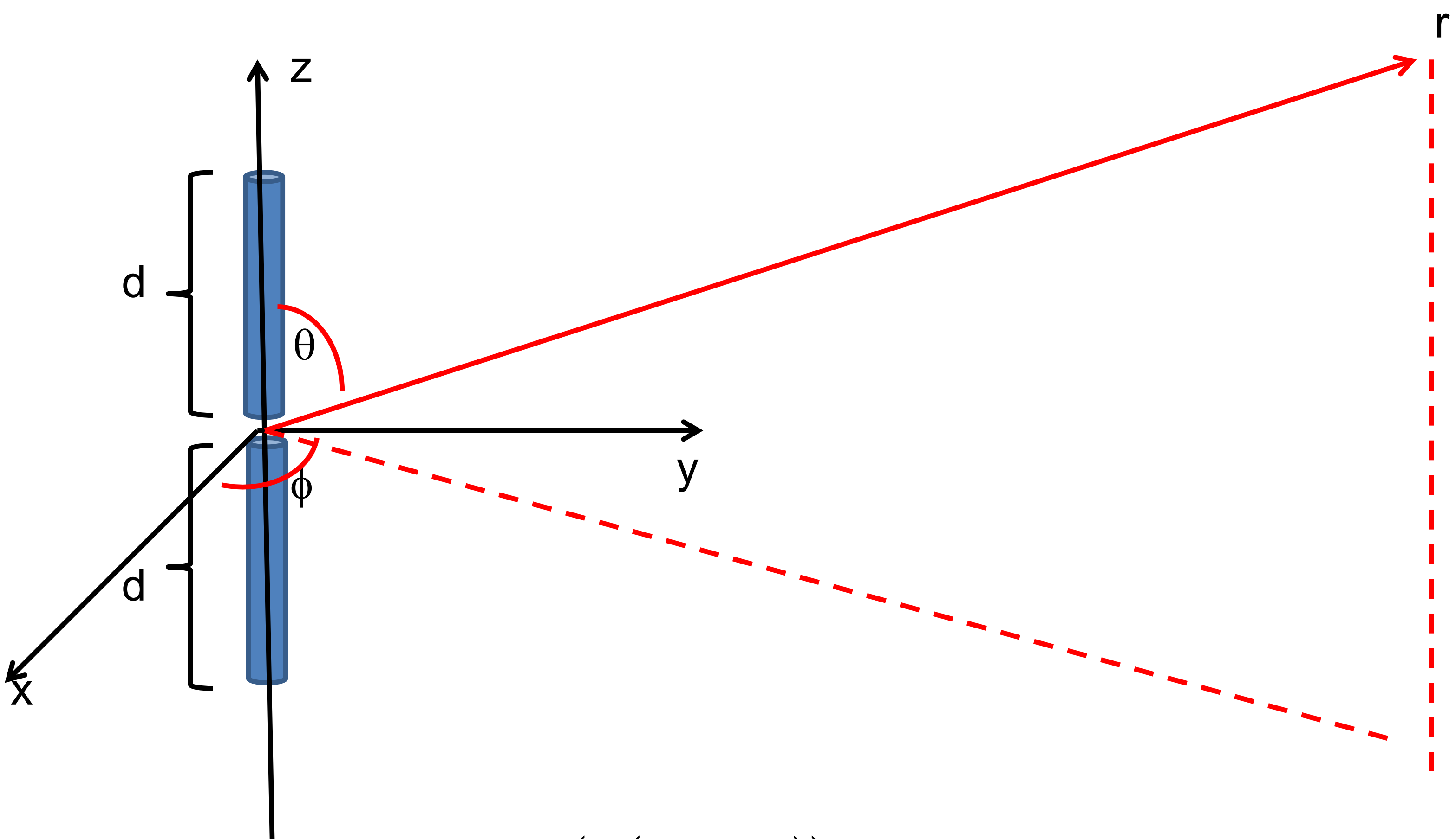
For $r >> r'$: $\qquad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \ldots$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\, \tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\, \tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Consider antenna source (center-fed)

Note – these notes differ from previous formulation d/2 ←→ d



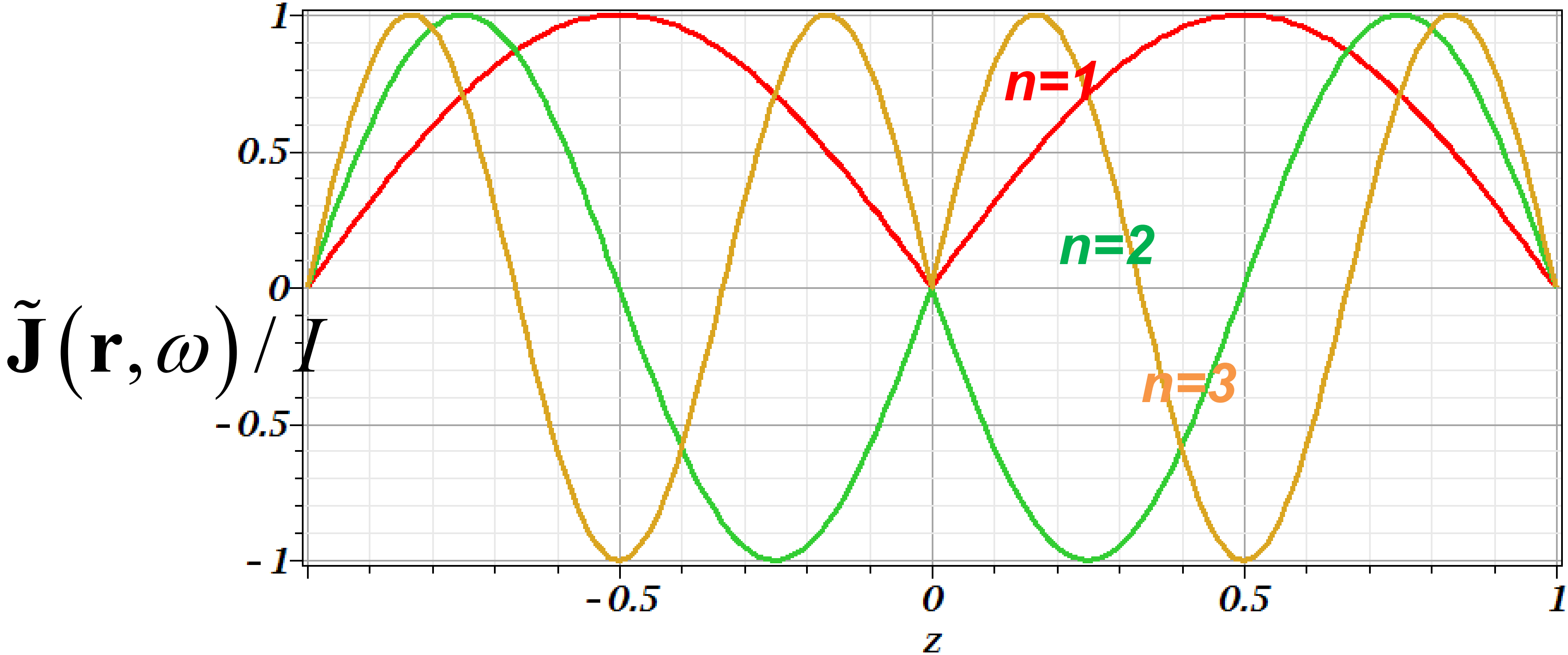
$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin\left(k\left(d-|z|\right)\right)\delta(x)\delta(y) \qquad \text{for} \quad -d\le z\le d$$

