PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 32:
Radiation by moving charges

Continue reading Chap. 14 – (Sec. 14.1-14.8 in JDJ)

1. Recap of results for synchrotron radiation from land-based sources
2. Synchrotron radiation from astronomical sources
3. Compton scattering

24	Mon: 03/18/2024	Chap. 9	Digression on Math methods and Radiation from localized oscillating sources	#19	03/25/2024
25	Wed: 03/20/2024	Chap. 9	Radiation from localized oscillating sources	#20	03/25/2024
26	Fri: 03/22/2024	Chap. 9 & 10	Radiation and scattering	#21	03/25/2024
27	Mon: 03/25/2024	Chap. 11	Special Theory of Relativity	#22	04/01/2024
28	Wed: 03/27/2024	Chap. 11	Special Theory of Relativity	#23	04/01/2024
29	Fri: 03/29/2024	Chap. 11	Special Theory of Relativity		
30	Mon: 04/01/2024	Chap. 14	Radiation from moving charges	#24	04/08/2024
31	Wed: 04/03/2024	Chap. 14	Radiation from accelerating charged particles	#25	04/08/2024
32	Fri: 04/05/2024	Chap. 14	Synchrotron radiation and Compton scattering	#26	04/08/2024
33	Mon: 04/08/2024	Chap. 15	Radiation from collisions of charged particles		
34	Wed: 04/10/2024	Chap. 13	Cherenkov radiation		
35	Fri: 04/12/2024		Special topic: E & M aspects of superconductivity		
	Mon: 04/15/2024		Presentations I		
	Wed: 04/17/2024		Presentations II		
	Fri: 04/19/2024		Presentations III		
36	Mon: 04/22/2024		Special topic: Quantum Effects in E & M		
37	Wed: 04/24/2024		Special topic: Quantum Effects in E & M		
38	Fri: 04/26/2024		Special topic: Quantum Effects in E & M		
39	Mon: 04/29/2024		Review		
40	Wed: 05/01/2024		Review		

PHY 712 -- Assignment #26

Assigned: 4/05/2024   Due: 4/08/2024

*Finish reading Chap. 14 (Sec. 14.1-14.8) in **Jackson** .*

1. *This problem concerns the Compton scattering of a photon having an initial momentum magnitude of p and a final momentum magnitude p' at an angle θ due to an electron of mass m, initially at rest, as discussed in lecture and on page 696 of **Jackson**. The wavelength of the photon before the collision is λ=h/p and after is λ'=h/p', where h is Planck's constant. Show that*

   $\lambda'-\lambda=(h/(mc))(1-\cos\theta)$.

Comment about units

Differential power (cgs Gaussian)

$$\frac{dP_r(t_r)}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\hat{\mathbf{r}}\times\left(\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}\right)^5}$$

e measured in Statcoulombs
Length measured in cm
Energy measured in ergs

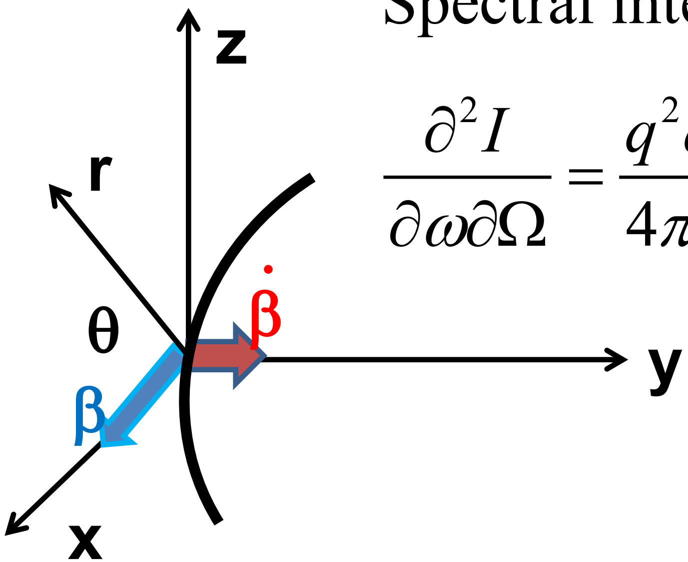
Differential power (SI)

$$\frac{dP_r(t_r)}{d\Omega} = \frac{e^2}{\left(4\pi\epsilon_0\right)4\pi c}\frac{\left|\hat{\mathbf{r}}\times\left(\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}\right)^5}$$

e measured in Coulombs
Length measured in m
Energy measured in joules

Main results from synchrotron radiation spectra from man made sources --

Spectral intensity relationship:



$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \; \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]\right|^2$$

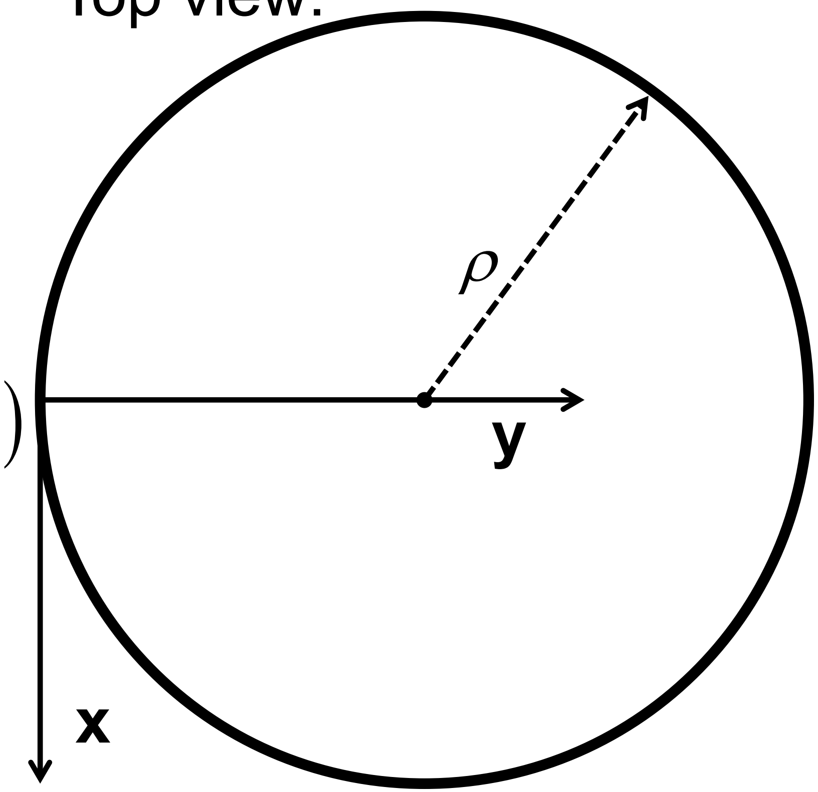
$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

