



PHY 712 Electrodynamics

10-10:50 PM MWF Olin 103

Notes for Lecture 30:

Radiation by moving charges

Chap. 14 – (Sec. 14.1-14.5 in JDJ)

- 1. Motion in a line**
- 2. Motion in a circle**
- 3. Spectral analysis of radiation**

24	Mon: 03/18/2024	Chap. 9	Digression on Math methods and Radiation from localized oscillating sources	#19	03/25/2024
25	Wed: 03/20/2024	Chap. 9	Radiation from localized oscillating sources	#20	03/25/2024
26	Fri: 03/22/2024	Chap. 9 & 10	Radiation and scattering	#21	03/25/2024
27	Mon: 03/25/2024	Chap. 11	Special Theory of Relativity	#22	04/01/2024
28	Wed: 03/27/2024	Chap. 11	Special Theory of Relativity	#23	04/01/2024
29	Fri: 03/29/2024	Chap. 11	Special Theory of Relativity		
30	Mon: 04/01/2024	Chap. 14	Radiation from moving charges	#24	04/08/2024
31	Wed: 04/03/2024	Chap. 14	Radiation from accelerating charged particles		
32	Fri: 04/05/2024	Chap. 14	Synchrotron radiation and Compton scattering		
33	Mon: 04/08/2024	Chap. 15	Radiation from collisions of charged particles		
34	Wed: 04/10/2024	Chap. 13	Cherenkov radiation		
35	Fri: 04/12/2024		Special topic: E & M aspects of superconductivity		
	Mon: 04/15/2024		Presentations I		
	Wed: 04/17/2024		Presentations II		
	Fri: 04/19/2024		Presentations III		
36	Mon: 04/22/2024		Special topic: Quantum Effects in E & M		
37	Wed: 04/24/2024		Special topic: Quantum Effects in E & M		
38	Fri: 04/26/2024		Special topic: Quantum Effects in E & M		
39	Mon: 04/29/2024		Review		
40	Wed: 05/01/2024		Review		

PHY 712 -- Assignment #24

Assigned: 4/01/2024 Due: 4/08/2024

Start reading Chap. 14 (Sec. 14.1-14.5) in **Jackson** .

1. Consider an electron moving at constant speed $\beta c \approx c$ in a circular trajectory of radius ρ . Its total energy is $E = \gamma m c^2 = 200 \text{ GeV}$. Estimate the value of the ratio of the energy lost during one full circle to its total energy. Assume that synchrotron radius in this case is $\rho = 10^3$ meters. Note if you use the expression for this process analyzed by **Jackson**, please explain the details.

When to sign up?

