PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 29:

Finish Chap. 11 and

begin Chap. 14 (Sec. 14.1-14.3)

A. **Electromagnetic field transformations & corresponding analysis of Liénard-Wiechert potentials for constant velocity sources**

B. **Radiation by moving charged particles**

24	Mon: 03/18/2024	Chap. 9	Digression on Math methods and Radiation from localized oscillating sources	[#19](#)	03/25/2024
25	Wed: 03/20/2024	Chap. 9	Radiation from localized oscillating sources	[#20](#)	03/25/2024
26	Fri: 03/22/2024	Chap. 9 & 10	Radiation and scattering	[#21](#)	03/25/2024
27	Mon: 03/25/2024	Chap. 11	Special Theory of Relativity	[#22](#)	04/01/2024
28	Wed: 03/27/2024	Chap. 11	Special Theory of Relativity	[#23](#)	04/01/2024
29	Fri: 03/29/2024	Chap. 11	Special Theory of Relativity		
30	Mon: 04/01/2024	Chap. 14	Radiation from moving charges		
31	Wed: 04/03/2024	Chap. 14	Radiation from accelerating charged particles		
32	Fri: 04/05/2024	Chap. 14	Synchrotron radiation and Compton scattering		
33	Mon: 04/08/2024	Chap. 15	Radiation from collisions of charged particles		
34	Wed: 04/10/2024	Chap. 13	Cherenkov radiation		
35	Fri: 04/12/2024		Special topic: E & M aspects of superconductivity		
	Mon: 04/15/2024		Presentations I		
	Wed: 04/17/2024		Presentations II		
	Fri: 04/19/2024		Presentations III		
36	Mon: 04/22/2024		Special topic: Quantum Effects in E & M		
37	Wed: 04/24/2024		Special topic: Quantum Effects in E & M		
38	Fri: 04/26/2024		Special topic: Quantum Effects in E & M		
39	Mon: 04/29/2024		Review		
40	Wed: 05/01/2024		Review		

When would you like to sign up for time/topic?

PHY 712 Presentation Schedule

Monday 4/15/2024

	Presenter Name	Topic
10:00-10:24		
10:25-10:50		

Wednesday 4/17/2024

	Presenter Name	Topic
10:00-10:24		
10:25-10:50		

Friday 4/19/2024

	Presenter Name	Topic
10:00-10:24		

Some Ideas for Computational Project

The purpose of the "Computational Project" is to provide an opportunity for you to study a topic of your choice in greater depth. The general guideline for your choice of project is that it should have something to do with electrodynamics, and there should be some degree of computation or analysis with the project. The completed project will include a short write-up and a ~15 min presentation to the class. You may design your own project or use one of the following list (which will be updated throughout the term).

- Evaluate the Ewald sum of various ionic crystals using Maple or a programing language. (Template available in Fortran code.)
- Work out the details of the finite difference or finite element methods.
- Work out the details of the hyperfine Hamiltonian as discussed in Chapter 5 of Jackson.
- Work out the details of Jackson problem 7.2 and related problems.
- Work out the details of reflection and refraction from birefringent materials.
- Analyze the Kramers-Kronig transform of some optical data or calculations.
- Determine the classical electrodynamics associated with an infrared or optical laser.
- Analyze the radiation intensity and spectrum from an interesting source such as an atomic or molecular transition, a free electron laser, etc.
- Work out the details of Jackson problem 14.15.

Comment: Some of you have been looking at textbooks (such as Zangwill) and sources available on the internet and finding different equations from those presented in these lecture notes and in Jackson.   That is a good thing in general, however please be aware that there are different units (SI for example) and different conventions for 4-vectors (some using different ordering of space and time, some using imaginary (i) for the time-like portion).   Since we are using Jackson for now, it will be good to make sure that you are OK with Jackson's equations and those in the lecture notes as well.

Solution of Maxwell's equations in the Lorentz gauge – review using SI units for now --

Liénard-Wiechert potentials and fields --

Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz **The Classical Theory of Fields**, Chapter 8.)

Consider the fields produced by the following source: a point charge *q* moving on a trajectory $R_q(t)$.

Charge density: $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \dfrac{d\mathbf{R}_q(t)}{dt}$.

