PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 35:

Some quantum effects in electrodynamics

Mon: Review of quantum eigenstates of EM fields and discussion of Glauber's coherent states

Wed: More general quantum states of EM fields and related correlations functions

Fri: More complicated quantum states of EM fields

24	Mon: 03/18/2024	Chap. 9	Digression on Math methods and Radiation from localized oscillating sources	#19	03/25/2024
25	Wed: 03/20/2024	Chap. 9	Radiation from localized oscillating sources	#20	03/25/2024
26	Fri: 03/22/2024	Chap. 9 & 10	Radiation and scattering	#21	03/25/2024
27	Mon: 03/25/2024	Chap. 11	Special Theory of Relativity	#22	04/01/2024
28	Wed: 03/27/2024	Chap. 11	Special Theory of Relativity	#23	04/01/2024
29	Fri: 03/29/2024	Chap. 11	Special Theory of Relativity		
30	Mon: 04/01/2024	Chap. 14	Radiation from moving charges	#24	04/08/2024
31	Wed: 04/03/2024	Chap. 14	Radiation from accelerating charged particles	#25	04/08/2024
32	Fri: 04/05/2024	Chap. 14	Synchrotron radiation and Compton scattering	#26	04/08/2024
	Mon: 04/08/2024	No class	Eclipse related absences		
33	Wed: 04/10/2024	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung	#27	04/22/2024
34	Fri: 04/12/2024		Special topic: E & M aspects of superconductivity		
	Mon: 04/15/2024		Presentations I		
	Wed: 04/17/2024		Presentations II		
	Fri: 04/19/2024		Presentations III		
35	Mon: 04/22/2024		Special topic: Quantum Effects in E & M		
36	Wed: 04/24/2024		Special topic: Quantum Effects in E & M		
37	Fri: 04/26/2024		Special topic: Quantum Effects in E & M		
38	Mon: 04/29/2024		Review		
39	Wed: 05/01/2024		Review		

Quantization of the Electromagnetic fields
Reference – PHY 742 – Chapter 17
in Professor Carlson's textbook

- **Review of the quantum harmonic oscillator**
- **Hamiltonian for electromagnetic energy and its eigenstates**
- **Properties of the quantized electromagnetic fields**
- **Coherent states**

Review of one-dimensional quantum harmonic oscillator in terms of momentum *P* and displacement *X* with spring constant $m\omega^2$

$$H\psi(x) = \left(\frac{P^2}{2m} + \frac{m\omega^2}{2}X^2\right)\psi(x) = E\psi(x)$$

Define:

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} X + i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} P$$

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} X - i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} P$$

Note that:

$$[a, a^\dagger] = 1$$

It can be shown that for functions --

$\psi_n \rightarrow |n\rangle$ where $n = 0, 1, 2, 3, \ldots.$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a^\dagger a|n\rangle = n|n\rangle$$

$$\Rightarrow H|n\rangle = \hbar\omega\left(\frac{1}{2} + a^\dagger a\right)|n\rangle = \hbar\omega\left(\frac{1}{2} + n\right)|n\rangle$$

Summary of results for the one dimensional quantum oscillator:

$$H|n\rangle = \hbar\omega\left(\frac{1}{2}+a^{\dagger}a\right)|n\rangle = \hbar\omega\left(\frac{1}{2}+n\right)|n\rangle$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Contributing to the discussion –

The creation and annihilation operators within the harmonic oscillator formalism seem to have been introduced by mathematical logic and found to have very interesting properties. In fact, as shown in Chapter 5, starting from the creation and annihilation operators, one can deduce the Harmonic Oscillator spectrum. These operators do not by themselves represent physical quantities and therefore do not "have" to be Hermitian. The matrix form of X and P in the basis of |n> is just one of many ways to represent these operators.

Further comments --

The harmonic oscillator states clearly have an associated quantum number *n.* It is convenient to call *n* a "phonon number" for the moment. We will generalize this notion in the context of electromagnetic fields.

How does this beautiful formalism lead to the notion of creation and annihilation operators?