PHY 712 Presentation Schedule

Monday 4/15/2024

	Presenter Name	Topic
10:00-10:24		
10:25-10:50		

Wednesday 4/17/2024

	Presenter Name	Topic
10:00-10:24		
10:25-10:50		

Friday 4/19/2024

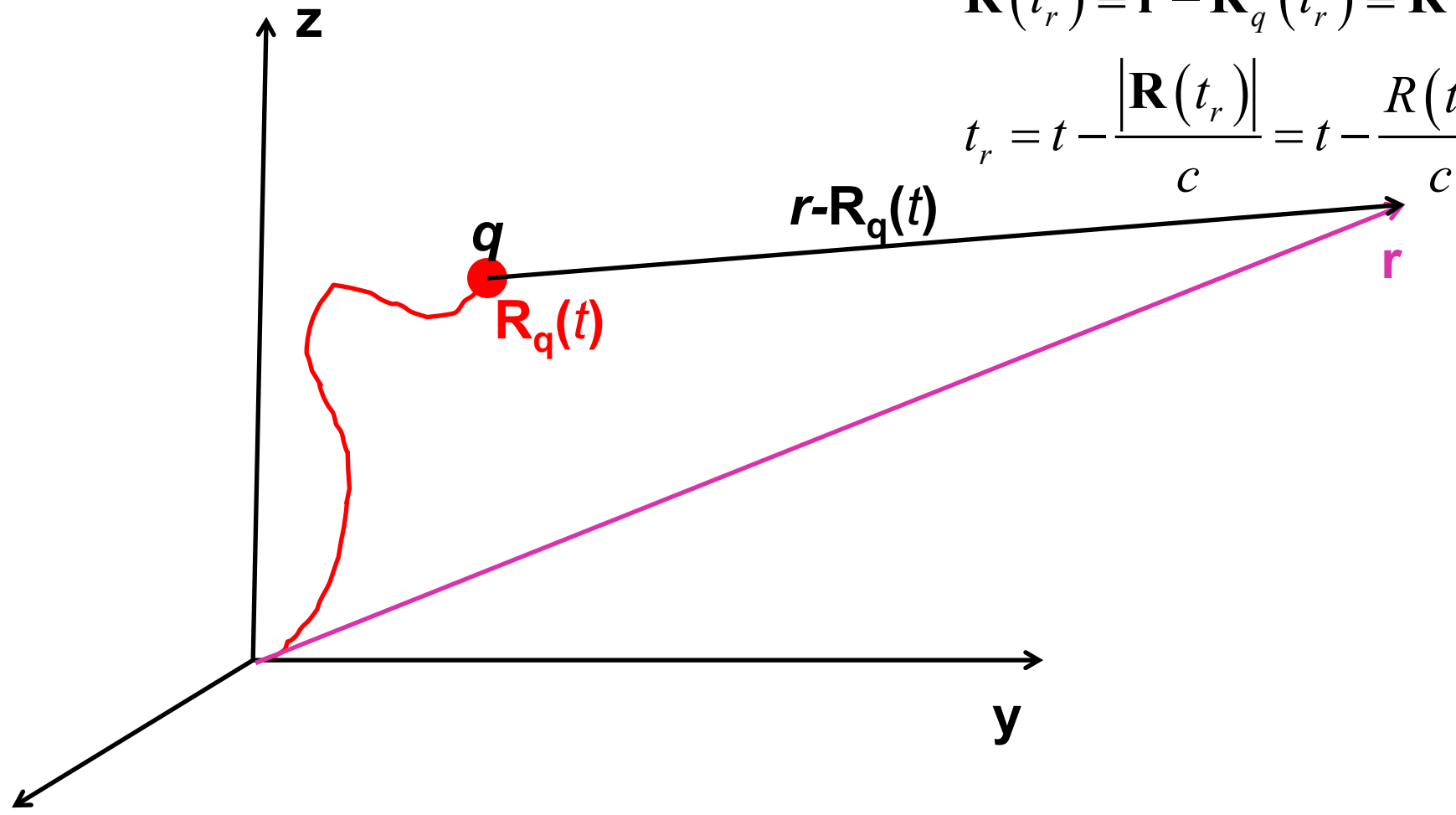
	Presenter Name	Topic
10:00-10:24		

Radiation from a moving charged particle

Variables (notation): $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$

$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

$t_r = t - \frac{|\mathbf{R}(t_r)|}{c} = t - \frac{R(t_r)}{c}$



Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\}\right) \right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}. \quad (21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \quad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \quad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^2}$$

Comment --

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\}\right) \right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}. \quad (21)$$

Note that (21) can be demonstrated by evaluating $\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)$

Other helpful identities:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Electric field far from source:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left\{ \mathbf{R} \times \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}$$

Note that all of the variables on the right hand side of the equations depend on t_r .

$$\text{Let } \hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \quad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t) \quad \mathbf{E} \times (\hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)) = \hat{\mathbf{R}} |\mathbf{E}|^2 - \mathbf{E} (\hat{\mathbf{R}} \cdot \mathbf{E})$$

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

Note: We have used the fact that

$\hat{\mathbf{R}} \cdot \mathbf{E}(\mathbf{r}, t) = 0$ which follows from the vector identities.

Power radiated

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

In the non-relativistic limit: $\beta \ll 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c} \left| \hat{\mathbf{R}} \times \left[\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}} \right] \right|^2 = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