$$\text{for}\;\; k\equiv\frac{\omega}{c}=\frac{n\pi}{d}; \qquad n=1,2,3....$$



Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\Big(k\big(d - |z|\big)\Big)\delta(x)\delta(y) \quad \text{for} \quad -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> d$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \int_{-d}^{d} dz' \, e^{-ikz'\cos\theta} \quad \sin\Big(k\big(d - |z'|\big)\Big)$$

Consider antenna source -- continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} I \int_{-d}^{d} dz\, e^{-ikz\cos\theta} \sin\left(k\left(d-|z|\right)\right)$$

$$= \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{kr} 2I \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin^2\theta} \right]$$

In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

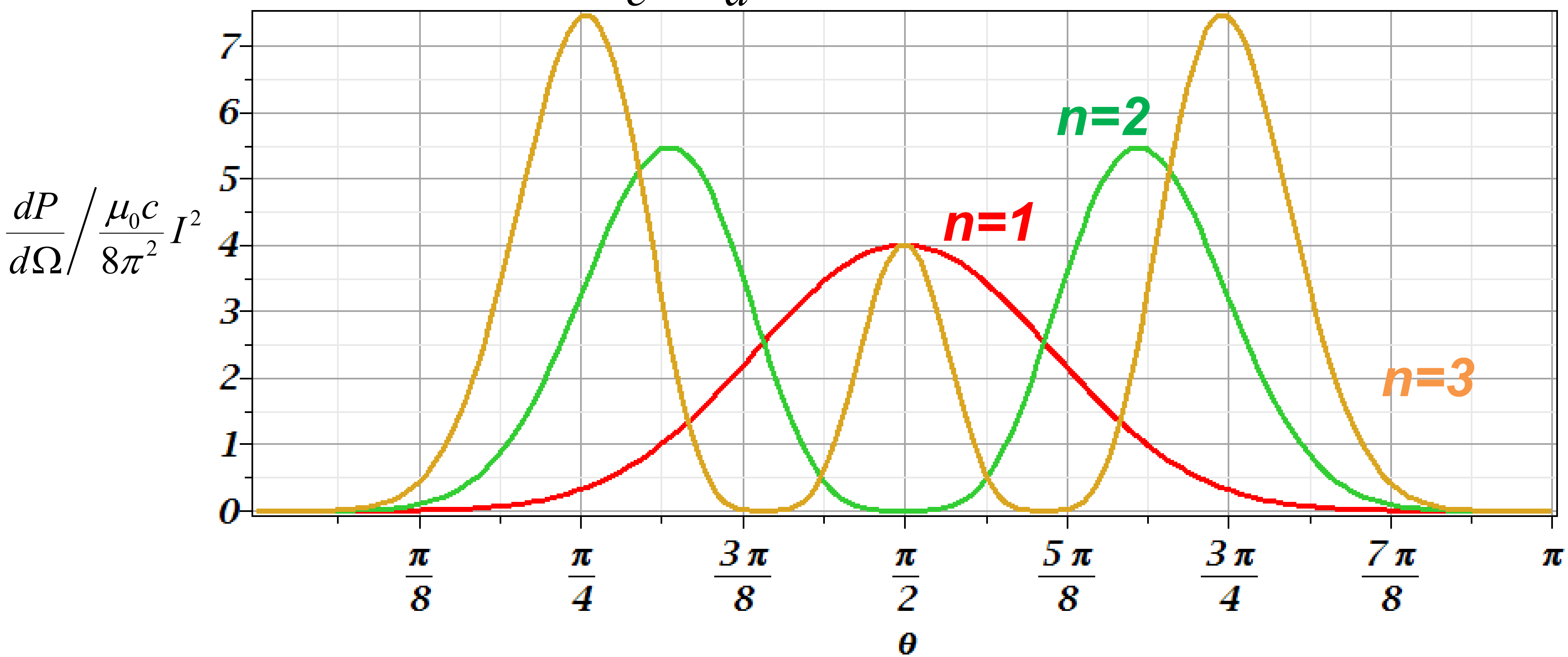
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) \times \tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c}{2\mu_0} r^2 \left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 \right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

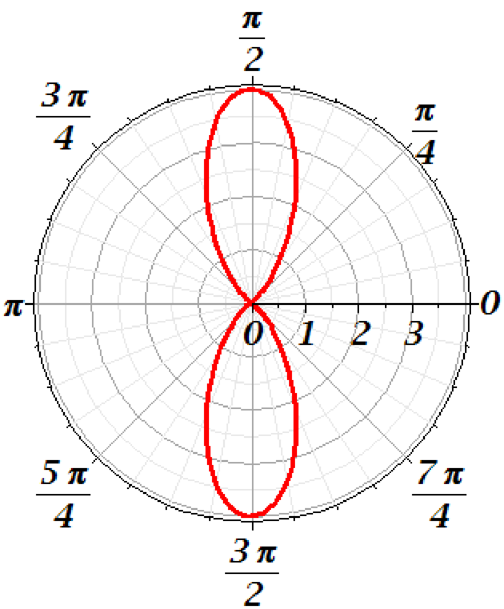
$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$

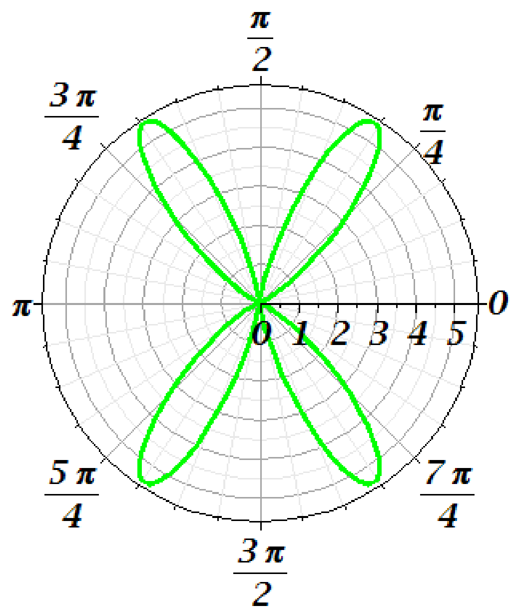


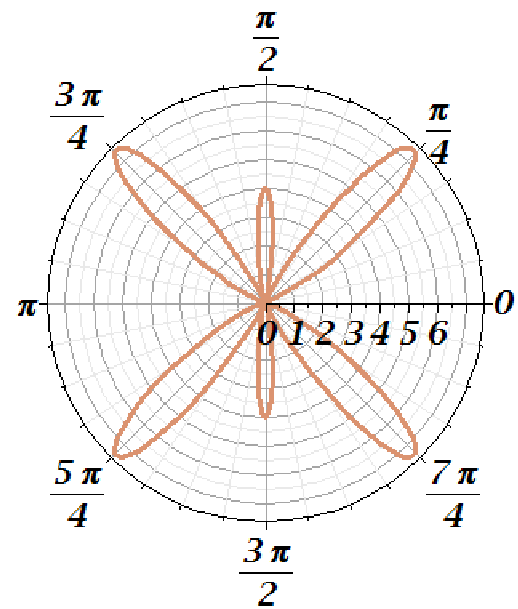
Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