The phonon number eigenvalues take the values $n = 0, 1, 2, ...$

$a|0\rangle = 0 \quad a|1\rangle = |0\rangle \quad a|2\rangle = \sqrt{2}|1\rangle$ ... interpretation of a as annihilation operator

$a^\dagger|0\rangle = |1\rangle \quad a^\dagger|1\rangle = \sqrt{2}|2\rangle \quad a^\dagger|2\rangle = \sqrt{3}|3\rangle$ ... interpretation of $a^\dagger$ as creation operator

It follows that $|n\rangle = \frac{1}{\sqrt{(n!)}}\left(a^\dagger\right)^n|0\rangle$

➔**We can "create" any phonon state from the ground state with this operator.**

Extension of these ideas to multiple independent harmonic oscillator modes

$$\omega \Rightarrow \{\omega_1, \omega_2, \omega_3 ....\}$$

$$a \Rightarrow \{a_1, a_2, a_3 ....\}$$

$$a^\dagger \Rightarrow \{a_1^\dagger, a_2^\dagger, a_3^\dagger ....\}$$

Here *1,2,..i,j...* denotes an arbitrary index referencing distinct modes.

Commutation relations: $\left[a_i, a_j\right] = 0$

Commutation relations: $\left[a_i^\dagger, a_j^\dagger\right] = 0$

Commutation relations: $\left[a_i, a_j^\dagger\right] = \delta_{ij}$

This result means that for a multiphonon state $\left|n_1, n_2 .... n_i ... n_j ... n_N\right\rangle$, the action of the creation operator works as follows:

$$a_i^\dagger a_j^\dagger \left|n_1, n_2 .... n_i ... n_j ... n_N\right\rangle = \sqrt{n_i + 1}\sqrt{n_j + 1}\left|n_1, n_2 .... (n_i + 1) ... (n_j + 1) ... n_N\right\rangle$$

Later, we will see how this formalism has the capability of keeping track of symmetry/antisymmetry properties of multi particle systems.

Favorite equations from classical electrodynamics

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Back to SI units

Recall the electromagnetic field energy --

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r\left(\left|\mathbf{E}(\mathbf{r},t)\right|^2 + c^2\left|\mathbf{B}(\mathbf{r},t)\right|^2\right)$$

It will be convenient to express Maxwell's equations and the electromagnetic field energy in terms of scalar and vector potentials:

$$\nabla\cdot\mathbf{B} = 0 \qquad \Rightarrow \quad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla\times\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0 \quad \Rightarrow \mathbf{E} + \frac{\partial\mathbf{A}}{\partial t} = -\nabla\Phi \quad \Rightarrow \mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

Far from sources, the remaining equations become:

$$\nabla\cdot\mathbf{E} = 0 \qquad \Rightarrow \quad \nabla^2\Phi + \frac{\partial\nabla\cdot\mathbf{A}}{\partial t} = 0$$

$$\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = 0 \quad \Rightarrow \nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{c^2}\left(\frac{\partial\nabla\Phi}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = 0$$

Further manipulations of Maxwell's equations in terms of scalar and vector potentials --

$$\nabla \cdot \mathbf{E} = 0 \qquad \Rightarrow \nabla^2 \Phi + \frac{\partial \nabla \cdot \mathbf{A}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = 0 \quad \Rightarrow \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial \nabla \Phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = 0$$

$$\Rightarrow \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{1}{c^2}\left(\frac{\partial \nabla \Phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = 0$$

$$\Rightarrow \left(\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla \underbrace{\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right)}_{\text{zero in Lorenz gauge}} = 0$$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \qquad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

Equations within the Lorenz gauge --

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0$$

It is further convenient to seek solutions with $\Phi \equiv 0 \quad \Rightarrow \nabla\cdot\mathbf{A} = 0$

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{B}=\nabla\times\mathbf{A}$$

Note that this is one of many possible choices and it turns out to be convenient.