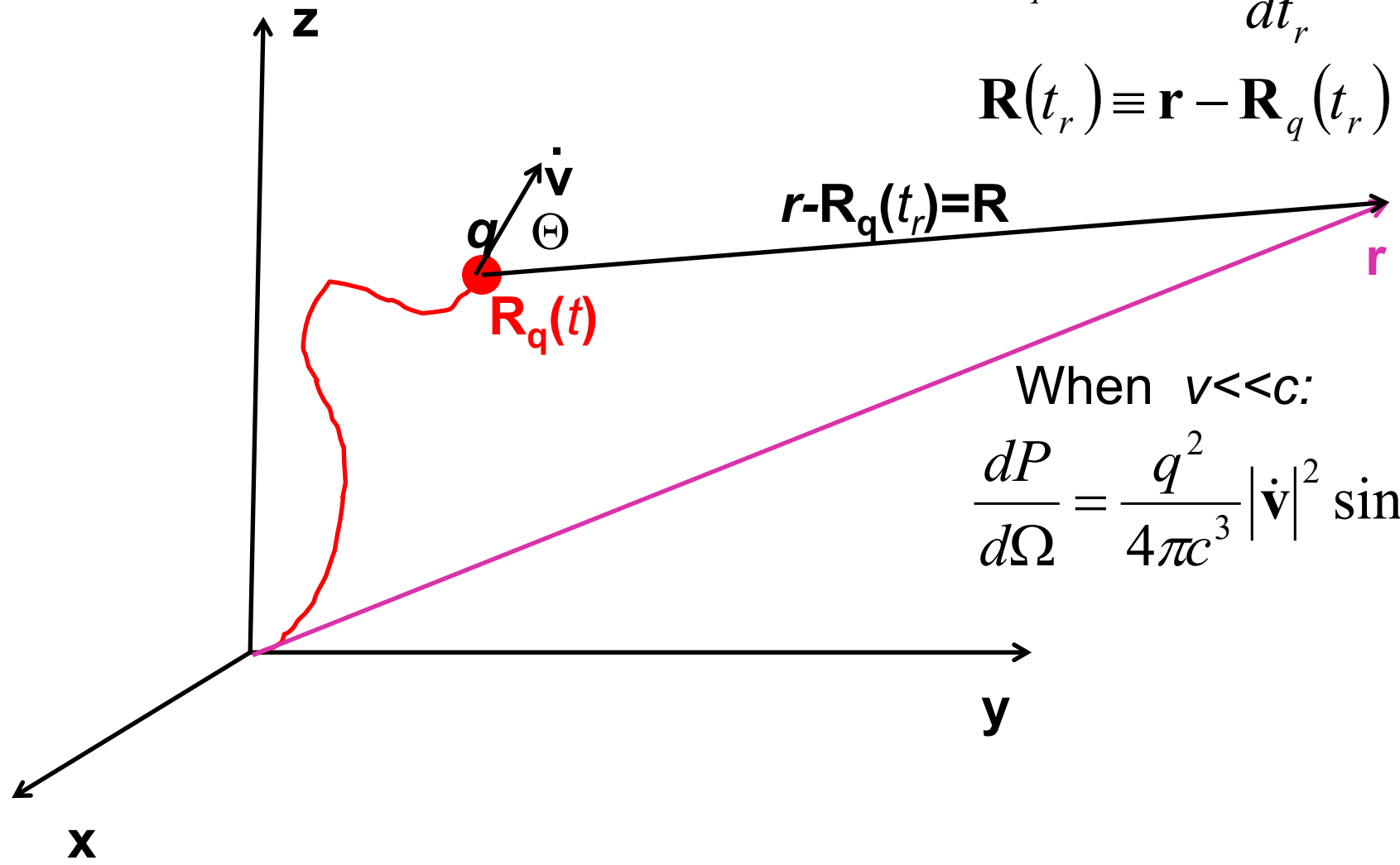


Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$



When $v \ll c$:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2$$

Radiation distribution in the relativistic case

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left| \frac{\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6} \right|^2 \Bigg|_{t_r = t - R/c}$$

This expression gives us the energy per unit field time t . We are often interested in the power per unit retarded time $t_r = t - R/c$:

$$\frac{dP_r(t)}{d\Omega} = \frac{dP(t)}{d\Omega} \frac{dt}{dt_r} \quad \frac{dt}{dt_r} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left| \frac{\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|^2 \Bigg|_{t_r = t - R/c}$$

Some details –

The power derived from the Poynting vector in terms of the field times is given by:

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6} \right|_{t_r = t - R/c}$$

The integrated power would be given by

$$W = \int dt \frac{dP(t)}{d\Omega} = \int dt_r \frac{dt}{dt_r} \frac{dP(t)}{d\Omega} \longrightarrow \frac{dP_r(t_r)}{d\Omega}$$

More comments

$$t_r = t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

$$t = t_r + \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

$$\frac{dt}{dt_r} = 1 + \left(-\frac{d\mathbf{R}_q(t_r)}{cdt_r} \right) \cdot \frac{\mathbf{r} - \mathbf{R}_q(t_r)}{|\mathbf{r} - \mathbf{R}_q(t_r)|} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

$$\rightarrow \frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \Bigg|_{t_r = t - R/c}$$



Why do you think it useful to measure the power as energy per unit retarded time P_r ?