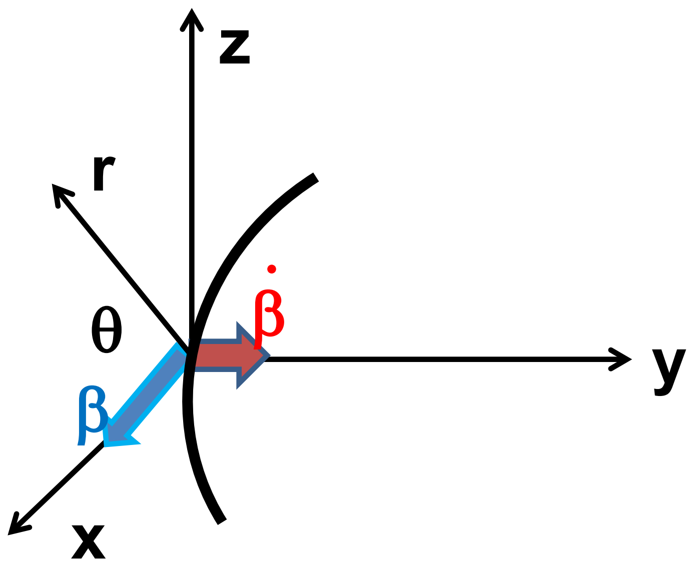
$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Top view:





$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

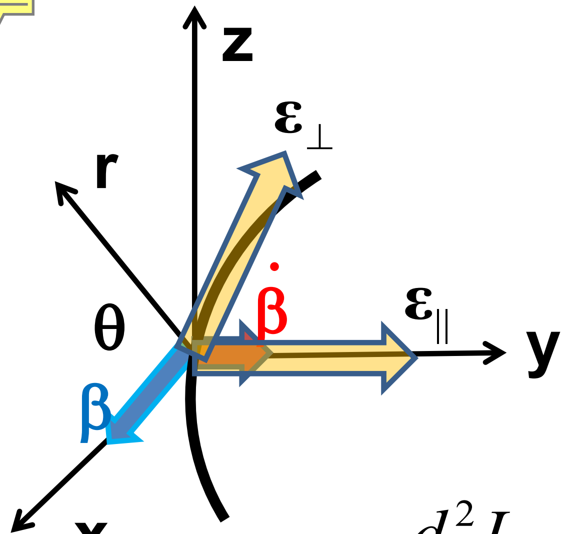
For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Note that we have previous shown that in the radiation zone, the Poynting vector is in the $\hat{\mathbf{r}}$ direction; we can then choose to analyze two orthogonal polarization directions:

$$\boldsymbol{\varepsilon}_{||} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{||}\sin(vt_r/\rho) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$



$$\boldsymbol{\varepsilon}_{||} = \hat{\mathbf{y}} \qquad\qquad \boldsymbol{\varepsilon}_\perp = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta}\right) =$$

$$\beta\left(-\boldsymbol{\varepsilon}_{||}\sin\left(vt_r/\rho\right) + \boldsymbol{\varepsilon}_\perp \sin\theta\cos\left(vt_r/\rho\right)\right)$$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \beta) e^{i\omega(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)} dt\right|^2$$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_{||}(\omega)|^2 + |C_\perp(\omega)|^2\right\}$$

$$C_{||}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

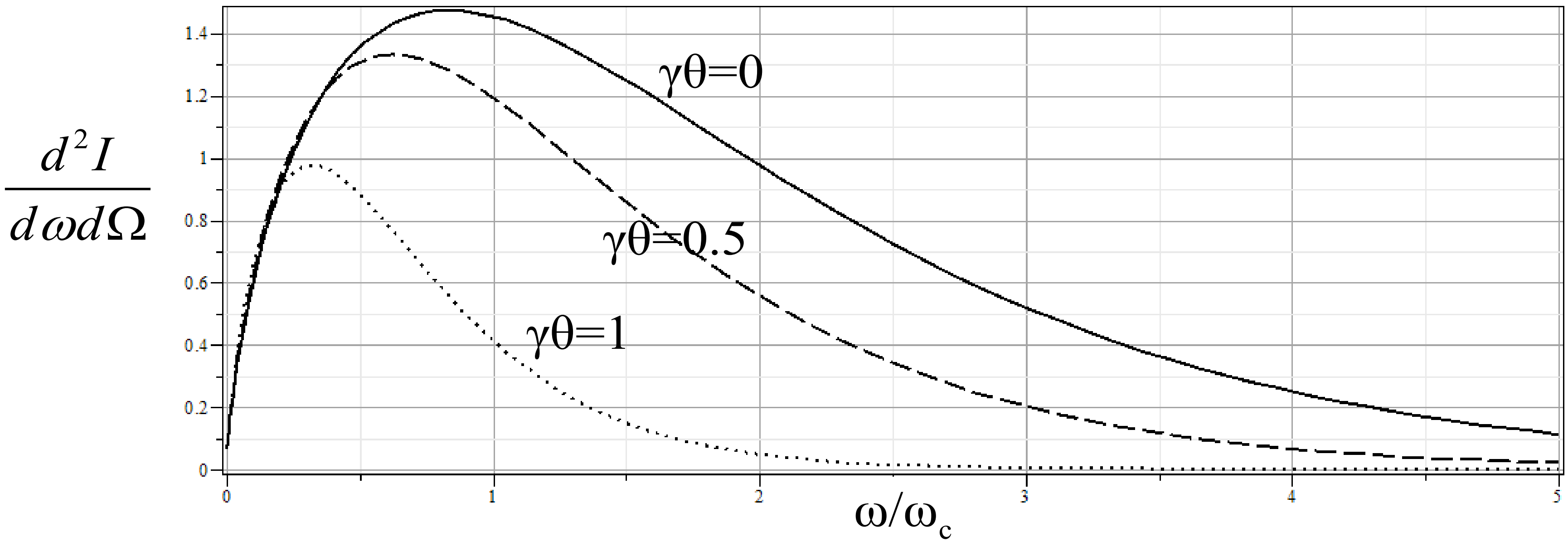
$$C_\perp(\omega) = \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1-1/(2\gamma^2))$) passing a beam line port. In addition, because of the design of the radiation ports, $\theta \approx 0$, and the relevant integration times t are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency $\omega_c \equiv \dfrac{3c\gamma^3}{2\rho}$.