Electromagnetic field energy for this choice --

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r\left(\left|\mathbf{E}(\mathbf{r},t)\right|^2 + c^2\left|\mathbf{B}(\mathbf{r},t)\right|^2\right)$$

$$= \frac{\epsilon_0}{2}\int d^3r\left(\left|\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2 + c^2\left|\nabla\times\mathbf{A}(\mathbf{r},t)\right|^2\right)$$

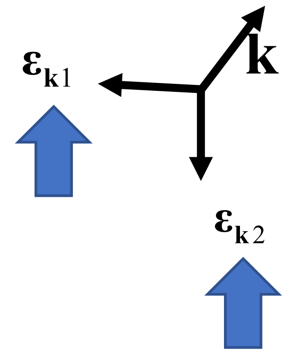
Plane wave solutions to electromagnetic waves in terms of vector potentials

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \qquad\qquad \nabla \cdot \mathbf{A} = 0$$

A pure plane wave takes the form

$$\mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) = A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} \qquad\qquad \omega_{\mathbf{k}} = |\mathbf{k}|c$$

$$\mathbf{k} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} = 0 \qquad\qquad \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma'} = \delta_{\sigma\sigma'}$$



3 mutually perpendicular vectors

These are unit polarization vectors.

For the pure plane wave, the following relations hold:

$$\frac{\partial \mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t)}{\partial t} = -i\omega_{\mathbf{k}} A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t}$$

$$\nabla \times \mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) = i\mathbf{k} \times A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t}$$

General form of vector potential as a superposition of plane waves:

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{V}\sum_{\mathbf{k}\sigma} \mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) = \frac{1}{V}\sum_{\mathbf{k}\sigma} A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}$$

Here V denotes the volume of the analysis system; different treatments put this factor in different ways.

Now we must evaluate the electromagnetic field energy --

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r \left(\left|\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2 + c^2 \left|\nabla \times \mathbf{A}(\mathbf{r},t)\right|^2 \right)$$

Because of the orthogonality of the plane waves, the result can be expressed as a sum over distinct plane wave modes:

$$E_{\text{field}} = \frac{\epsilon_0}{2V}\sum_{\mathbf{k}\sigma} \left|A_{\mathbf{k}\sigma}\right|^2 \left(\omega_{\mathbf{k}}^2 + c^2 \left|\mathbf{k}\right|^2\right)$$

Note that we can use the identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

Electromagnetic field energy --

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r \left(\left|\mathbf{E}(\mathbf{r},t)\right|^2 + c^2 \left|\mathbf{B}(\mathbf{r},t)\right|^2 \right)$$

In terms of the vector potential, using the Lorenz gauge with $\Phi = 0$:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}$$

where $\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$ and $\nabla \cdot \mathbf{A} = 0$

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r \left(\left|\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2 + c^2 \left|\nabla \times \mathbf{A}(\mathbf{r},t)\right|^2 \right)$$

Some details, with more care to use real functions --

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2V}\sum_{\mathbf{k}\sigma}\left(\mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) + \mathbf{A}^*_{\mathbf{k}\sigma}(\mathbf{r},t)\right) = \frac{1}{2V}\sum_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(A_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} + A^*_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Electromagnetic field energy --

$$E_{\text{field}} = \frac{\epsilon_0}{2V}\int d^3r\left(\left|\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2 + c^2\left|\nabla\times\mathbf{A}(\mathbf{r},t)\right|^2\right)$$

Note that the plane waves are distributed throughout the analysis volume

such that the following orthogonality holds. $\frac{1}{V}\int d^3r\ e^{i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}} = \delta_{\mathbf{k}\mathbf{k}'}$

Also recall that $\omega_{\mathbf{k}} = |\mathbf{k}|c$ and average out all high frequency contributions

to the field energy -- $E_{\text{field}} = \frac{\epsilon_0}{4V}\sum_{\mathbf{k}\sigma}\left(A_{\mathbf{k}\sigma}A^*_{\mathbf{k}\sigma} + A^*_{\mathbf{k}\sigma}A_{\mathbf{k}\sigma}\right)\left(\omega_{\mathbf{k}}^2 + c^2|\mathbf{k}|^2\right)$

$$E_{\text{field}} = \frac{\epsilon_0}{2V}\sum_{\mathbf{k}\sigma}\omega_{\mathbf{k}}^2\left(A_{\mathbf{k}\sigma}A^*_{\mathbf{k}\sigma} + A^*_{\mathbf{k}\sigma}A_{\mathbf{k}\sigma}\right)$$

In the next slide, we will "jump" to quantizing the electromagnetic field using the analogy of the harmonic oscillator Hamiltonian. In fact, the analogy has nothing to do with the physics of the harmonic oscillator other than their particle symmetry as Bose particles.