1. Jackson likes to torture us.
2. There should be no difference.
3. ???

Radiation distribution in the relativistic case -- continued

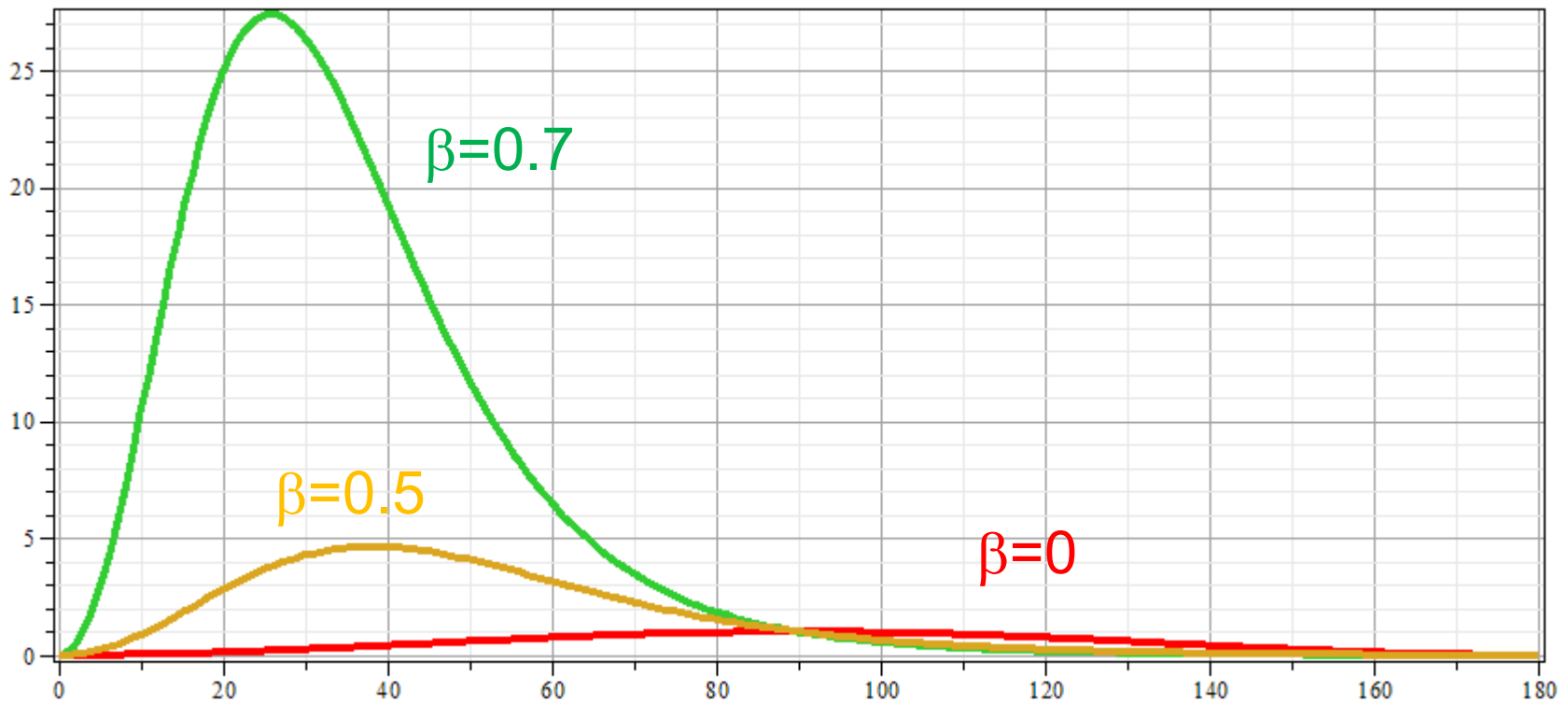
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \Bigg|_{t_r = t - R/c}$$

For linear acceleration: $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}) \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

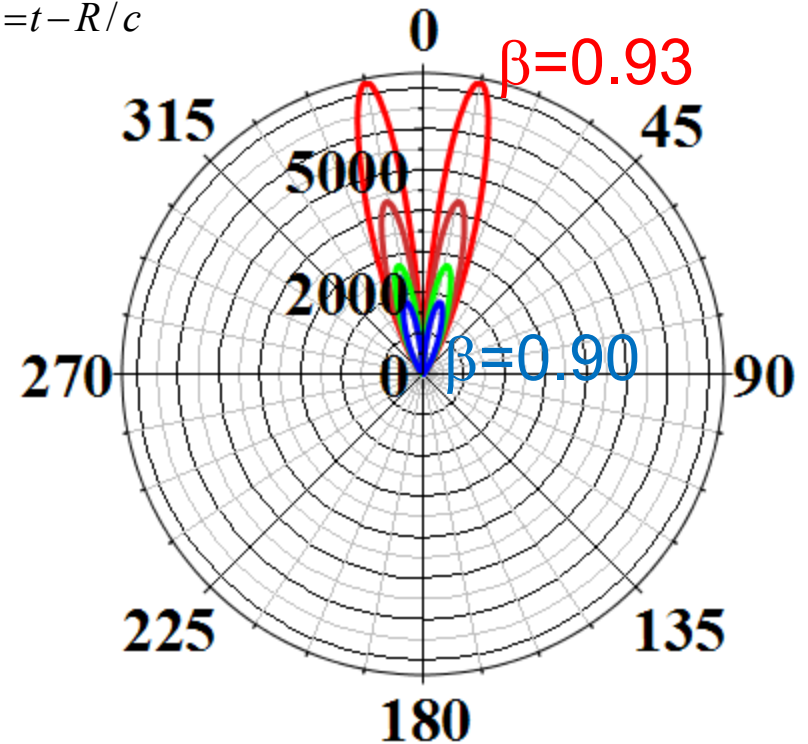
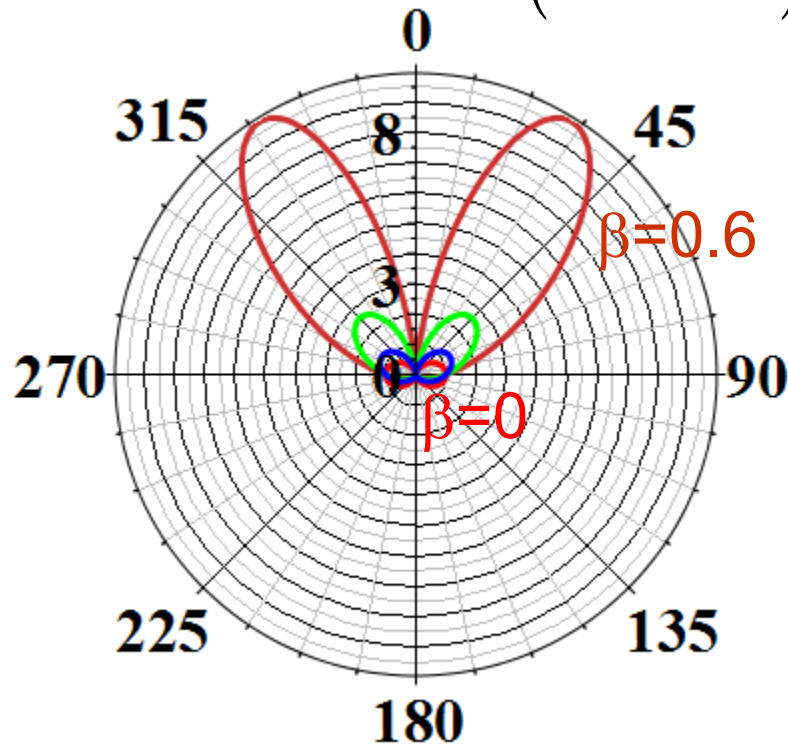
Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$