$R_q(t)$

q

Solution of Maxwell's equations in the Lorenz gauge -- continued

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \int\int d^3r' dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|} \delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2} \int\int d^3r' dt' \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt' \, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1-\dfrac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation:
$$\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

Solution of Maxwell's equations in the Lorenz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1-\frac{v^2}{c^2}\right)-\frac{v}{c}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)+\mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2}-\frac{\mathbf{v}\cdot\mathbf{R}}{Rc}-\frac{\dot{\mathbf{v}}\cdot R}{c^2}\right)-\frac{\dot{\mathbf{v}}R}{c^2}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\left(\mathbf{R}\times\left\{\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right)-\frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

Convert to cgs Gaussian units:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c} \right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right)\left(1 - \frac{v^2}{c^2} \right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c} \left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c} \right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2} \right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c} \right)^2} \right]$$

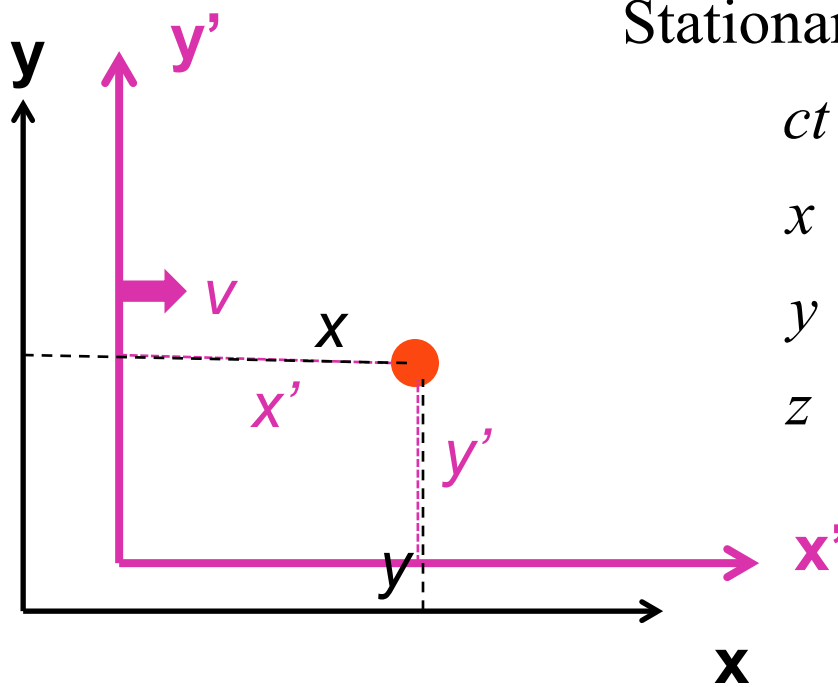$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Note that this analysis is carried out in a single frame of reference.   Now we resume our discussion about transforming values between two different inertial frames of reference.

Lorentz transformations

Convenient notation :

$$\beta_v \equiv \frac{v}{c}$$

$$\gamma_v \equiv \frac{1}{\sqrt{1 - \beta_v{}^2}}$$

Stationary frame		Moving frame
ct	$=$	$\gamma(ct' + \beta x')$
x	$=$	$\gamma(x' + \beta ct')$
y	$=$	y'
z	$=$	z'

Lorentz transformations -- continued

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\boldsymbol{\mathcal{L}}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{\mathcal{L}}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \boldsymbol{\mathcal{L}}_v \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad\qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{\mathcal{L}}_v^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice :

$$c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

Field strength tensor $\qquad F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F'^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

Transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}_v{}^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v{}^{\delta\beta} \qquad\qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y + \beta_v B'_z\right) & -\gamma_v\left(E'_z - \beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z + \beta_v E'_y\right) & \gamma_v\left(B'_y - \beta_v E'_z\right) \\ \gamma_v\left(E'_y + \beta_v B'_z\right) & \gamma_v\left(B'_z + \beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z - \beta_v B'_y\right) & -\gamma_v\left(B'_y - \beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \boldsymbol{\mathcal{L}}_v^{-1\alpha\gamma} F^{\gamma\delta} \boldsymbol{\mathcal{L}}_v^{-1\delta\beta} \qquad \boldsymbol{\mathcal{L}}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Summary of results:

$$E'_x = E_x \qquad\qquad B'_x = B_x$$

$$E'_y = \gamma_v\left(E_y - \beta_v B_z\right) \qquad\qquad B'_y = \gamma_v\left(B_y + \beta_v E_z\right)$$

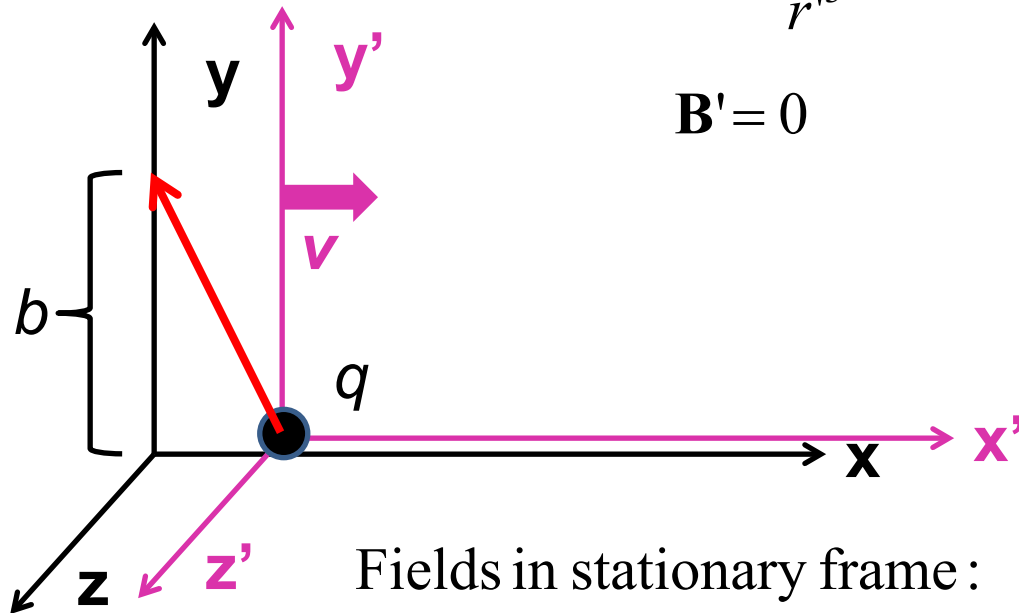$$E'_z = \gamma_v\left(E_z + \beta_v B_y\right) \qquad\qquad B'_z = \gamma_v\left(B_z - \beta_v E_y\right)$$

Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame :

$$E_x = E'_x \qquad\qquad B_x = B'_x$$
$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$
$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

Example:

Fields in moving frame :

$$\mathbf{E'} = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B'} = 0$$
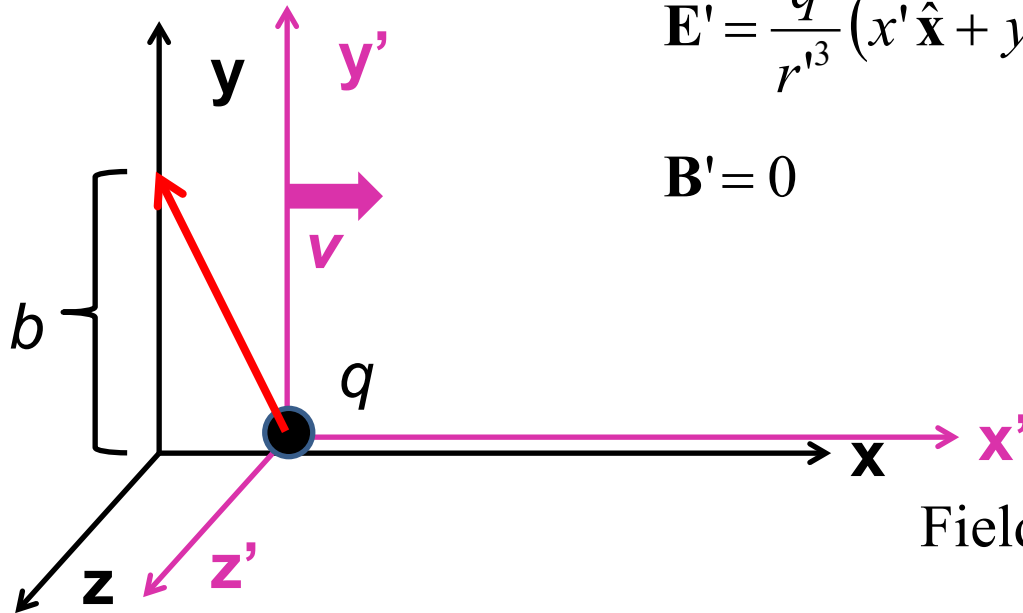
Fields in stationary frame :