Max Planck 1858-1947

Historical importance of the formula for Blackbody radiation

A blackbody means an idealized opaque (non-reflective) material which can absorb and emit electromagnetic radiation. If the body has an equilibrium temperature T, the energy associated with the blackbody is <U>. Using statistical mechanics and the assumption of quantized electromagnetic radiation, Planck showed that the black body internal energy and its distribution is given by in terms of frequency *f*:

$$\left\langle U\right\rangle = \frac{Vh^4}{\pi^2\hbar^3c^3}\int df\; f^3\,\frac{1}{e^{\beta hf}-1} = \frac{8\pi Vh}{c^3}\int_0^\infty df\,\frac{f^3}{e^{\beta hf}-1}$$

Figure from:
An Introduction to Thermal Physics, by Daniel V. Schroeder (Addison Wesley, 2000 and now Oxford University Press)

Showing frequency distribution of blackbody radiation from the big bang.

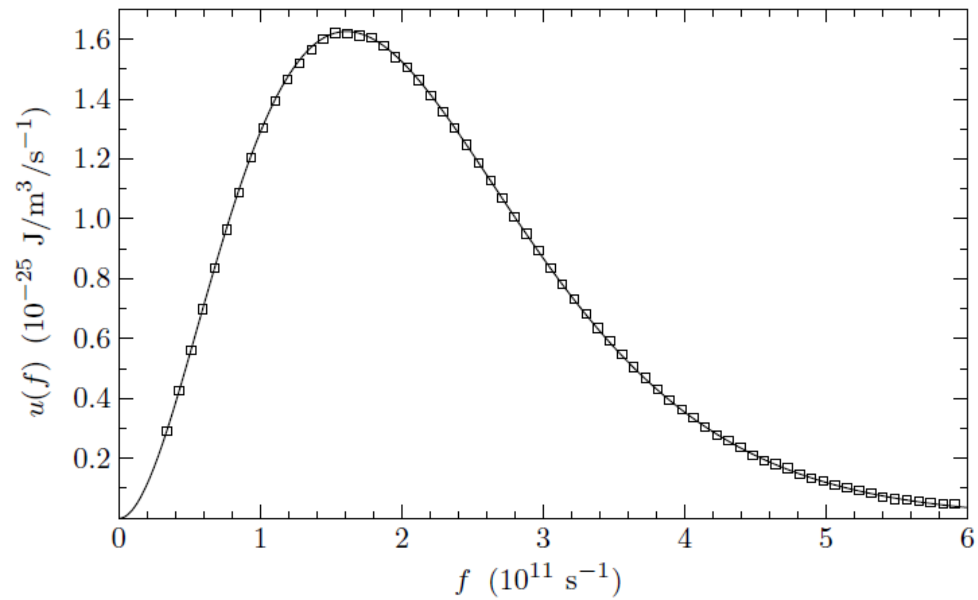


Figure 7.20. Spectrum of the cosmic background radiation, as measured by the *Cosmic Background Explorer* satellite. Plotted vertically is the energy density per unit frequency, in SI units. Note that a frequency of 3×10^{11} s^{-1} corresponds to a wavelength of $\lambda = c/f = 1.0$ mm. Each square represents a measured data point. The point-by-point uncertainties are too small to show up on this scale; the size of the squares instead represents a liberal estimate of the uncertainty due to systematic effects. The solid curve is the theoretical Planck spectrum, with the temperature adjusted to 2.735 K to give the best fit. From J. C. Mather et al., *Astrophysical Journal Letters* **354**, L37 (1990); adapted courtesy of NASA/GSFC and the COBE Science Working Group. Subsequent measurements from this experiment and others now give a best-fit temperature of 2.728 ± 0.002 K. Copyright ©2000, Addison-Wesley.