Polar plots:

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$



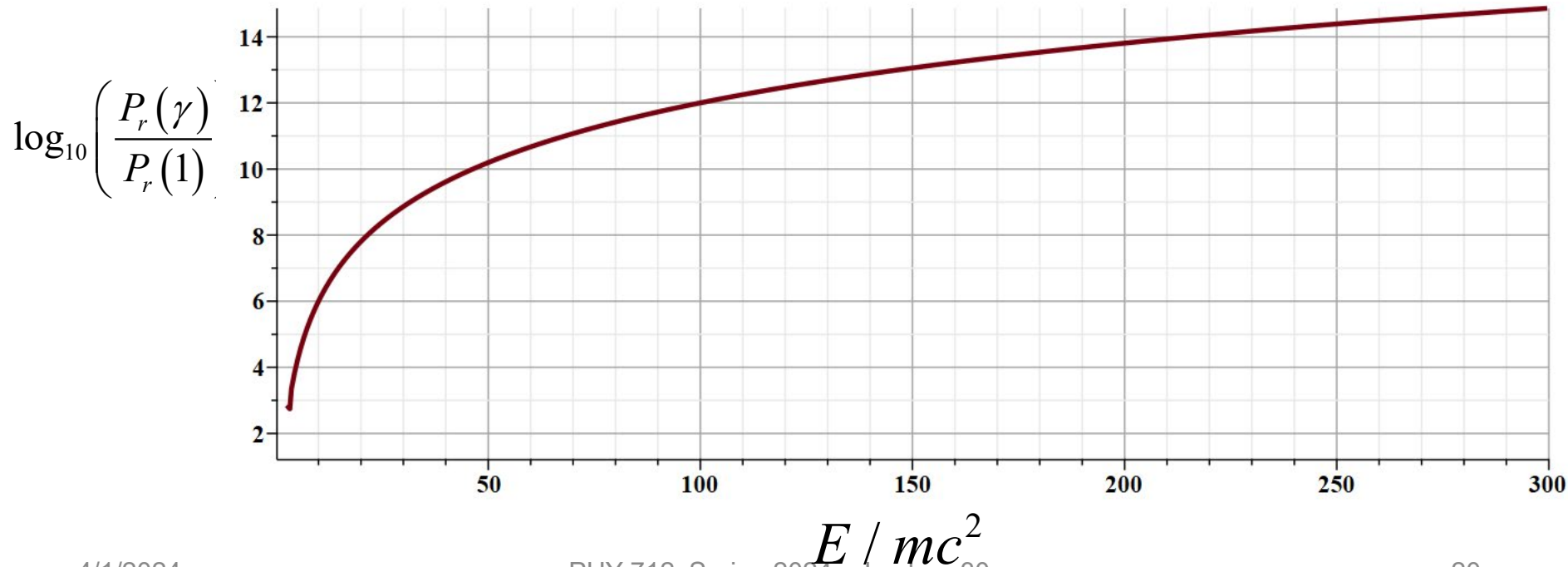
Note – two separate plots are introduced in order to see the drastic change of scale at values of β close to 1.