For $kd = n\pi$:

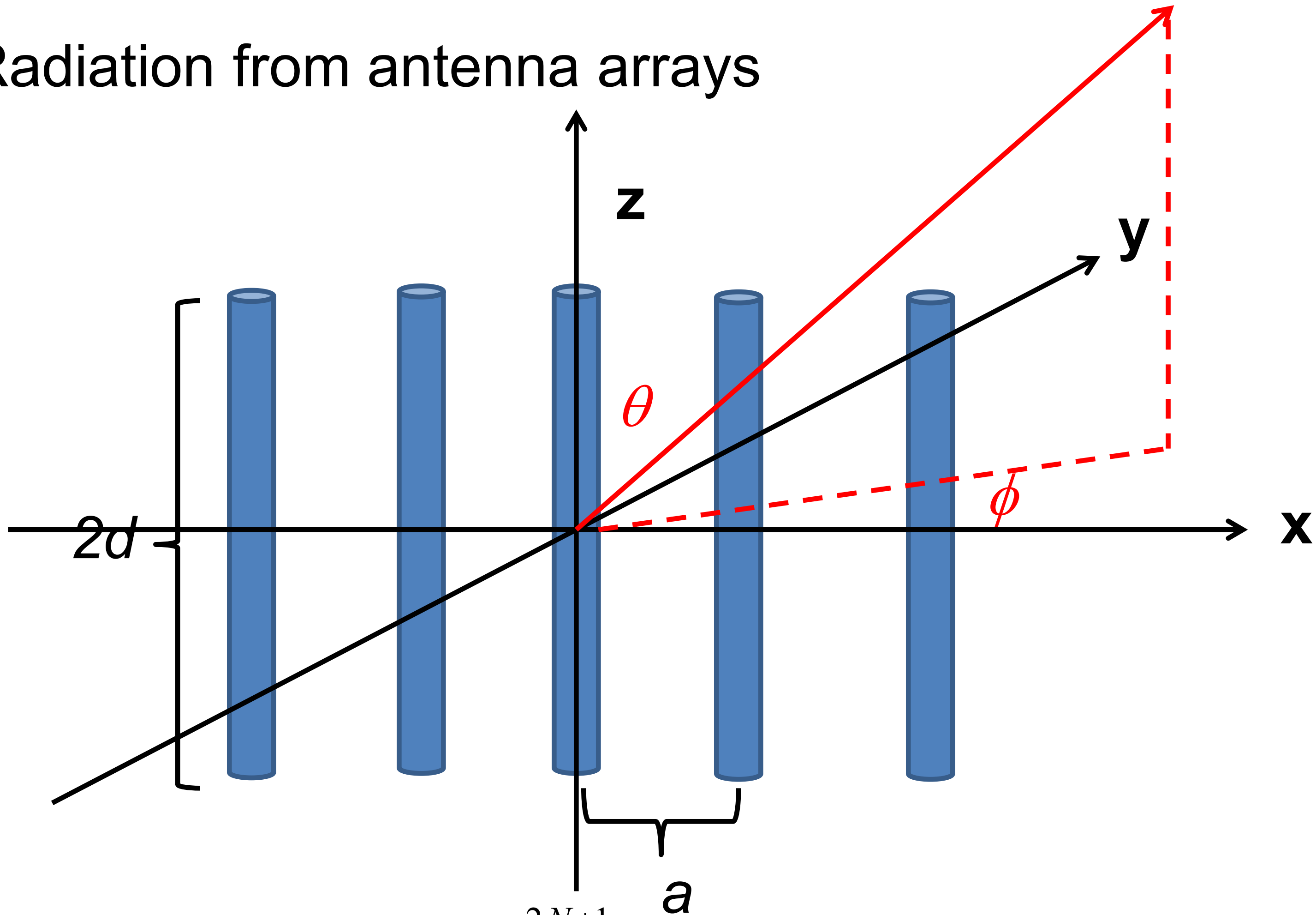






n=1 ***n=2*** ***n=3***

Radiation from antenna arrays



$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d-|z|\right)\right)\sum_{j=1}^{2N+1}\delta\left(x-\left(N+1-j\right)a\right)\delta\left(y\right) \quad \text{for } -d \leq z \leq d$$

$$k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1,2,3....$$

Note that these antennas are all "in phase".

Radiation from antenna arrays -- continued

Vector potential from array source :

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin\left(k\left(d-|z|\right)\right)\sum_{j=1}^{2N+1}\delta\left(x-(N+1-j)a\right)\delta(y)\quad \text{for } -d\leq z\leq d$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}\right)I\int_{-d}^{d}dz\, e^{-ikz\cos\theta}\quad\sin\left(k\left(d-|z|\right)\right)$$

$$\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}=\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}$$

Digression – summation of a geometric series

$$\sum_{j=-N}^{N} e^{-iAj} = e^{-iA}\sum_{j=-N}^{N} e^{-iAj} + e^{iAN} - e^{-iA(N+1)}$$

$$\sum_{j=-N}^{N} e^{-iAj} = \frac{e^{iAN} - e^{-iA(N+1)}}{1-e^{-iA}} = \frac{e^{iA/2}}{e^{iA/2}}\frac{e^{iAN} - e^{-iA(N+1)}}{1-e^{-iA}}$$

$$= \frac{2i\sin\left(A\left(N+1/2\right)\right)}{2i\sin(A/2)}$$

$$= \frac{\sin\left(A\left(N+1/2\right)\right)}{\sin(A/2)}$$

$$\sum_{j=-N}^{N} e^{-ikaj\sin\theta\cos\varphi} = \frac{\sin\left(\frac{1}{2}ka\left(2N+1\right)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}$$