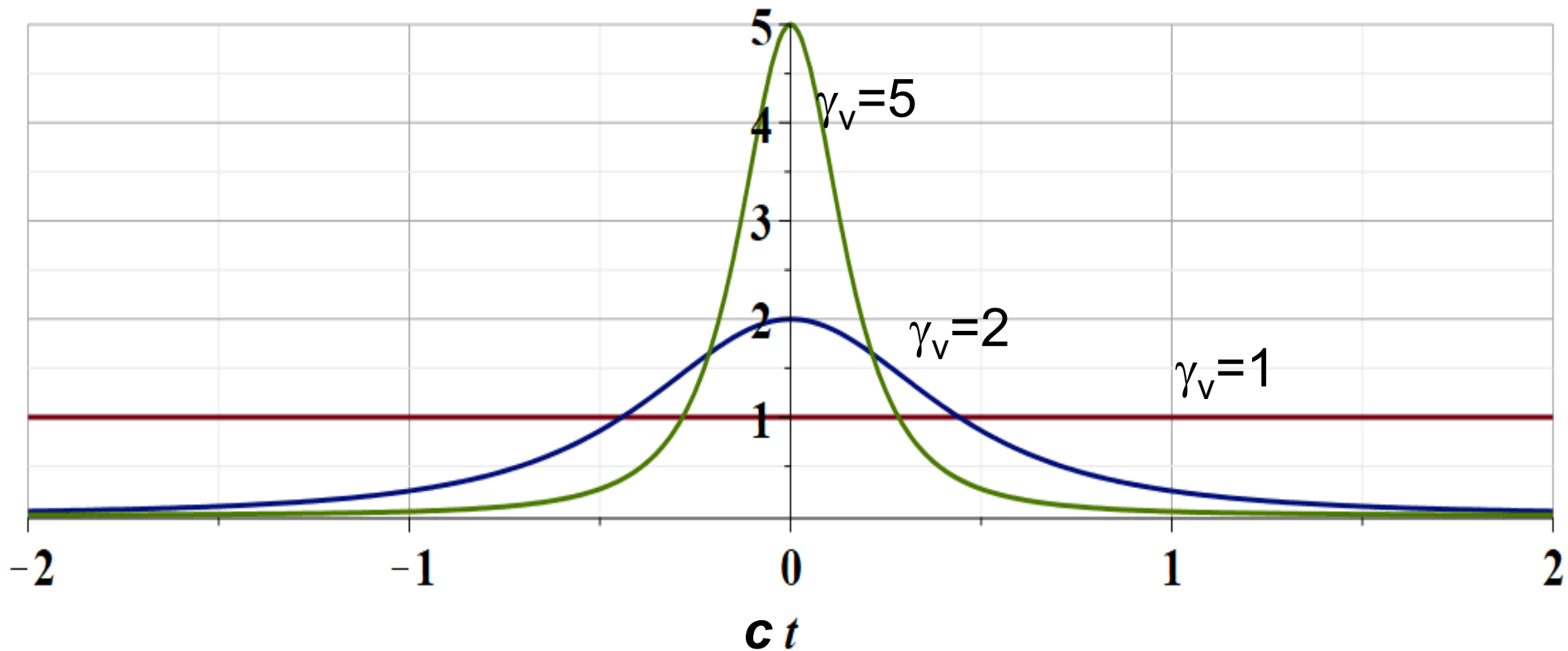$$E_x = E'_x = \frac{q\left(-vt'\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

y   y'

v

b

q

x   x'

z   z'
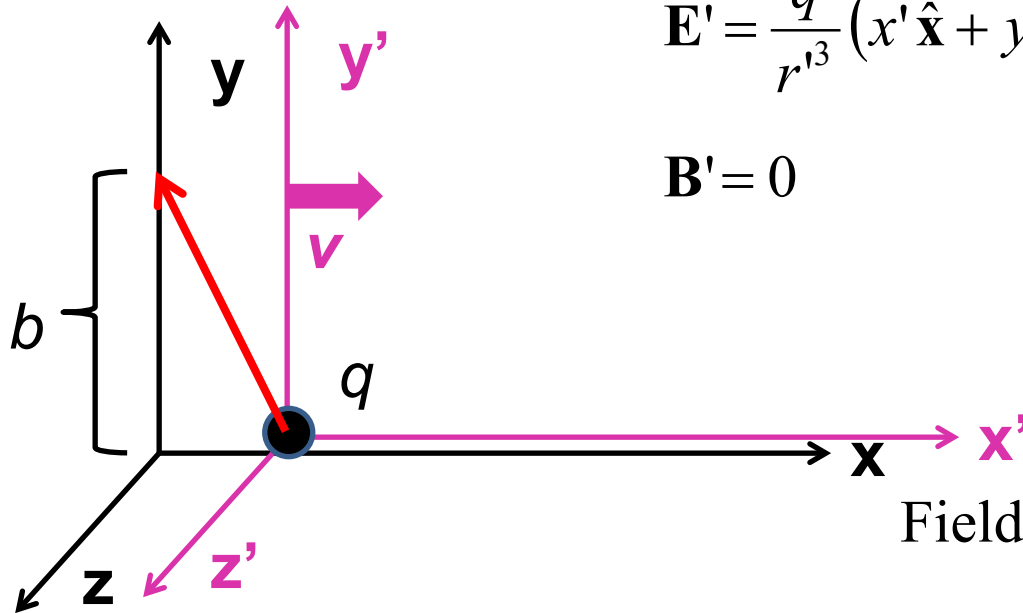
Fields in stationary frame :

$$E_x = E'_x = \frac{q\left(-v\gamma_v t\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

Expression in terms of consistent coordinates

$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$E_y = \frac{q(\gamma_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}} = \frac{q(\gamma_v b)}{\left((\gamma_v^2 - 1)c^2 t^2 + b^2\right)^{3/2}} = B_z / (\gamma_v \beta_v)$$

Plot with *q=1;*   *b=1* γ_v as given

Examination of this system from the viewpoint of the the Liénard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Question – Why would you want to use the Liénard-Wiechert potentials?