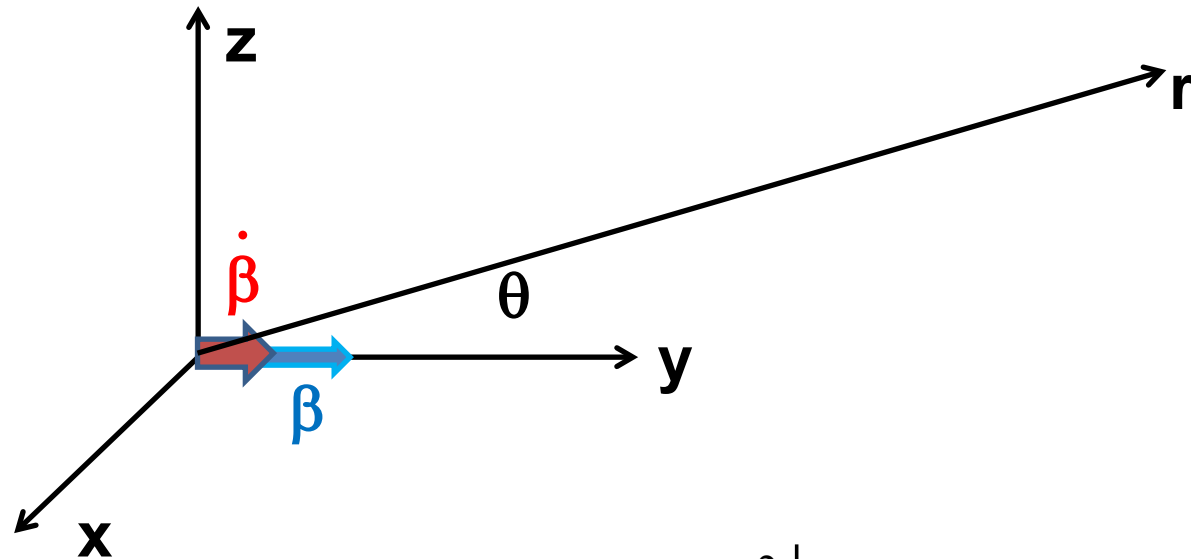
Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} = \frac{E}{mc^2}$$



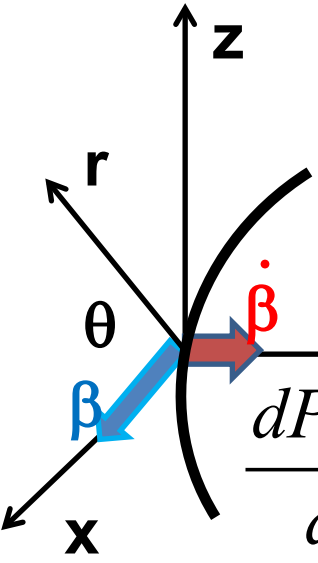
Power distribution for linear acceleration -- continued



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

Power distribution for circular acceleration

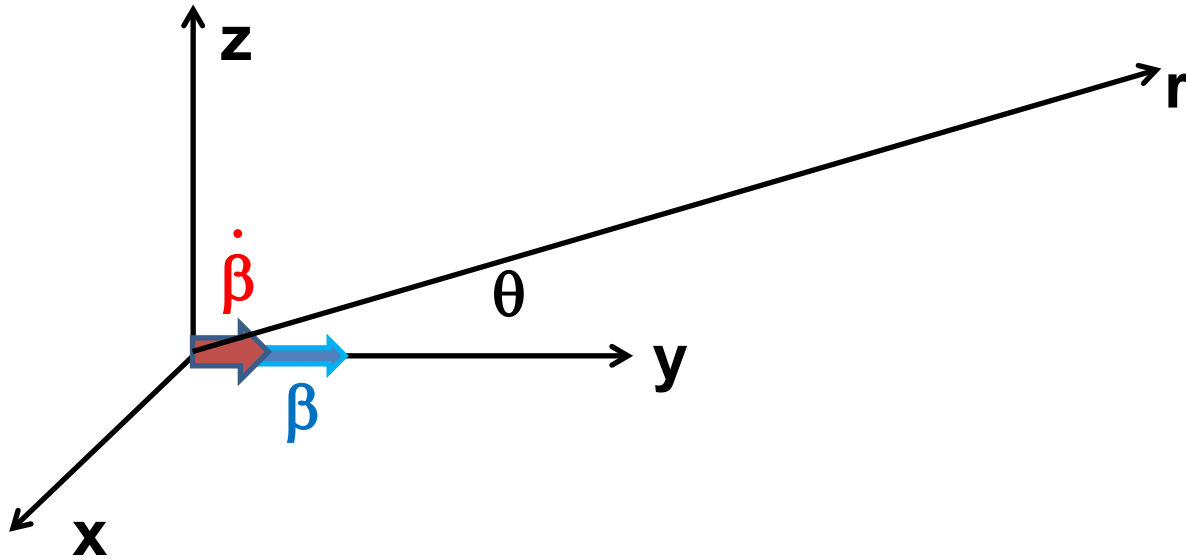


$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \left. \frac{|\dot{\boldsymbol{\beta}}|^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2 - (\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}})^2 (1 - \beta^2)}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c}$$