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

By plotting the intensity as a function of ω, we see that the intensity is largest near $\omega \approx \omega_c$. The plot below shows the intensity as a function of ω/ω_c for $\gamma\theta=0$, 0.5 and 1:

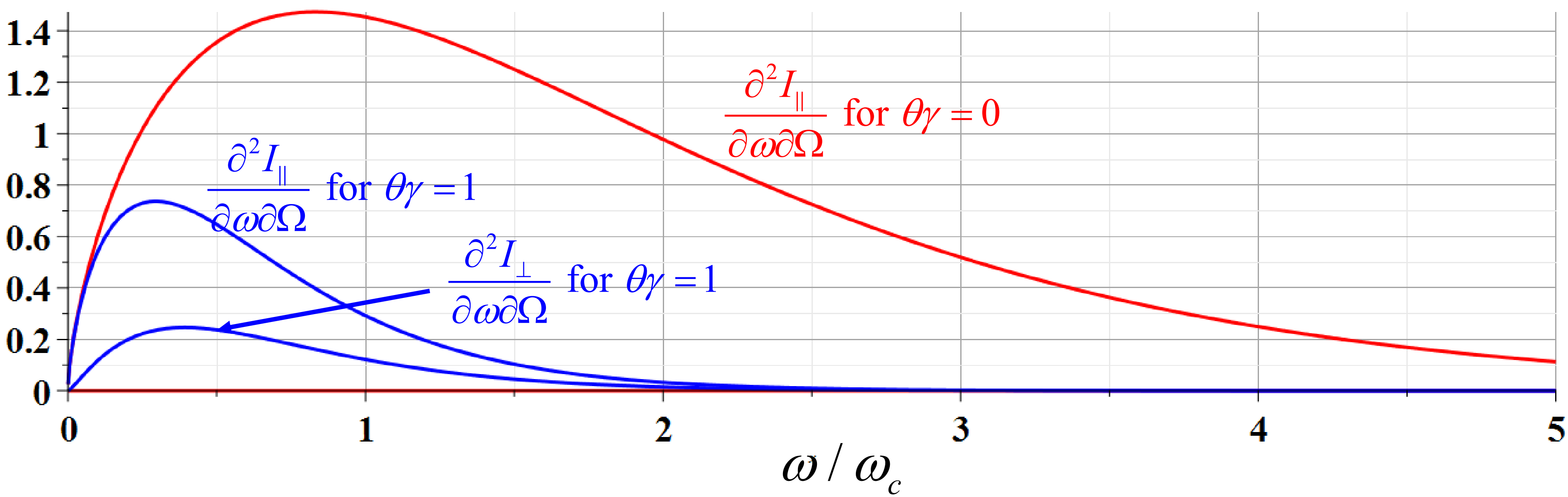


More details

$$\frac{d^2 I}{d\omega d\Omega} = \frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega}$$

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$



https://lightsources.org/lightsources-of-the-world/

Comment on light source facilities and the newer free electron laser technology

SCIENCE'S COMPASS • REVIEW

REVIEW: LASER TECHNOLOGY

Free-Electron Lasers: Status and Applications

Patrick G. O'Shea[1] and Henry P. Freund[2]

A free-electron laser consists of an electron beam propagating through a periodic magnetic field. Today such lasers are used for research in materials science, chemical technology, biophysical science, medical applications, surface studies, and solid-state physics. Free-electron lasers with higher average power and shorter wavelengths are under development. Future applications range from industrial processing of materials to light sources for soft and hard x-rays.

tions at wavelengths down to 1 Å, and this is illustrated by the peak brilliance of a wide range of the present-day synchrotrons and the projected performance of FELs (Fig. 3) (7). Consequently, applications in the x-ray region will undergo an upheaval similar to that which followed the invention of the laser at visible wavelengths.