1. They are extremely complicated.   It is best to avoid them at all costs?
2. The Lorentz transformations were bad enough?
3. There are some circumstances for which the Lorentz transformations become very complicated (such as when acceleration is involved)
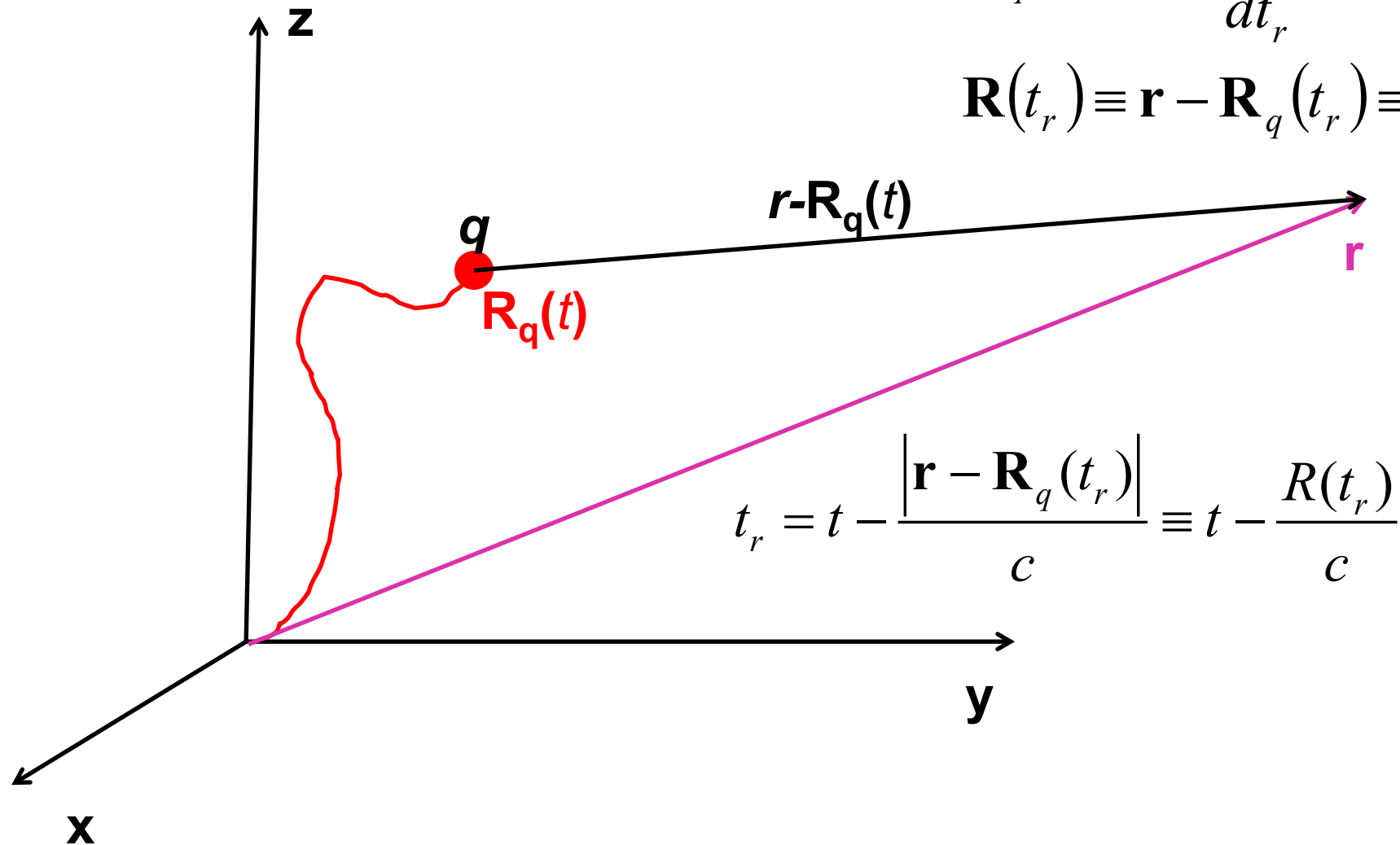
Analysis using a single reference frame --
Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

z

q

r-**R**$_\mathbf{q}$(*t*)

r

R$_\mathbf{q}$(*t*)

$$t_r = t - \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c} \equiv t - \frac{R(t_r)}{c}$$

y

x

Examination of this system from the viewpoint of the the Liénard-Wiechert potentials –(Gaussian units)

Note that for our example there are no acceleration terms.

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

For our example:

$$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}} \qquad \mathbf{r} = b\hat{\mathbf{y}}$$