Electromagnetic field energy expression:

$$E_{\text{field}} = \frac{\epsilon_0}{2V} \sum_{\mathbf{k}\sigma} \omega_{\mathbf{k}}^2 \left(A_{\mathbf{k}\sigma} A_{\mathbf{k}\sigma}^* + A_{\mathbf{k}\sigma}^* A_{\mathbf{k}\sigma} \right)$$

Here $A_{\mathbf{k}\sigma}$ represents the amplitude of the vector potential.

Big leap --

Suppose that $A_{\mathbf{k}\sigma} \rightarrow C_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$ $\qquad A_{\mathbf{k}\sigma}^* \rightarrow C_{\mathbf{k}\sigma}^* a_{\mathbf{k}\sigma}^\dagger$

where $C_{\mathbf{k}\sigma}$ is a constant and $a_{\mathbf{k}\sigma}$ is an annihilation operator

$$E_{\text{field}} = \frac{\epsilon_0}{2V} \sum_{\mathbf{k}\sigma} \omega_{\mathbf{k}}^2 \left| C_{\mathbf{k}\sigma} \right|^2 \left(a_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^\dagger + a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} \right)$$

More leaping -- $\qquad C_{\mathbf{k}\sigma} = \sqrt{\frac{V\hbar}{\epsilon_0 \omega_{\mathbf{k}}}}$

$$E_{\text{field}} = \frac{1}{2} \sum_{\mathbf{k}\sigma} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^\dagger + a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} \right) = \sum_{\mathbf{k}\sigma} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} + \frac{1}{2} \right)$$

Here $a_{\mathbf{k}\sigma}$ and $a^{\dagger}_{\mathbf{k}\sigma}$ are "borrowed" from the Harmonic oscillator formalism.

Commutation relations: $\left[a_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$ $\quad \left[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'}\right] = 0$ $\quad \left[a^{\dagger}_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'}\right] = 0$

$$H_{\text{field}} = \frac{1}{2}\sum_{\mathbf{k}\sigma} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}\sigma} a^{\dagger}_{\mathbf{k}\sigma} + a^{\dagger}_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \right) = \sum_{\mathbf{k}\sigma} \hbar\omega_{\mathbf{k}} \left(a^{\dagger}_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \frac{1}{2} \right)$$

From the analogy of the Harmonic oscillator, the eigenstates of the EM Field Hamiltonian are integers $n_{\mathbf{k}\sigma}$:

$$H_{\text{field}} \left| n_{\mathbf{k}\sigma} \right\rangle = \sum_{\mathbf{k}'\sigma'} \hbar\omega_{\mathbf{k}'} \left(a^{\dagger}_{\mathbf{k}'\sigma'} a_{\mathbf{k}'\sigma'} + \frac{1}{2} \right) \left| n_{\mathbf{k}\sigma} \right\rangle = \left(\hbar\omega_{\mathbf{k}} n_{\mathbf{k}\sigma} + \sum_{\mathbf{k}'\sigma'} \frac{\hbar\omega_{\mathbf{k}'}}{2} \right) \left| n_{\mathbf{k}\sigma} \right\rangle$$

$$H^{\text{fixed}}_{\text{field}} \left| n_{\mathbf{k}\sigma} \right\rangle = \sum_{\mathbf{k}'\sigma'} \left(\hbar\omega_{\mathbf{k}'} a^{\dagger}_{\mathbf{k}'\sigma'} a_{\mathbf{k}'\sigma'} \right) \left| n_{\mathbf{k}\sigma} \right\rangle = \hbar\omega_{\mathbf{k}} n_{\mathbf{k}\sigma} \left| n_{\mathbf{k}\sigma} \right\rangle$$