Ultraviolet FEL oscillators using electron

DOI: 10.1126/science.1055718

Free-Electron Lasers: Status and Applications

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SCIENCE VOL 292 8 JUNE 2001

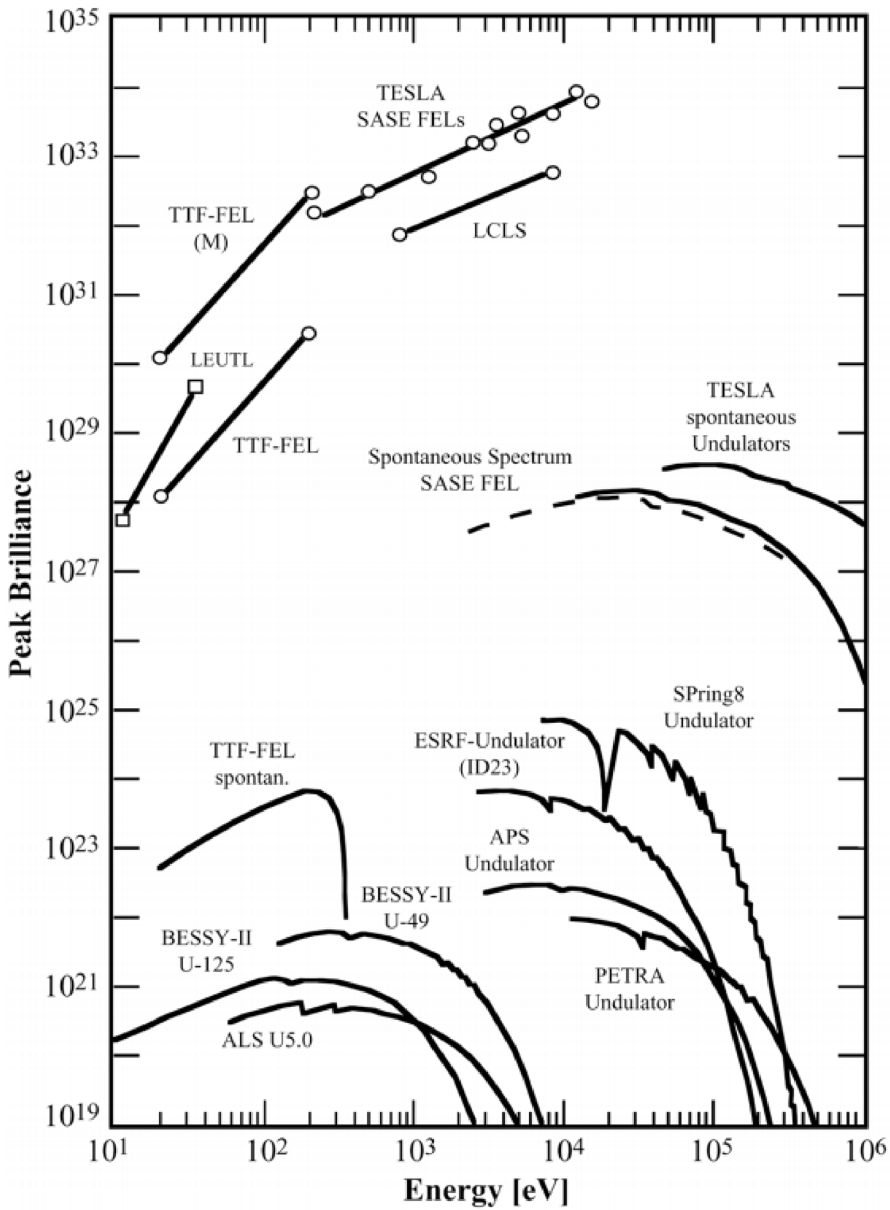


Fig. 3. Peak brilliance of x-ray FELs and undulators for spontaneous radiation at the TESLA Test Facility, in comparison with synchrotron radiation sources. Brilliance is expressed as photons s^{-1} $mrad^{-2}$ mm^{-2} per 0.1% bandwidth. For comparison, the spontaneous spectrum of x-ray FEL undulators is also shown. The label TTF-FEL indicates design values for the FEL at the TESLA Test Facility, with (M) for the planned seeded version (*28*).

Additional references on Free Electron Lasers

https://doi.org/10.1016/j.xinn.2021.100097

5 J.M.J. Madey

Stimulated emission of bremsstrahlung in a periodic magnetic field

J. Appl. Phys., 42 (1971), pp. 1906-1913, 10.1063/1.1660466

View PDF View Record in Scopus Google Scholar

6 D.A.G. Deacon, L.R. Elias, J.M.J. Madey, G.J. Ramian, H.A. Schwettman, T.I. Smith

First operation of a free-electron laser

Phys. Rev. Lett., 38 (1977), pp. 892-894, 10.1103/PhysRevLett.38.892

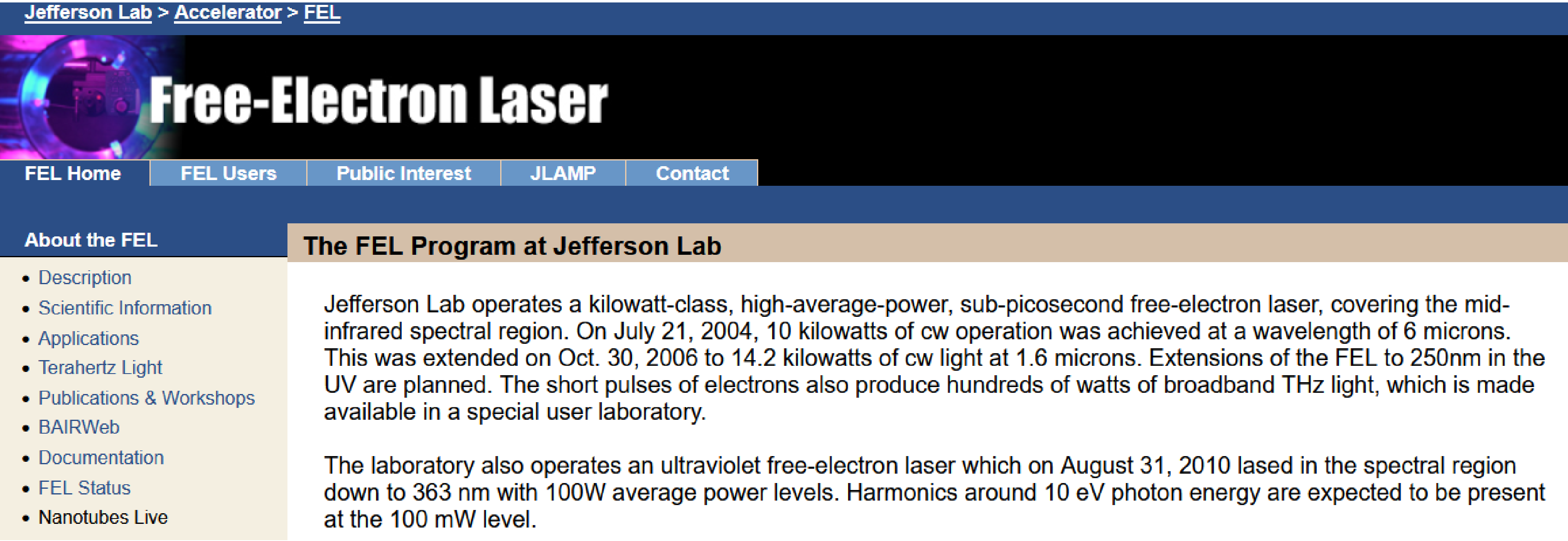
View PDF Google Scholar

Components of the FEL

1. Electrons moving in circular paths
2. Self-amplified spontaneous emission (SASE)
3. +++

Often designed for the X-ray range

Jefferson Free Electron Laser Lab in VA is designed in the microwave←→infrared range



Jefferson Lab > Accelerator > FEL

Free-Electron Laser

FEL Home | FEL Users | Public Interest | JLAMP | Contact

About the FEL

- Description
- Scientific Information
- Applications
- Terahertz Light
- Publications & Workshops
- BAIRWeb
- Documentation
- FEL Status
- Nanotubes Live

The FEL Program at Jefferson Lab

Jefferson Lab operates a kilowatt-class, high-average-power, sub-picosecond free-electron laser, covering the mid-infrared spectral region. On July 21, 2004, 10 kilowatts of cw operation was achieved at a wavelength of 6 microns. This was extended on Oct. 30, 2006 to 14.2 kilowatts of cw light at 1.6 microns. Extensions of the FEL to 250nm in the UV are planned. The short pulses of electrons also produce hundreds of watts of broadband THz light, which is made available in a special user laboratory.