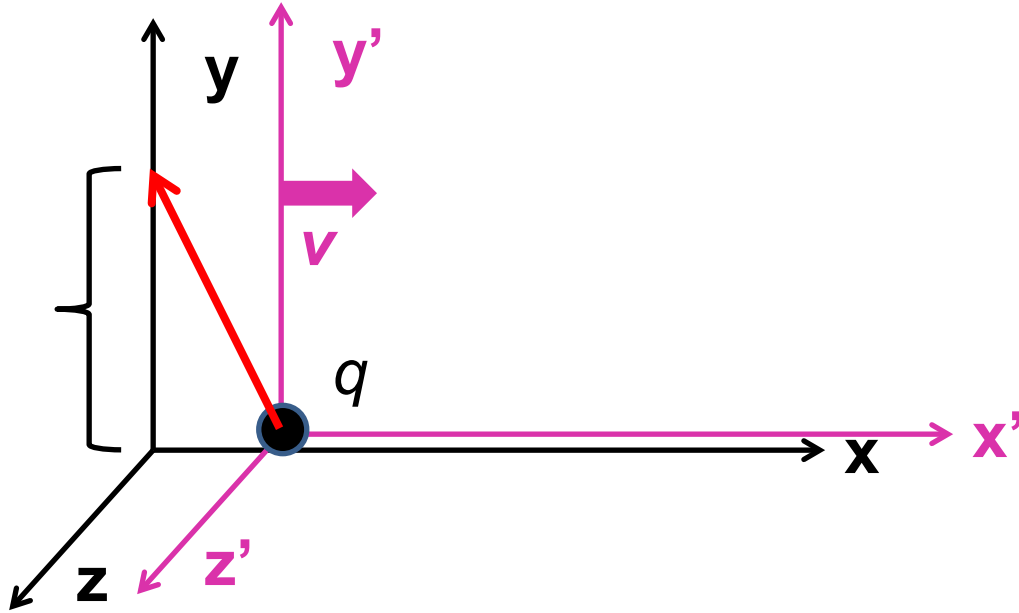$$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}} \qquad R = \sqrt{v^2 t_r^2 + b^2}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{v} = v\hat{\mathbf{x}} \qquad t_r = t - \frac{R}{c}$$

This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(0,b,0,t) = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(x,y,z,t) = \mathbf{B}(0,b,0,t) = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

Example geometry

Trajectory within stationary frame — $\mathbf{R}_q(t_r) = v t_r \hat{\mathbf{x}}$ $\mathbf{r} = b\hat{\mathbf{y}}$

This choice allows us to analyze the Liénard-Wiechert approach (within the "stationary" reference frame) of the same phenomenon solved previously using the Lorentz transformation.   Because of the geometry E_z is zero here.

Why take this example?
1. Complete waste of time since we already know the answer.
2. If we get the same answer as we did using the Lorentz transformation, we will feel more confident in applying this approach to study electromagnetic fields resulting from more complicated trajectories.

Note that it might be advisable to derive the details of the analysis for yourselves.

Some details

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

For our example:

$$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}} \qquad \mathbf{r} = b\hat{\mathbf{y}}$$

$$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}} \qquad R = \sqrt{v^2t_r^2 + b^2}$$

$$\mathbf{v} = v\hat{\mathbf{x}} \qquad t_r = t - \frac{R}{c}$$

t_r must be a solution to a quadratic equation:

$$t_r - t = -\frac{R}{c} \qquad \Rightarrow \qquad t_r^2 - 2\gamma^2 t t_r + \gamma^2 t^2 - \gamma^2 b^2 / c^2 = 0$$

with the physical solution:

$$t_r = \gamma\left(\gamma t - \frac{\sqrt{(v\gamma t)^2 + b^2}}{c}\right)$$

Note that $(t_r - t)^2 = \dfrac{R^2}{c^2} = \dfrac{v^2t_r^2 + b^2}{c^2}$

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$$

Some details continued:

Now we can express R as:

$$R = \gamma\left(-\beta v \gamma t + \sqrt{(v\gamma t)^2 + b^2}\right)$$

and the related quantities:

$$\mathbf{R} - \mathbf{v}R/c = -vt\hat{\mathbf{x}} + b\hat{\mathbf{y}}$$

$$R - \mathbf{v}\cdot\mathbf{R}/c = \frac{\sqrt{(v\gamma t)^2 + b^2}}{\gamma}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right] = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right] = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$
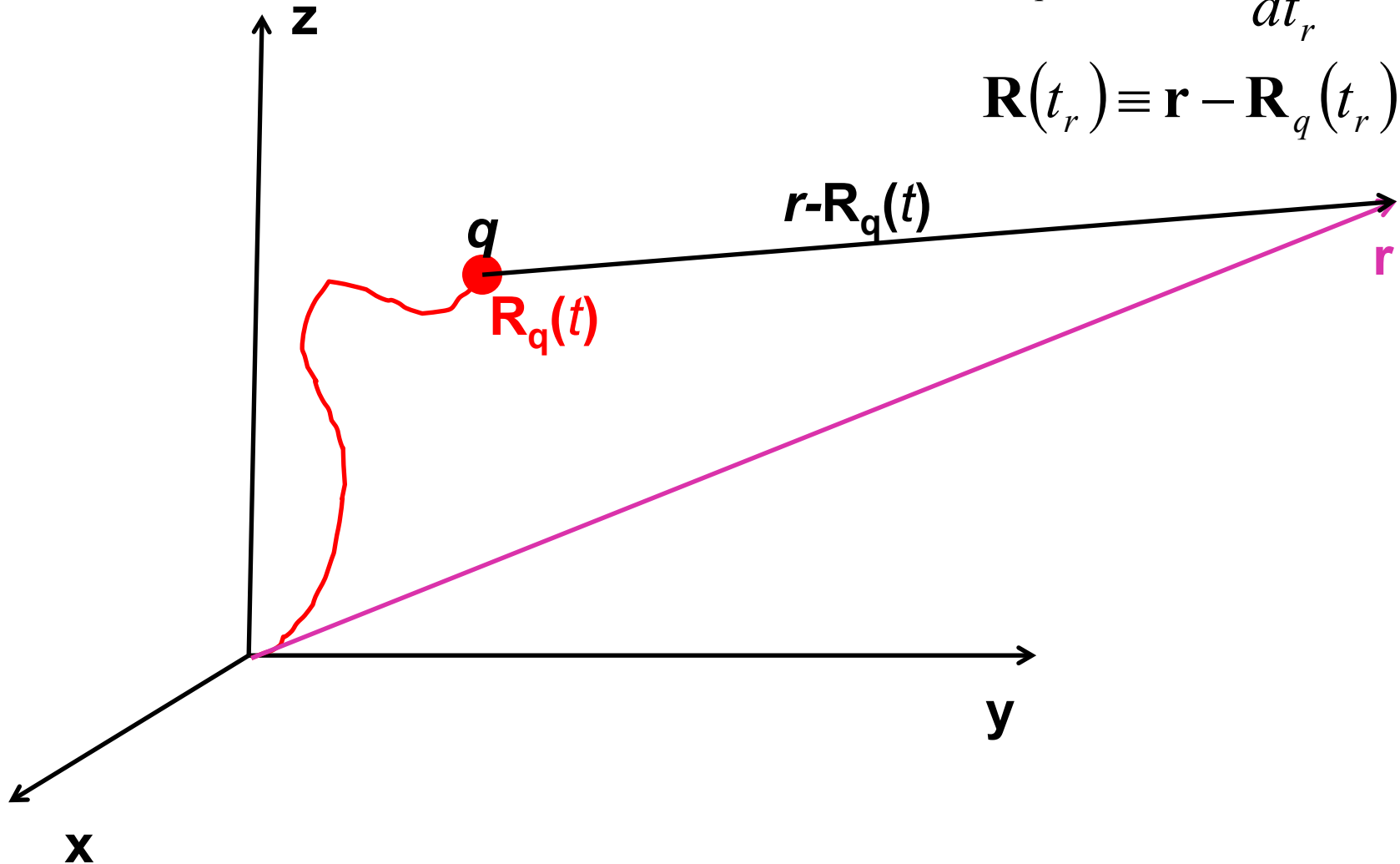
EM fields from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right].$$

(19)

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c} \left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2} \right].$$

(20)

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{R}.$$

(21)

Notation:

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \qquad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\,2}}$$

Electric field far from source – keeping only dominant terms

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Let $\quad \hat{\mathbf{R}} \equiv \dfrac{\mathbf{R}}{R} \qquad \boldsymbol{\beta} \equiv \dfrac{\mathbf{v}}{c} \qquad \dot{\boldsymbol{\beta}} \equiv \dfrac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\left(\mathbf{E} \times \mathbf{B}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^6}$$