Radiation from antenna arrays -- continued

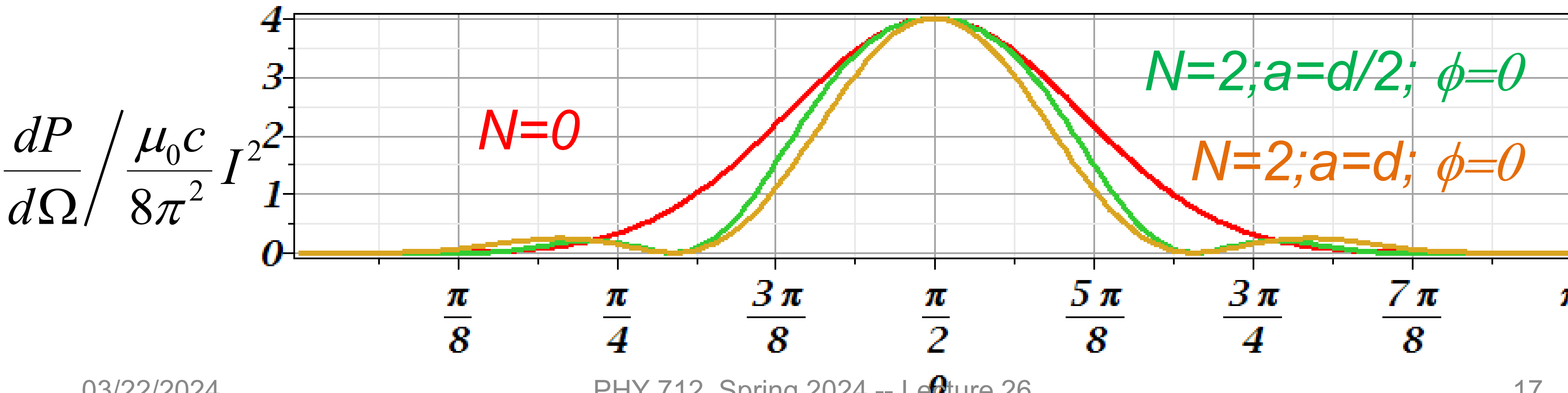
In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

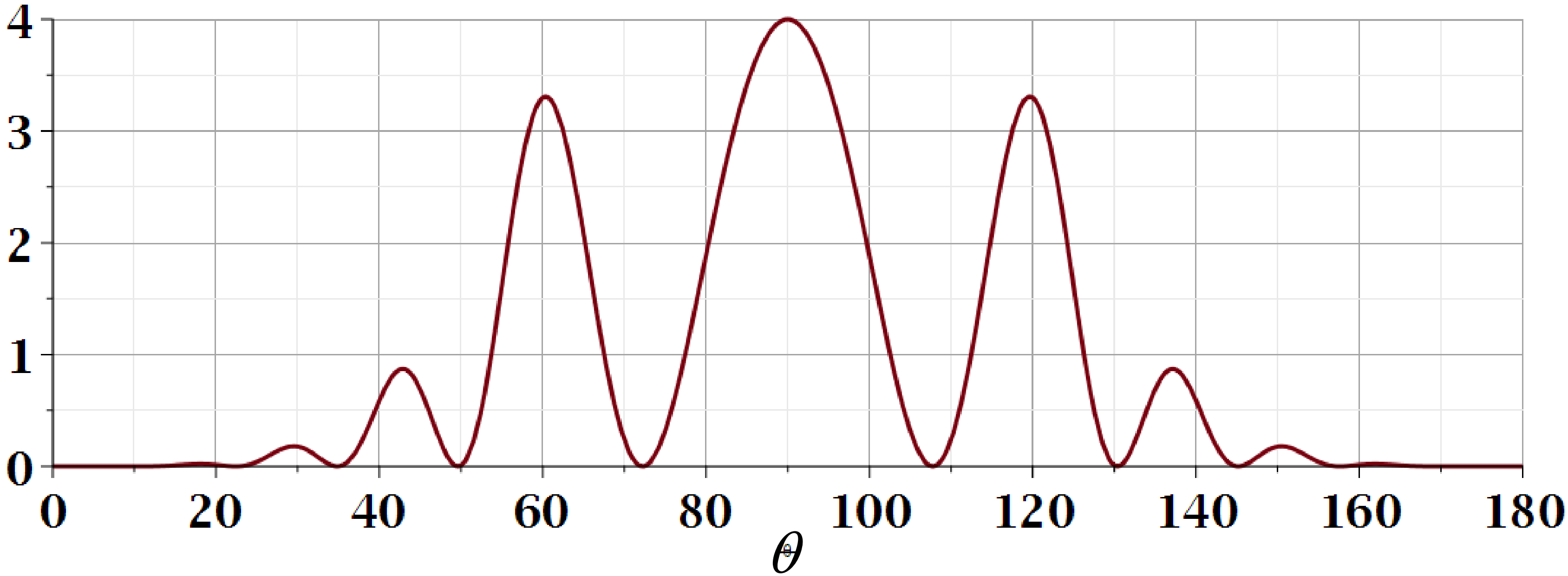
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) \times \tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c r^2}{2\mu_0}\left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta}\right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}\right]^2$$



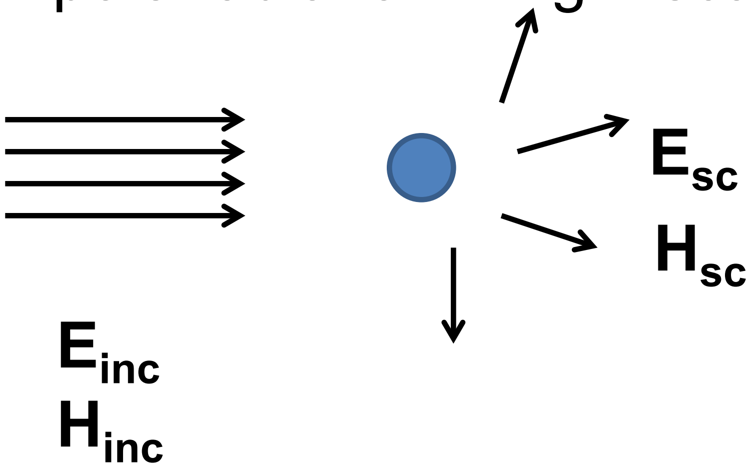
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)} \right]^2$$

Example for $\phi = 0, N = 10, \quad kd = \pi = 2ka$



Additional amplitude patterns can be obtained by controlling relative phases of antennas.