The laboratory also operates an ultraviolet free-electron laser which on August 31, 2010 lased in the spectral region down to 363 nm with 100W average power levels. Harmonics around 10 eV photon energy are expected to be present at the 100 mW level.

Back to synchrotron case --

The above analysis applies to a class of man-made facilities dedicated to producing intense radiation in the continuous spectrum. For more specific information on man-made synchrotron sources, the following web page is useful: http://www.als.lbl.gov/als/synchrotron_sources.html.

Synchrotron radiation is also produced by astronomical sources as analyzed by Julian Schwinger –

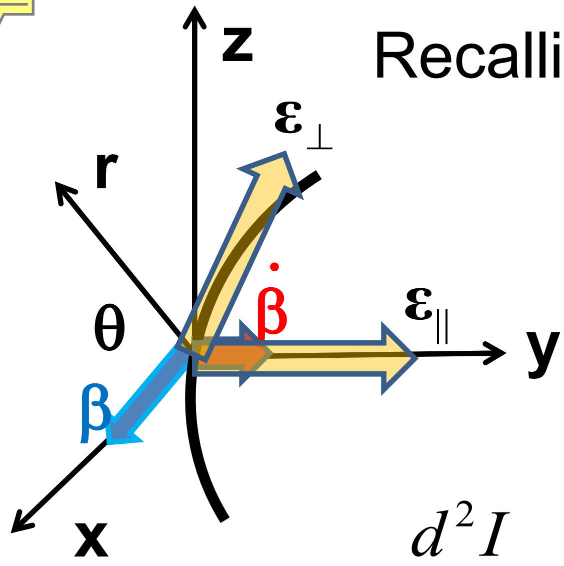
PHYSICAL REVIEW VOLUME 75, NUMBER 12 JUNE 15, 1949

On the Classical Radiation of Accelerated Electrons

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts
(Received March 8, 1949)

This paper is concerned with the properties of the radiation from a high energy accelerated electron, as recently observed in the General Electric synchrotron. An elementary derivation of the total rate of radiation is first presented, based on Larmor's formula for a slowly moving electron, and arguments of relativistic invariance. We then construct an expression for the instantaneous power radiated by an electron moving along an arbitrary, prescribed path. By casting this result into various forms, one obtains the angular distribution, the spectral distribution, or the combined angular and spectral distributions of the radiation. The method is based on an examination of the rate at which the electron irreversibly transfers energy to the electromagnetic field, as determined by half the difference of retarded and advanced electric field intensities. Formulas are obtained for an arbitrary charge-current distribution and then specialized to a point charge. The total radiated power and its angular distribution are obtained for an arbitrary trajectory. It is found that the direction of motion is a strongly preferred direction of emission at high energies. The spectral distribution of the radiation depends upon the detailed motion over a time interval large compared to the period of the radiation. However, the narrow cone of radiation generated by an energetic electron indicates that only a small part of the trajectory is effective in producing radiation observed in a given direction, which also implies that very high frequencies are emitted. Accordingly, we evaluate the spectral and angular distributions of the high frequency radiation by an energetic electron, in their dependence upon the parameters characterizing the instantaneous orbit. The average spectral distribution, as observed in the synchrotron measurements, is obtained by averaging the electron energy over an acceleration cycle. The entire spectrum emitted by an electron moving with constant speed in a circular path is also discussed. Finally, it is observed that quantum effects will modify the classical results here obtained only at extraordinarily large energies.

DOI:https://doi.org/10.1103/PhysRev.75.1912



Recalling general results of analysis --

$$\boldsymbol{\varepsilon}_{\parallel} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta}\right) =$$

$$\beta\left(-\boldsymbol{\varepsilon}_{\parallel}\sin\left(vt_r/\rho\right) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos\left(vt_r/\rho\right)\right)$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\beta)e^{i\omega(t-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)}dt\right|^2$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2\right\}$$

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt\sin(vt/\rho)e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt\sin\theta\cos(vt/\rho)e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

Useful identity involving Bessel functions

$$e^{-iA\sin\alpha} = \sum_{m=-\infty}^{\infty} J_m(A)e^{-im\alpha}$$ Here $J_m(A)$ is a Bessel function of integer order m.

In our case, $A = \frac{\omega\rho}{c}\cos\theta$ and $\alpha = \frac{vt}{\rho}$.

$$C_{||}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\int_{-\infty}^{\infty} dt e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)2\pi\delta(\omega - m\frac{v}{\rho}).$$

Some details for last step --

$$e^{-iA\sin\alpha} = \sum_{m=-\infty}^{\infty} J_m(A)e^{-im\alpha}$$ Here $J_m(A)$ is a Bessel function of integer order m.

In our case, $A = \frac{\omega\rho}{c}\cos\theta$ and $\alpha = \frac{vt}{\rho}$.

$$\int_{-\infty}^{\infty} dt e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))} = \sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\int_{-\infty}^{\infty} dt e^{i\omega t(\omega - m\frac{v}{\rho}))}$$

$$= \sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)2\pi\delta(\omega - m\frac{v}{\rho}).$$

Astronomical synchrotron radiation -- continued:

Note that:

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m\frac{v}{\rho})t} = 2\pi\delta(\omega - m\frac{v}{\rho}).$$

$$\Rightarrow C_{\|}(\omega) = 2\pi i \sum_{m=-\infty}^{\infty} J_m'\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}),$$

where $J_m'(A) \equiv \frac{dJ_m(A)}{dA}$

Similarly:

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin\theta \cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta \sin(vt/\rho))}$$

$$= 2\pi \frac{\tan\theta}{v/c} \sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}).$$

Some details:

$$C_\perp(\omega) = \int_{-\infty}^{\infty} dt \sin\theta \cos(vt/\rho) \mathrm{e}^{i\omega(t - \frac{\rho}{c}\cos\theta \sin(vt/\rho))}$$