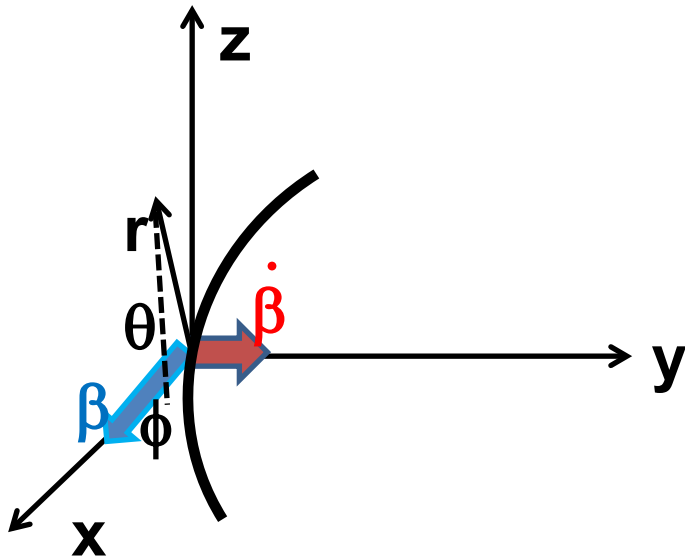
$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^4$$

Summary of results --For linear acceleration --



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\dot{\boldsymbol{\beta}}|^2 \left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5} \Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{\left(1 - \beta \cos(\theta)\right)^3} \left(1 - \frac{\cos^2 \theta \sin^2 \phi}{\gamma^2 \left(1 - \beta \cos(\theta)\right)^2}\right)$$

Angular integrals for the two cases –

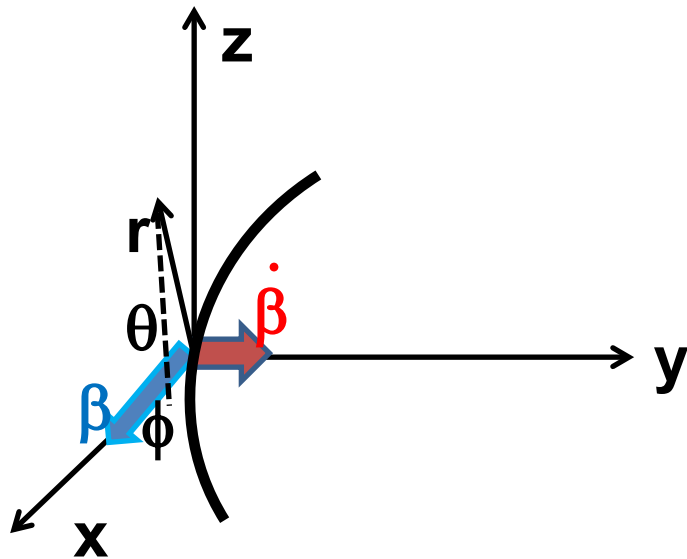
Linear acceleration

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = 2\pi \int \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta d\sin \theta}{(1 - \beta \cos \theta)^5} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6$$

Circular acceleration

$$\begin{aligned} P_r(t_r) &= \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \int d\phi d\sin \theta \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{(1 - \beta \cos(\theta))^3} \left(1 - \frac{\cos^2 \theta \sin^2 \phi}{\gamma^2 (1 - \beta \cos(\theta))^2} \right) \\ &= \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^4 \end{aligned}$$

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\dot{\boldsymbol{\beta}}|^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2 - (\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}})^2 (1 - \beta^2)}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{(1 - \beta \cos(\theta))^3} \left(1 - \frac{\cos^2 \theta \sin^2 \phi}{\gamma^2 (1 - \beta \cos(\theta))^2} \right)$$

Spectral composition of electromagnetic radiation

Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left| \frac{\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6} \right|^2$$

Note: Here we are finding power wrt to the field time frame.

$t_r = t - R/c$

$$\equiv |\mathbf{a}(t)|^2$$

where

$$\mathbf{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c} \left| \frac{\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|^2}$$

$t_r = t - R/c$

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Fourier amplitude :

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \quad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{a}}(\omega) e^{-i\omega t}$$

Parseval's theorem

Marc-Antoine Parseval des Chênes 1755-1836

<http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html>

Checking:

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t}$$

Amplitude in time:

$$\mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{a}}(\omega) e^{-i\omega t}$$

$$\begin{aligned} \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 &= \int_{-\infty}^{\infty} dt \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{a}}(\omega) e^{-i\omega t} \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \tilde{\mathbf{a}}^*(\omega') e^{i\omega' t} \right) \\ &= \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{a}}(\omega) \int_{-\infty}^{\infty} d\omega' \tilde{\mathbf{a}}^*(\omega') \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} dt e^{i(\omega' - \omega)t} \\ &= \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{a}}(\omega) \int_{-\infty}^{\infty} d\omega' \tilde{\mathbf{a}}^*(\omega') \delta(\omega' - \omega) = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2 \end{aligned}$$



Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Note that: $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2 = \int_0^{\infty} d\omega \left(|\tilde{\mathbf{a}}(\omega)|^2 + |\tilde{\mathbf{a}}(-\omega)|^2 \right) \equiv \int_0^{\infty} d\omega \frac{\partial^2 I}{\partial \Omega \partial \omega}$$

$$\frac{\partial^2 I}{\partial \Omega \partial \omega} \equiv 2 |\tilde{\mathbf{a}}(\omega)|^2$$

What is the significance of $\frac{\partial^2 I}{\partial \Omega \partial \omega}$?