Dipole radiation in light scattering by small (dielectric) particles

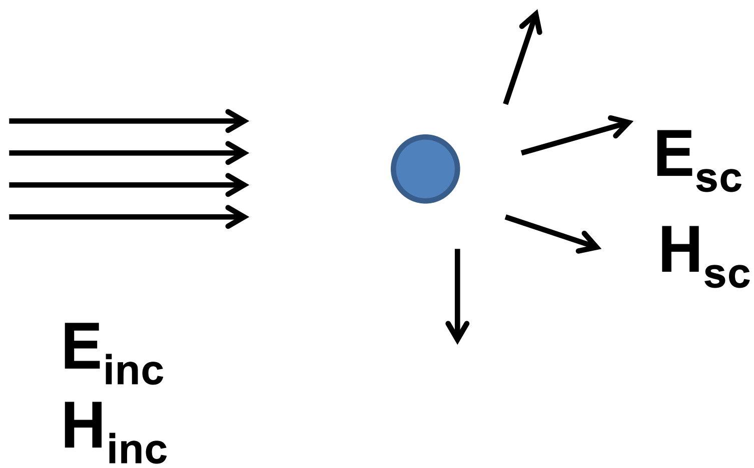


$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}} \qquad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0}k^2\frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_\mathbf{0}\cdot\mathbf{r}}$$

$$\mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_\mathbf{0} \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation:

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right)$$

$$\mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}} \times \mathbf{E}_{\text{sc}}$$

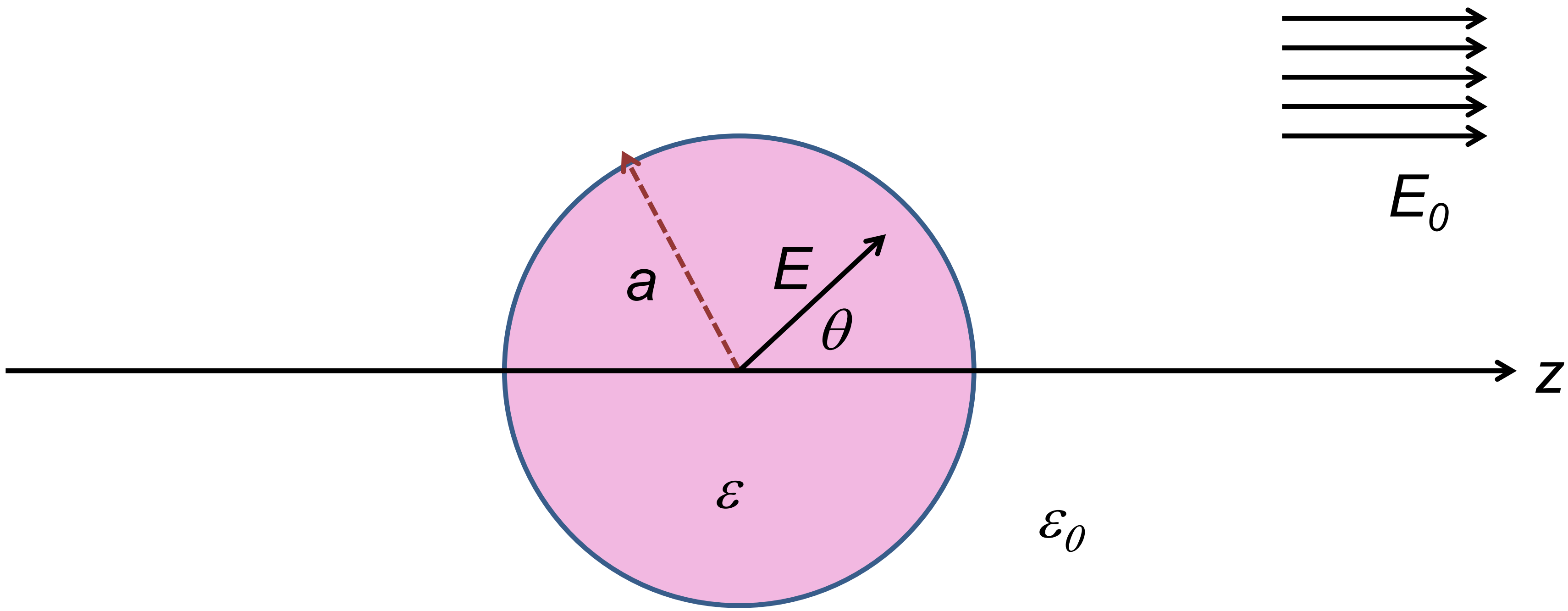
Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_\mathbf{0},\hat{\mathbf{v}}_0\right) = \frac{r^2\hat{\mathbf{r}}\cdot\left\langle \mathbf{S}_{sc}\right\rangle_{avg}}{\hat{\mathbf{k}}_\mathbf{0}\cdot\left\langle \mathbf{S}_{inc}\right\rangle_{avg}}$$

$$= \frac{r^2\left|\hat{\mathbf{v}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{\mathbf{v}}\cdot\mathbf{p}\right|^2$$

Recall previous analysis for electrostatic case:

Boundary value problems in the presence of dielectrics – example:



$$\text{At } r = a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$: $$\varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \rightarrow \infty$ $\quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon / \varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon / \varepsilon_0 - 1}{2 + \varepsilon / \varepsilon_0} \right) a^3 E_0$$

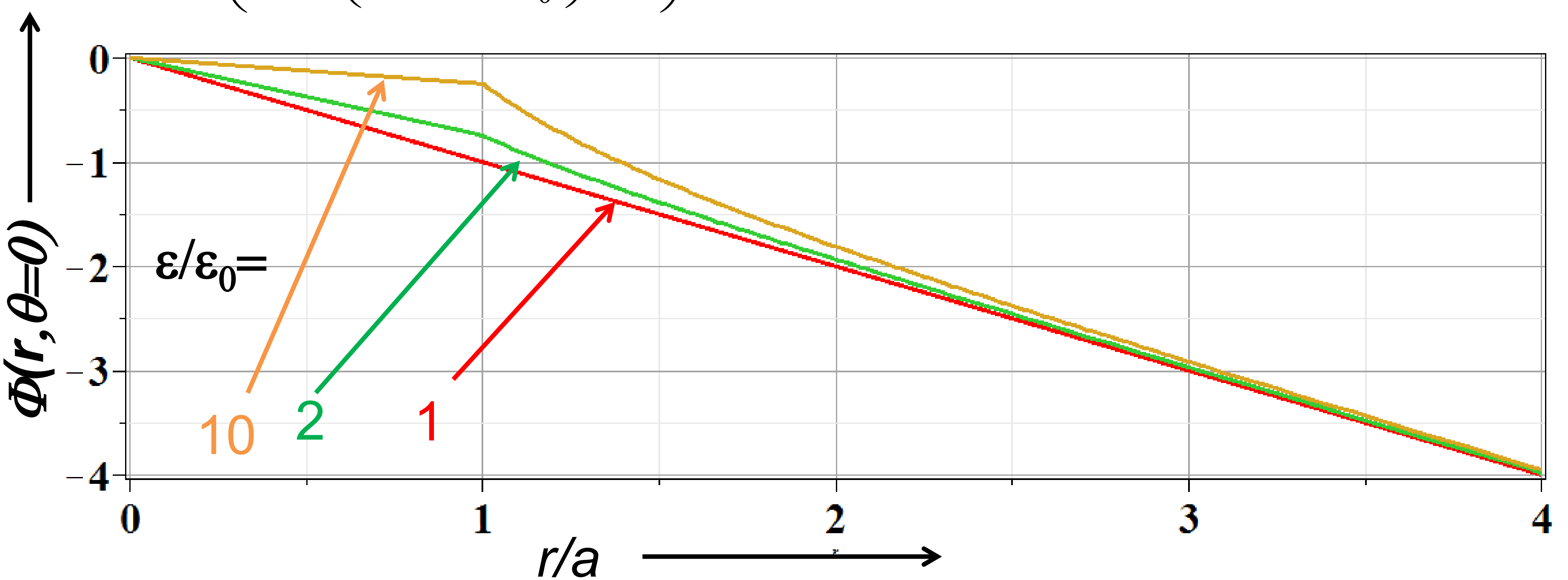
Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0\cos\theta$$