Note -- $\cos(vt/\rho)\mathrm{e}^{-i\omega\frac{\rho}{c}\cos\theta\sin(vt/\rho)} = \dfrac{c}{-i\omega v \cos\theta}\dfrac{d}{dt}\mathrm{e}^{-i\omega\frac{\rho}{c}\cos\theta\sin(vt/\rho)}$

Integrating by parts and assuming vanishing boundary values:

$$C_\perp(\omega) = \frac{c \ \sin\theta}{v \cos\theta}\int_{-\infty}^{\infty} dt \ \mathrm{e}^{i\omega(t - \frac{\rho}{c}\cos\theta \sin(vt/\rho))}$$

$$= 2\pi \frac{\tan\theta}{v/c}\sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}).$$

Astronomical synchrotron radiation -- continued:

In both of the expressions, the sum over *m* includes both negative and positive values. However, only the positive values of ω and therefore positive values of *m* are of interest. Using the identity: $J_{-m}(A) = (-1)^m J_m(A)$, the result becomes:

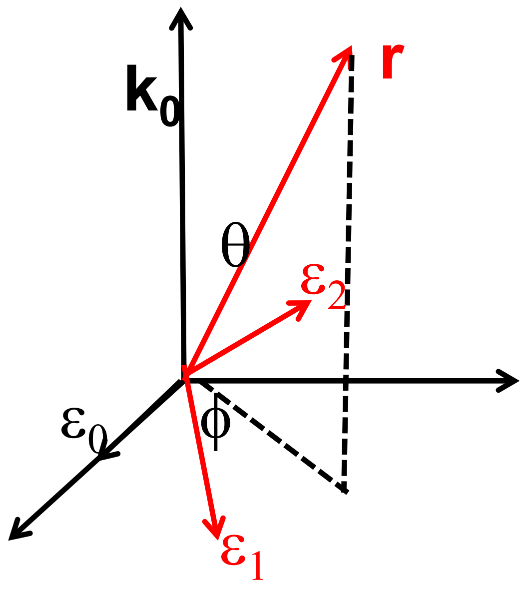
$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{c}\mathcal{S},$$

$$\text{where } \mathcal{S} \equiv 2\sum_{m=1}^{\infty}\delta(\omega - m\frac{v}{\rho})\left\{\left[J_m'\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2 + \frac{\tan^2\theta}{v^2/c^2}\left[J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2\right\}$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text (with $\theta \rightarrow \theta + \pi/2$).

Back to fundamental processes – Thompson and Compton scattering (see section 14.8 in Jackson)

Some details of scattering of electromagnetic waves incident on a particle of charge q and mass m_q

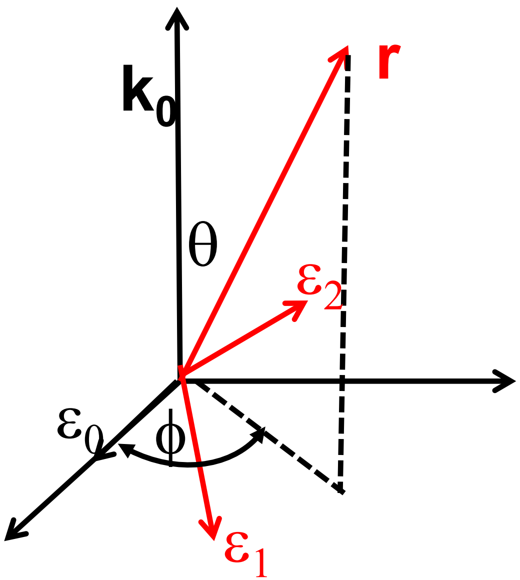


Incident electric field:

$$\mathbf{E}_{inc}(\mathbf{r},t) = \Re\left(\boldsymbol{\varepsilon}_0 E_0 e^{i\mathbf{k_0}\cdot\mathbf{r}-i\omega t}\right)$$

Thompson scattering -- classical picture

Some details of scattering of electromagnetic waves incident on a particle of charge q and mass m_q



Incident electomagnetic wave:

$\mathbf{k}_0$ propagation direction

$\boldsymbol{\varepsilon}_0$ polarization direction

$$\mathbf{E}_{inc}(\mathbf{r},t) = \Re\left(\boldsymbol{\varepsilon}_0 E_0 e^{i\mathbf{k_0}\cdot\mathbf{r}-i\omega t}\right)$$

Scattered radiation:

$\mathbf{r}$ observed position

$\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$ polarization directions

Thompson scattering – non relativistic approximation

Power radiated in direction $\hat{\mathbf{r}}$ by charged particle with acceleration $\dot{\mathbf{v}}$:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \dot{\mathbf{v}}\right)\right|^2$$

Suppose that the acceleration $\dot{\mathbf{v}}$ of a particle (charge q and mass m_q)

is caused by an electric field: $\mathbf{E}(\mathbf{r},t) = \Re\left(\boldsymbol{\varepsilon}_0 E_0 e^{i\mathbf{k}_0\cdot\mathbf{r}-i\omega t}\right)$

$$\dot{\mathbf{v}} = \frac{q}{m_q}\Re\left(\boldsymbol{\varepsilon}_0 E_0 e^{i\mathbf{k}_0\cdot\mathbf{r}-i\omega t}\right)$$

Time averaged power: $$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi}\left(\frac{q^2}{m_q c^2}\right)^2 \left|E_0\right|^2 \left|\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\varepsilon}_0\right)\right|^2$$

What assumptions are made to conclude that

$$\dot{\mathbf{v}} = \frac{q}{m_q} \Re\left(\boldsymbol{\varepsilon}_0 E_0 e^{i\mathbf{k_0}\cdot\mathbf{r} - i\omega t} \right) \quad ?$$

Is it always true?