Note: We have used the fact that

$$\hat{\mathbf{R}} \cdot \mathbf{E}(\mathbf{r},t) = 0$$

Power radiated

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

In the non-relativistic limit: $\quad \beta << 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$
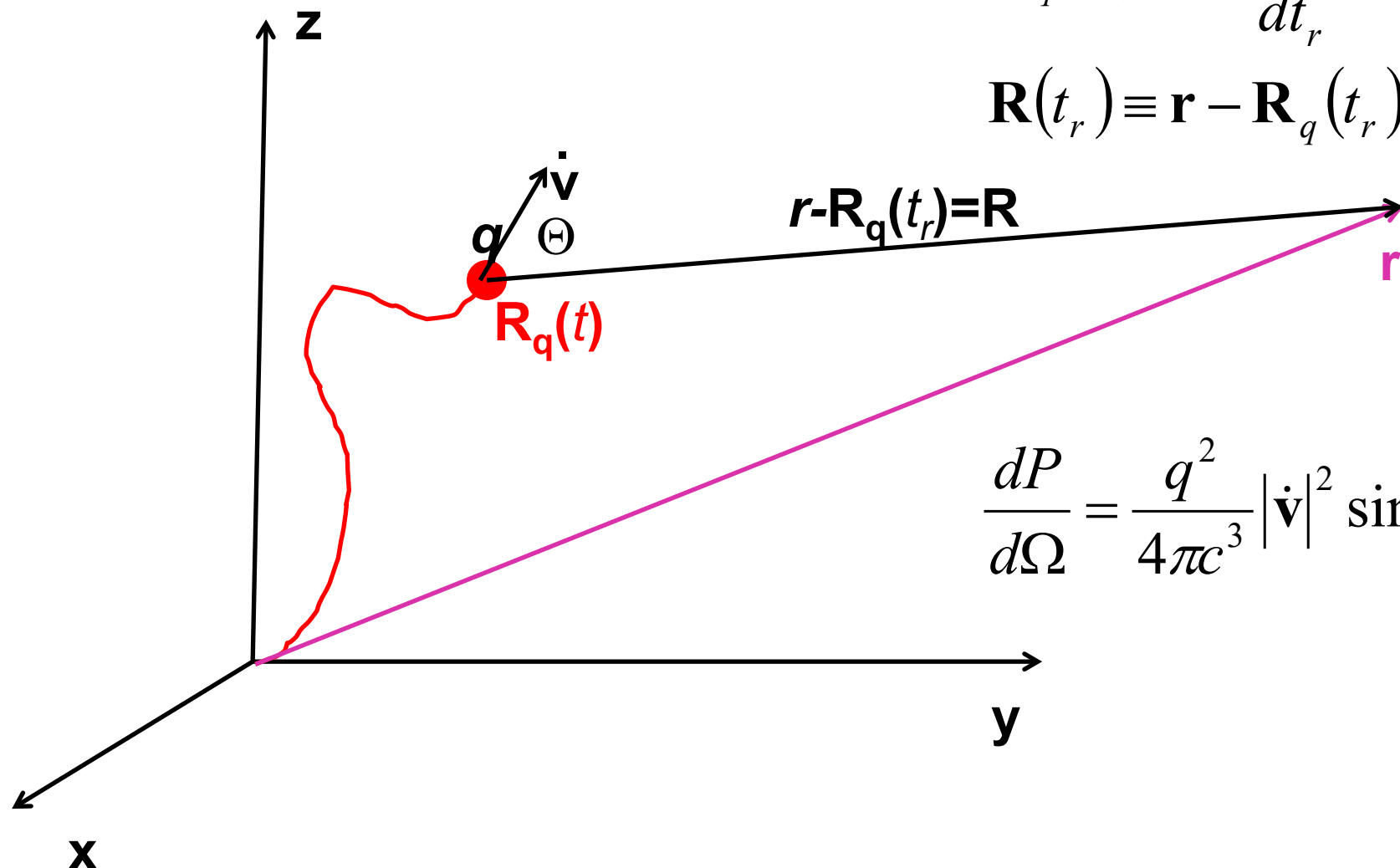
Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

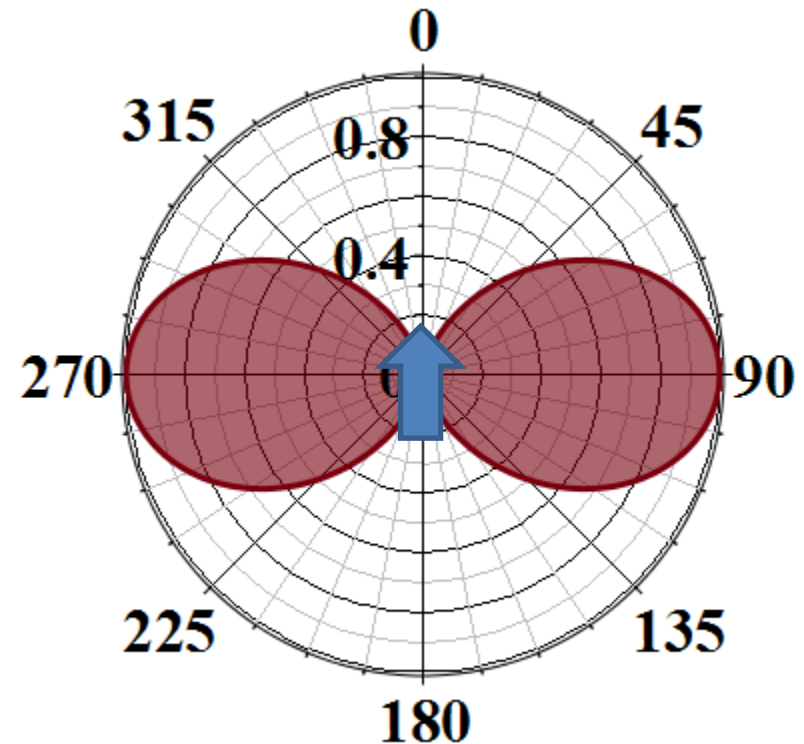$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

z

v̇

q Θ

r-**R$_q$**(*t$_r$*)=**R**

r

R$_q$(*t*)

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2 \sin^2\Theta$$

y

x

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2 \sin^2\Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2$$

Blue arrow indicates the particle acceleration direction

What do you think will happen when the particle velocities become larger with respect to the speed of light in vacuum?

1. The radiation pattern will be essentially the same.
2. The radiation pattern will be quite different.