Induced dipole moment:

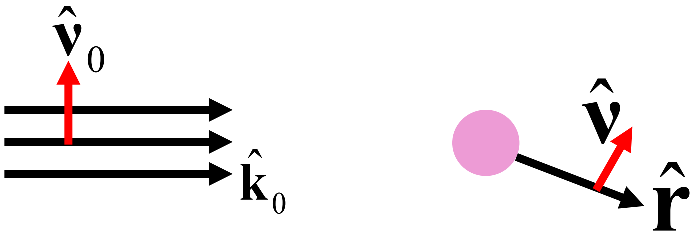
$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\mathbf{E}_0$$



Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:

Note polarization notation change for clarity.



$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_{inc}$$

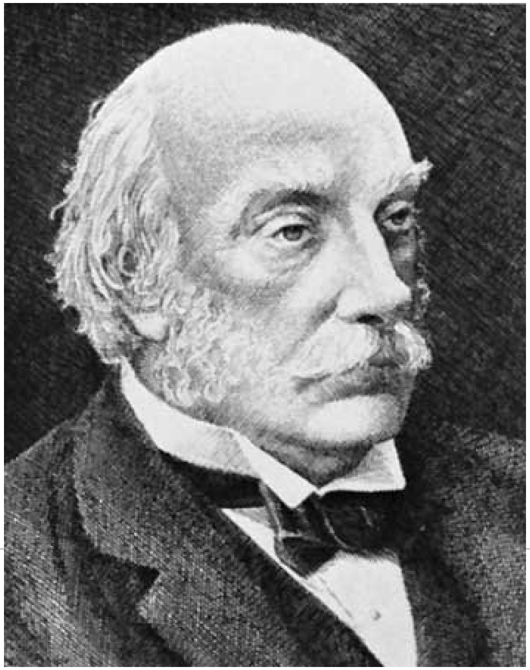
$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0\right) = \frac{r^2 \left|\hat{\mathbf{v}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\mathbf{v}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\mathbf{v}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_0\right|^2$$

https://www.britannica.com/biography/John-William-Strutt-3rd-Baron-Rayleigh



WRITTEN BY: R. Bruce Lindsay

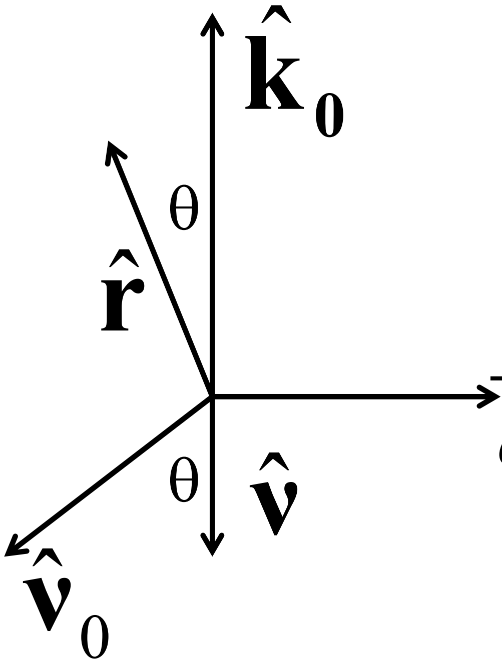
See Article History

Alternative Titles: John William Strutt, 3rd Baron Rayleigh of Terling Place

Lord Rayleigh, in full **John William Strutt, 3rd Baron Rayleigh of Terling Place**, (born November 12, 1842, Langford Grove, Maldon, Essex, England—died June 30, 1919, Terling Place, Witham, Essex), English physical scientist who made fundamental discoveries in the fields of acoustics and optics that are basic to the theory of wave propagation in fluids. He received the Nobel Prize for Physics in 1904 for his successful isolation of argon, an inert atmospheric gas.

Scattering by dielectric sphere with permittivity ε and radius *a*:

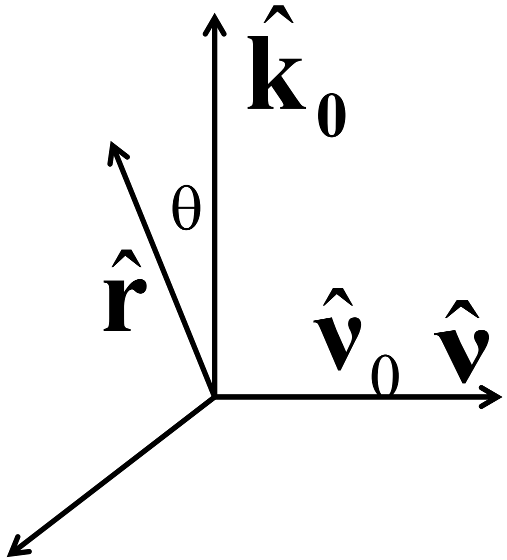
For $\mathbf{E}_{\mathbf{inc}}$ polarized in scattering plane:



$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right) == k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$= k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \cos^2\theta$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{inc}$ polarized perpendicular to scattering plane:

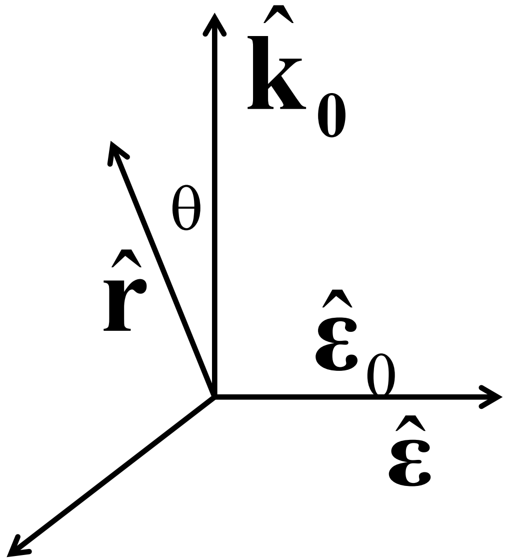
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2$$