Comment on acceleration

Lorentz force: $$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$$

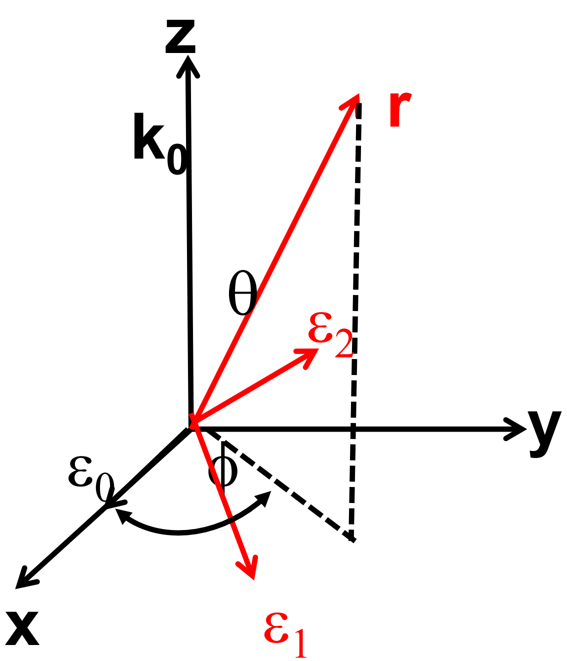
For $v << c$, the dominate force on a charged particle is from the electric field. According to Newton:

$$m_q \frac{d\mathbf{v}}{dt} \equiv m_q \dot{\mathbf{v}} = q\mathbf{E}(\mathbf{r},t) = q\boldsymbol{\varepsilon}_0 E_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$$

Thompson scattering – non relativistic approximation -- continued

Time averaged power: $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} \left(\frac{q^2}{m_q c^2} \right)^2 |E_0|^2 \left| \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\varepsilon}_0) \right|^2$

$$\hat{\mathbf{r}} = \sin\theta \left(\cos\phi \, \hat{\mathbf{x}} + \sin\phi \, \hat{\mathbf{y}} \right) + \cos\theta \hat{\mathbf{z}}$$



Polarization of incident light: $\boldsymbol{\varepsilon}_0 = \hat{\mathbf{x}}$

Polarization of scattered light:

$$\boldsymbol{\varepsilon}_1 = \cos\theta(\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi) - \hat{\mathbf{z}}\sin\theta$$

$$\boldsymbol{\varepsilon}_2 = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

Are these polarizations unique?

Note that we are associating the vector $\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \dot{\mathbf{v}} \right)$ with the polarization of the light. Why?

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \left(1 - \frac{v^2}{c^2} \right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]. \tag{19}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2} \right]. \tag{20}$$

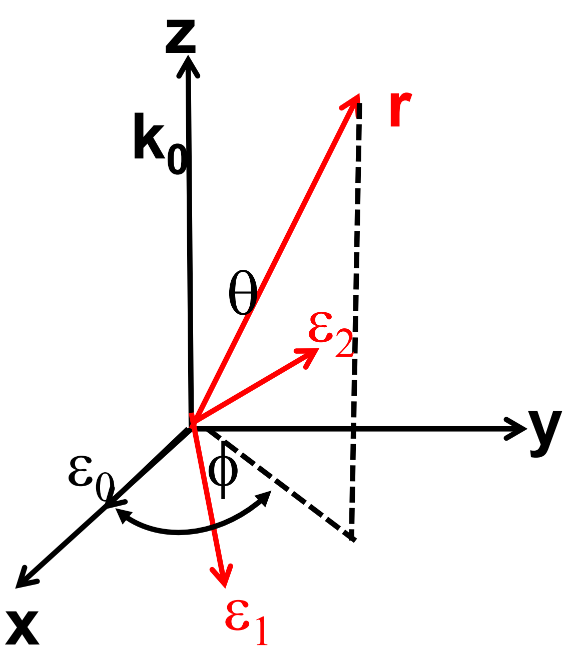
In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{R}. \tag{21}$$

Thompson scattering – non relativistic approximation -- continued

Time averaged power: $$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi}\left(\frac{q^2}{m_q c^2}\right)^2 |E_0|^2 \left|\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\varepsilon}_0\right)\right|^2$$

$$\hat{\mathbf{r}} = \sin\theta\left(\cos\phi\ \hat{\mathbf{x}} + \sin\phi\ \hat{\mathbf{y}}\right) + \cos\theta\hat{\mathbf{z}}$$



Polarization of incident light: $\boldsymbol{\varepsilon}_0 = \hat{\mathbf{x}}$

Polarization of scattered light:

$$\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\varepsilon}_0\right) = \hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\boldsymbol{\varepsilon}_0\right) - \boldsymbol{\varepsilon}_0 \quad \text{(perpendicular to } \hat{\mathbf{r}}\text{)}$$

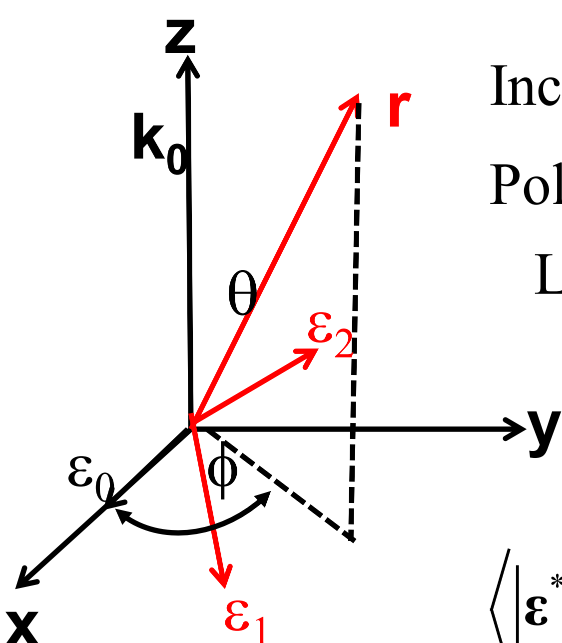
denote scattered light polarization by $\boldsymbol{\varepsilon}^*$

$$\boldsymbol{\varepsilon}^*\cdot\left(\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\varepsilon}_0\right)\right) = -\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0$$

Thompson scattering – non relativistic approximation -- continued

Time averaged power: $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} \left(\frac{q^2}{m_q c^2} \right)^2 |E_0|^2 \left| \hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\varepsilon}_0 \right) \right|^2$

$$\hat{\mathbf{r}} = \sin\theta \left(\cos\phi \ \hat{\mathbf{x}} + \sin\phi \ \hat{\mathbf{y}} \right) + \cos\theta \hat{\mathbf{z}}$$