1. It is purely a mathematical construct
2. It can be measured



Spectral composition of electromagnetic radiation -- continued

For our case:
$$\mathbf{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \Bigg|_{t_r = t - R/c}$$

Fourier amplitude:

$$\begin{aligned} \tilde{\mathbf{a}}(\omega) &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} \mathbf{a}(t) \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt e^{i\omega t} \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \Bigg|_{t_r = t - R/c} \end{aligned}$$

Spectral composition of electromagnetic radiation -- continued

Fourier amplitude :

$$\begin{aligned}\tilde{\mathbf{a}}(\omega) &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r=t-R/c} e^{i\omega t} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{dt}{dt_r} \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r=t-R/c} e^{i\omega(t_r+R(t_r)/c)} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2} \right|_{t_r=t-R/c} e^{i\omega(t_r+R(t_r)/c)}\end{aligned}$$

Spectral composition of electromagnetic radiation -- continued

Exact expression :

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Recall: $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$ $\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

For $r \gg R_q(t_r)$ $R(t_r) \approx r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)$ where $\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}$

At the same level of approximation: $\hat{\mathbf{R}} \approx \hat{\mathbf{r}}$



Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Approximate expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega(r/c)} \int_{-\infty}^{\infty} dt_r \frac{\left| \hat{\mathbf{r}} \times \left[(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \frac{\left| \hat{\mathbf{r}} \times \left[(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

Example – radiation from a collinear acceleration burst

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \frac{\left| \hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \right|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \Bigg|^2$$

Suppose that $\dot{\boldsymbol{\beta}} = \begin{cases} \frac{\hat{\boldsymbol{\beta}} \Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$

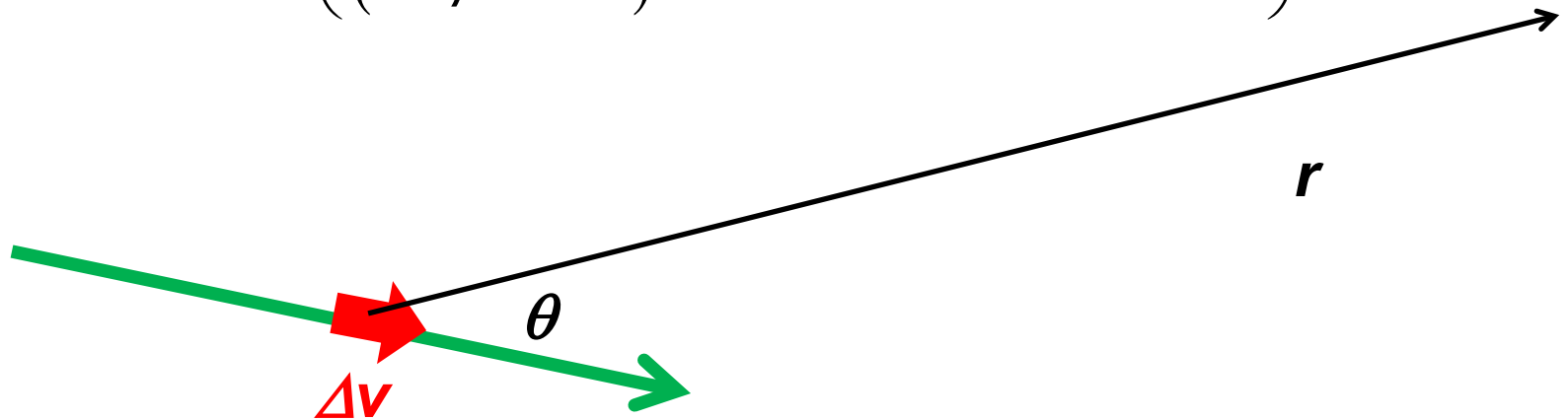
$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left| \frac{\left| \hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \hat{\boldsymbol{\beta}}] \right| \Delta v}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2 \tau} \right|^2 \left| \int_0^\tau dt_r e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \boldsymbol{\beta} t_r)} \right|^2 \quad \text{Let } \boldsymbol{\beta} \cdot \hat{\mathbf{r}} = \beta \cos \theta$$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left(\frac{\Delta v \sin \theta}{(1 - \beta \cos \theta)^2} \frac{\sin(\omega\tau(1 - \beta \cos \theta)/2)}{(\omega\tau(1 - \beta \cos \theta)/2)} \right)^2$$

Example:

$$\text{Suppose that } \dot{\boldsymbol{\beta}} = \begin{cases} \frac{\hat{\boldsymbol{\beta}} \Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left(\frac{\Delta v \sin \theta}{(1 - \beta \cos \theta)^2} \frac{\sin(\omega\tau(1 - \beta \cos \theta) / 2)}{(\omega\tau(1 - \beta \cos \theta) / 2)} \right)^2$$



Example: “Bremsstrahlung” radiation



Spectral composition of electromagnetic radiation -- continued

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}} \times \left[(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} = \frac{d}{dt_r} \left(\frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta})}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})} \right)$$

Integration by parts and assumptions about the integration limit behaviors shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}(t_r)) \right] e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$