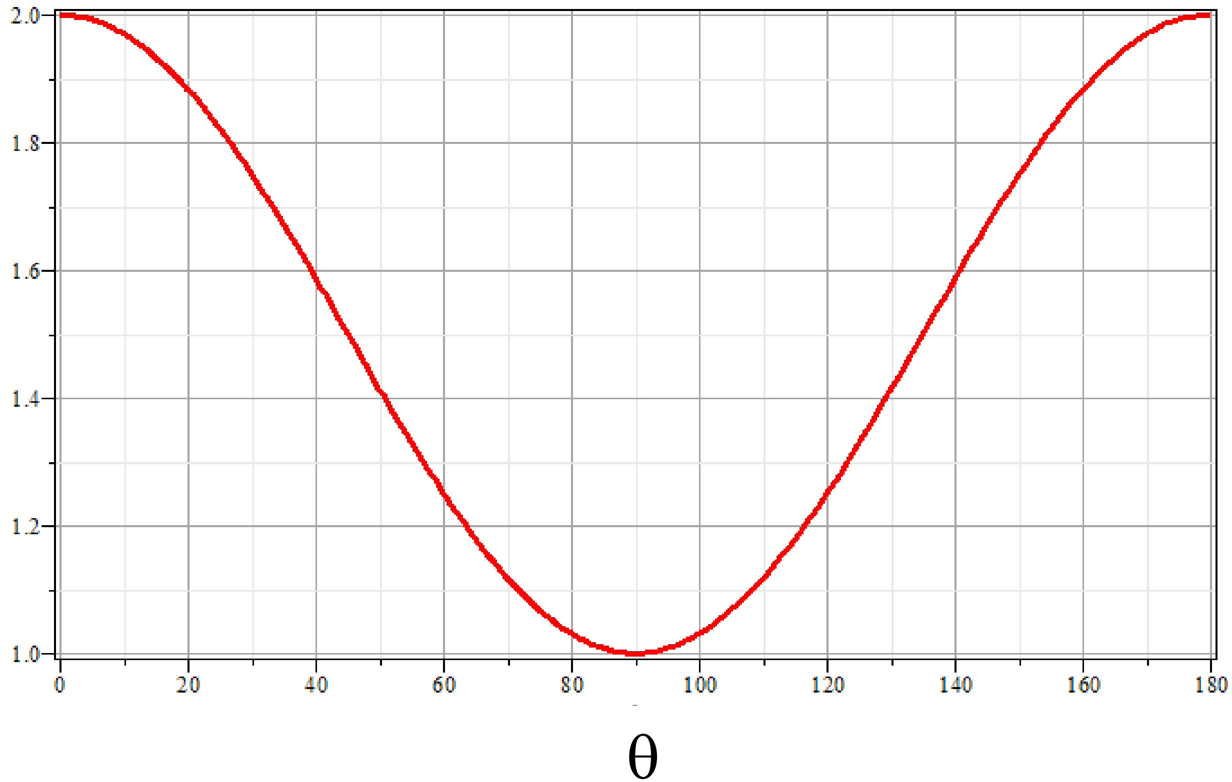
Assuming both incident polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$

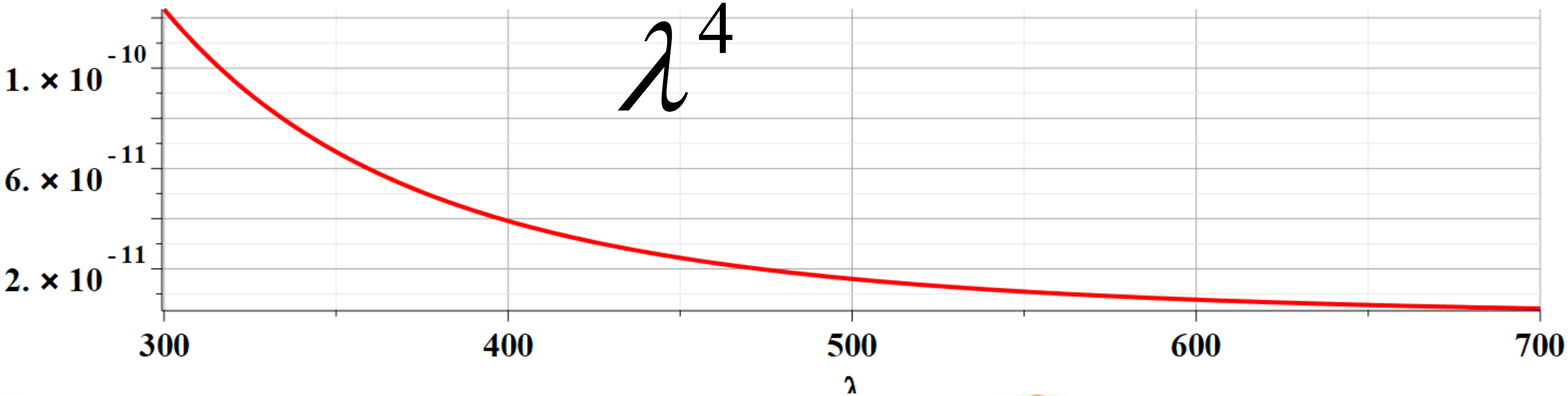
Scattering by dielectric sphere with permittivity ε and radius *a*:

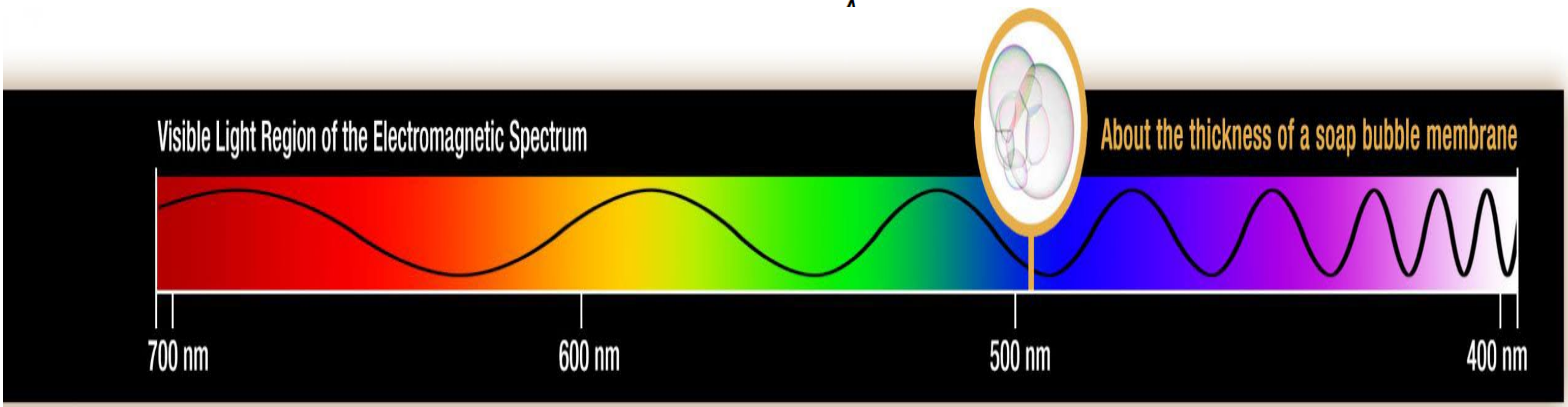


$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right) = \frac{k^4 a^6}{2}\left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left(\cos^2\theta + 1\right)$$