Uncontrolled energy shift

Some additional comments on the "fixed" solution --

EM Field Hamiltonian acting on eigenstate $\left|n_{\mathbf{k}\sigma}\right\rangle$:

$$H_{\text{field}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'} \hbar\omega_{\mathbf{k}'}\left(a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'} + \frac{1}{2}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \hbar\omega_{\mathbf{k}}n_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle + \underbrace{\sum_{\mathbf{k}'\sigma'}\frac{\hbar\omega_{\mathbf{k}'}}{2}}_{\text{Troublesome term}}\left|n_{\mathbf{k}\sigma}\right\rangle$$

$$H^{\text{fixed}}_{\text{field}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'}\left(\hbar\omega_{\mathbf{k}'}a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \hbar\omega_{\mathbf{k}}n_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle$$

Comment: For the phonon case which served as our model, the notion of zero point motion makes physical sense. For the electromagnetic Hamiltonian the role of the equivalent concept is not quite clear (at least to me). We need to be careful when we see divergent energies to distinguish physical processes from mathematical issues.

Creation and annihilation operators:

$$a_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}}\left|n_{\mathbf{k}\sigma}-1\right\rangle$$

$$a^{\dagger}_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}+1}\left|n_{\mathbf{k}\sigma}+1\right\rangle$$

Quantum mechanical form of vector potential in real space --

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} + a^{\dagger}_{\mathbf{k}\sigma}e^{-\left(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t\right)}\right)$$

Note: We are assuming that the polarization vector is real.

Quantum mechanical form of vector potential --

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Electric field:

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}\quad\Rightarrow\mathbf{E}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Magnetic field:

$$\mathbf{B}=\nabla\times\mathbf{A}\quad\Rightarrow\mathbf{B}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

$$\mathbf{E}(\mathbf{r},t) = i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

$$\mathbf{B}(\mathbf{r},t) = i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

What is the expectation value of the E field for a pure eigenstate |n> of the electromagnetic Hamiltonian?

1. **A complex (non zero) number**
2. **Zero**
3. **Infinity**

What is the expectation value of the B field for a pure eigenstate |n> of the electromagnetic Hamiltonian?

1. **A complex (non zero) number**
2. **Zero**
3. **Infinity**

➔In fact, these are non-trivial questions

At this point, we might wonder how the classical and quantum pictures of the EM field can be reconciled --

An interesting picture comes from a particular linear combination of quantum states of a single mode ($k\sigma$) arising for example in a laser

How does a quantum mechanical E or B field exist? Consider a linear combination of pure photon states --

VOLUME 10, NUMBER 3 PHYSICAL REVIEW LETTERS 1 FEBRUARY 1963

PHOTON CORRELATIONS*

Roy J. Glauber
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(Received 27 December 1962)

In 1956 Hanbury Brown and Twiss[1] reported that the photons of a light beam of narrow spectral width have a tendency to arrive in correlated pairs. We have developed general quantum mechanical methods for the investigation of such correlation effects and shall present here results for the distribution of the number of photons counted in an incoherent beam. The fact that photon correlations are enhanced by narrowing the spectral bandwidth has led to a prediction[2] of large-scale correlations to be observed in the beam of an optical maser. We shall indicate that this prediction is misleading and follows from an inappropriate model of the maser beam. In considering these problems we shall outline a method of describing the photon field which appears particularly well suited to the discussion of experiments performed with light beams, whether coherent or incoherent.

The correlations observed in the photoionization processes induced by a light beam were given a simple semiclassical explanation by Purcell,[3] who made use of the methods of microwave noise theory. More recently, a number of papers have been written examining the correlations in considerably greater detail. These papers[2,4-6] retain the assumption that the electric field in a light beam can be described as a classical Gaussian stochastic process. In actuality, the behavior of the photon field is considerably more

Gauber's coherent state: $\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}}\left|n\right\rangle$

Here α represents a complex amplitude

It is possible to prove the following identies for the coherent states:

1. $\left\langle c_\alpha \middle| c_\alpha \right\rangle = 1$
2. $\left\langle c_\alpha \middle| a \middle| c_\alpha \right\rangle = \alpha$
3. $\left\langle c_\alpha \middle| a^\dagger \middle| c_\alpha \right\rangle = \alpha^*$
4. $\left|\left\langle c_\alpha \middle| c_\beta \right\rangle\right|^2 = e^{-|\alpha-\beta|^2}$

Summary of previous results for the electromagnetic Hamiltonian

In terms of the operators $a_{\mathbf{k}\sigma}$ and $a^\dagger_{\mathbf{k}\sigma}$ operators for wavevector $\mathbf{k}$ and polarization σ.