Incident light polarization: $\boldsymbol{\varepsilon}_0 = \hat{\mathbf{x}}$

Polarization of scattered light: $\boldsymbol{\varepsilon}^*$

Linear combination of

$$\boldsymbol{\varepsilon}_1 = \cos\theta(\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi) - \hat{\mathbf{z}}\sin\theta$$

$$\boldsymbol{\varepsilon}_2 = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

$$\left\langle \left| \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \right|^2 \right\rangle = \left\langle \left| \boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_0 \right|^2 \right\rangle + \left\langle \left| \boldsymbol{\varepsilon}_2 \cdot \boldsymbol{\varepsilon}_0 \right|^2 \right\rangle = \left\langle \cos^2\theta \cos^2\phi \right\rangle + \left\langle \sin^2\phi \right\rangle$$

$$= \frac{1}{2}\cos^2\theta + \frac{1}{2}$$

Thompson scattering – non relativistic approximation -- continued

Time averaged power with polarization $\boldsymbol{\varepsilon}^*$:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi}\left(\frac{q^2}{m_q c^2}\right)^2 \left|E_0\right|^2 \left|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0\right|^2$$

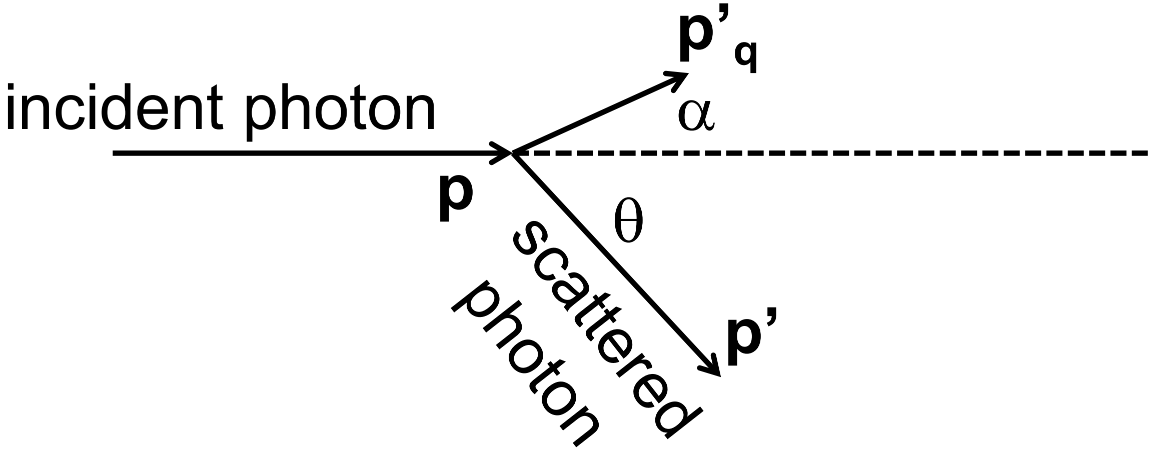
Scattered light may be polarized parallel to incident field or polarized with an angle θ so that the time and polarization averaged cross section is given by:

$$\left\langle \left|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0\right|^2 \right\rangle_\phi = \left\langle \left|\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_0\right|^2 \right\rangle_\phi + \left\langle \left|\boldsymbol{\varepsilon}_2 \cdot \boldsymbol{\varepsilon}_0\right|^2 \right\rangle_\phi = \frac{1}{2}\cos^2\theta + \frac{1}{2}$$

Averaged cross section: $$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{q^2}{m_q c^2}\right)^2 \frac{1}{2}\left(1+\cos^2\theta\right)$$

This formula is appropriate in the X-ray scattering of electrons or soft γ-ray scattering of protons

Thompson scattering – relativistic and quantum modifications



Conservation of momentum and energy:

$$p = p'\cos\theta + p'_q \cos\alpha \qquad\qquad pc = \hbar\omega$$

$$0 = p'\sin\theta - p'_q \sin\alpha \qquad\qquad p'c = \hbar\omega'$$

$$\hbar\omega + m_q c^2 = \hbar\omega' + \sqrt{p'^2_q c^2 + \left(m_q c^2\right)^2}$$

$$\frac{p'}{p} = \frac{1}{1 + \frac{\hbar\omega}{m_q c^2}\left(1 - \cos\theta\right)}$$

Some details --

$$p = p'\cos\theta + p'_q \cos\alpha \qquad 0 = p'\sin\theta - p'_q \sin\alpha$$

$$\left(p'_q\right)^2 = \left(p - p'\cos\theta\right)^2 + \left(p'\sin\theta\right)^2 = p^2 - 2pp'\cos\theta + p'^2$$

$$\left(\hbar\omega + m_q c^2 - p'c\right)^2 = \left(p'^2_q c^2 + \left(m_q c^2\right)^2\right) = p^2c^2 - 2pp'c^2\cos\theta + p'^2c^2 + \left(m_q c^2\right)^2$$

$$p^2c^2 - 2pp'c^2 + p'^2c^2 + 2m_q c^2\left(pc - p'c\right) + \left(m_q c^2\right)^2$$

$$= p^2c^2 - 2pp'c^2\cos\theta + p'^2c^2 + \left(m_q c^2\right)^2$$

Simplifying: $$-2pp'c^2 + 2m_q c^2\left(pc - p'c\right) = -2pp'c^2\cos\theta$$

$$pp'c^2\left(1-\cos\theta\right) = \left(pc - p'c\right)$$

$$\frac{\hbar\omega}{m_q c^2}\left(1-\cos\theta\right) = \frac{p}{p'} - 1$$

$$\Rightarrow \frac{p'}{p} = \frac{1}{1 + \frac{\hbar\omega}{m_q c^2}\left(1-\cos\theta\right)}$$

In fact, the more accurate treatment by Klein and Nishina gives

Klein-Nishina formula

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{q^2}{m_q c^2} \right)^2 \left(\frac{p'}{p} \right)^2 \frac{1}{2} \left(\frac{p'}{p} + \frac{p}{p'} - \sin^2\theta \right)$$

where: $$\frac{p'}{p} = \frac{1}{1 + \frac{\hbar\omega}{m_q c^2}\left(1 - \cos\theta\right)}$$

Note that for $\frac{\hbar\omega}{m_q c^2} << 1$ this simplies to $\frac{p'}{p} = 1$

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{q^2}{m_q c^2} \right)^2 \frac{1}{2} \left(1 + \cos^2\theta \right)$$ as previously derived

Modified Thompson scattering cross section