$$\frac{1}{\lambda^4}$$





Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations

in terms of $\mathbf{E}$ and $\mathbf{H}$ fields with time dependence $e^{-i\omega t}$:

$$\nabla \times \mathbf{E} = ikZ_0\mathbf{H} \qquad \nabla \times \mathbf{H} = -ik\mathbf{E} / Z_0$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

where $k \equiv \omega / c$ and $Z_0 \equiv \sqrt{\mu_0 / \epsilon_0}$

Decoupled equations:

$$\left(\nabla^2 + k^2\right)\mathbf{E} = 0 \qquad \left(\nabla^2 + k^2\right)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0}\nabla \times \mathbf{E} \qquad \mathbf{E} = \frac{iZ_0}{k}\nabla \times \mathbf{H}$$

Multipole expansion of electromagnetic fields -- continued

Note that:

$$\left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{E}\right) = 0 \qquad \left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{H}\right) = 0$$

Convenient operators for angular momentum analysis

Define: $\mathbf{L} \equiv \frac{1}{i}\left(\mathbf{r}\times\nabla\right)$

Note that $\mathbf{r}\cdot\mathbf{L} = 0$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$$

Multipole expansion of electromagnetic fields -- continued

Magnetic multipole field:

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{M} \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{M} = 0$$

$$\mathbf{L} \cdot \mathbf{E}_{lm}^{M} = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Electric multipole field:

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{E} \equiv -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{E} = 0$$

$$\mathbf{L} \cdot \mathbf{H}_{lm}^{E} = l(l+1) f_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for $l > 0$)

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi)$$

Orthogonality conditions:

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\mathbf{X}_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\left(\mathbf{r}\times\mathbf{X}_{lm}(\theta,\phi)\right) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm}\left[a_{lm}^{E} f_l(kr)\mathbf{X}_{lm}(\theta,\phi) - \frac{i}{k}a_{lm}^{M}\nabla\times\left(g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right)\right]$$

$$\mathbf{E} = \sum_{lm}\left[\frac{i}{k}a_{lm}^{E}\nabla\times\left(f_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right) + a_{lm}^{M} g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right]$$

Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a^E_{lm}\mathbf{X}_{lm}(\theta,\phi)\times\hat{\mathbf{r}} + a^M_{lm}\mathbf{X}_{lm}(\theta,\phi)\right]\right|^2$$

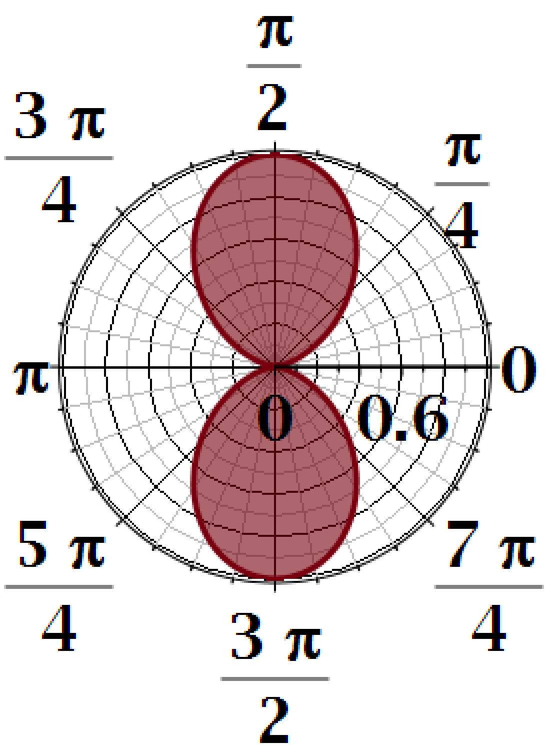
For a pure multipole radiation with either a^E_{lm} or a^M_{lm} :

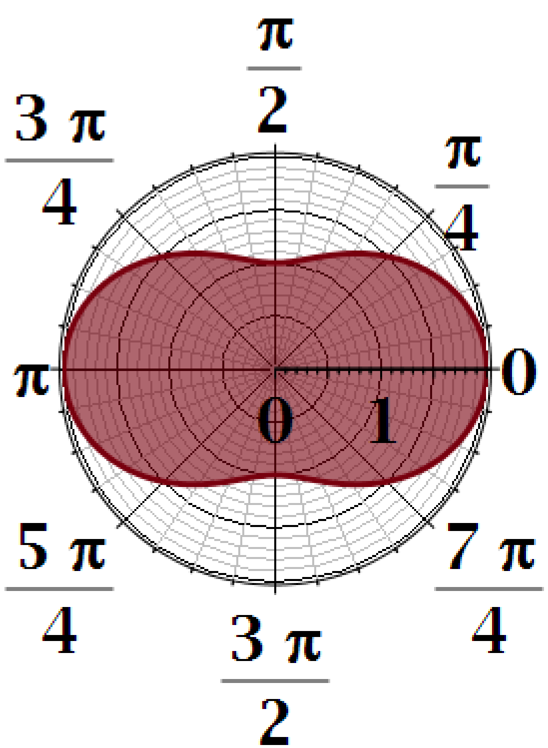
$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_{lm}\right|^2\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2$$

$$\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2 = \frac{1}{2l(l+1)}\left(2m^2\left|Y_{lm}\right|^2 + (l+m)(l-m+1)\left|Y_{l(m-1)}\right|^2 + (l-m)(l+m+1)\left|Y_{l(m+1)}\right|^2\right)$$

For example: $l=1$

$$\left|\mathbf{X}_{10}(\theta,\phi)\right|^2=\frac{3}{8\pi}\sin^2\theta \qquad \left|\mathbf{X}_{11}(\theta,\phi)\right|^2=\left|\mathbf{X}_{1-1}(\theta,\phi)\right|^2=\frac{3}{16\pi}\left(1+\cos^2\theta\right)$$





For example: $l = 2$

$$\left|\mathbf{X}_{20}(\theta,\phi)\right|^2 = \frac{15}{8\pi}\sin^2\theta\,\cos^2\theta \qquad \left|\mathbf{X}_{21}(\theta,\phi)\right|^2 = \frac{5}{16\pi}\left(1-3\cos^2\theta+4\cos^4\theta\right) \qquad \left|\mathbf{X}_{22}(\theta,\phi)\right|^2 = \frac{5}{16\pi}\left(1-\cos^4\theta\right)$$