With commutation relations: $\left[a_{\mathbf{k}\sigma}, a^\dagger_{\mathbf{k}'\sigma'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$ $\left[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'}\right] = 0$ $\left[a^\dagger_{\mathbf{k}\sigma}, a^\dagger_{\mathbf{k}'\sigma'}\right] = 0$

The eigenstates of the EM Field Hamiltonian (omitting diverging term) are integers $n_{\mathbf{k}\sigma}$:

$$H^{\text{fixed}}_{\text{field}} \left| n_{\mathbf{k}\sigma} \right\rangle = \sum_{\mathbf{k}'\sigma'} \left(\hbar\omega_{\mathbf{k}'} a^\dagger_{\mathbf{k}'\sigma'} a_{\mathbf{k}'\sigma'} \right) \left| n_{\mathbf{k}\sigma} \right\rangle = \hbar\omega_{\mathbf{k}} n_{\mathbf{k}\sigma} \left| n_{\mathbf{k}\sigma} \right\rangle$$

It is convenient to define the photon number operator

$\mathbf{N}_{\mathbf{k}'\sigma'} \equiv a^\dagger_{\mathbf{k}'\sigma'} a_{\mathbf{k}'\sigma'}$ such that $\mathbf{N}_{\mathbf{k}\sigma} \left| n_{\mathbf{k}\sigma} \right\rangle = n_{\mathbf{k}\sigma} \left| n_{\mathbf{k}\sigma} \right\rangle$

Properties of the creation and annihilation operators:

$$a_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}}\left|n_{\mathbf{k}\sigma}-1\right\rangle$$

$$a^{\dagger}_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}+1}\left|n_{\mathbf{k}\sigma}+1\right\rangle$$

Quantum mechanical form of vector potential --

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} + a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Note: We are assuming that the polarization vector is real.

Quantum mechanical form of vector potential and corresponding fields --

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} + a^{\dagger}_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t)} \right)$$

Electric field:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad \Rightarrow \mathbf{E}(\mathbf{r},t) = i\sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} - a^{\dagger}_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t)} \right)$$

Magnetic field:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \Rightarrow \mathbf{B}(\mathbf{r},t) = i\sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} - a^{\dagger}_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t)} \right)$$

Embarassing/puzzling expectation values --

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{A}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle=0$$

Electric field:

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}\quad\Rightarrow\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{E}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle=0$$

Magnetic field:

$$\mathbf{B}=\nabla\times\mathbf{A}\quad\Rightarrow\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{B}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle=0$$

In order to compare the classical treatment to the quantum approach we need to calculate expectation values of the observables. In addition to mean value of an observable, its statistical properties are also of interest, particularly the variance and the standard deviation (its square root) which is defined in terms of the average of the squared value of the observable and the average value itself:

$$\text{Standard deviation:} \quad \Delta V \equiv \sqrt{\left\langle V^2 \right\rangle - \left|\left\langle V \right\rangle\right|^2}$$

The next few slides review the relationship of this variance to observables in quantum mechanics which have non trivial commutation relationships and thus have built in variance values.

Digression -- Commutator formalism in quantum mechanics

Definition:

Given two Hermitian operators A and B, their commutator is

$$[A,B] \equiv AB - BA$$

Theorem:

Given Hermitian operators A, B, C such that

$$[A,B] = iC,$$

it follows that $$\Delta A \Delta B \geq \frac{1}{2}\left|\langle C \rangle\right|$$

Proof --

Note that:

$$[A,B]^\dagger = \left(iC\right)^\dagger$$

$$\left(AB - BA\right)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -iC^\dagger$$

$$= BA - AB = -iC$$

Calculation of the variance:

$$\left(\Delta A\right)^2 \equiv \left\langle \psi \left| \left(A - \left\langle A \right\rangle\right)^2 \right| \psi \right\rangle$$

$$= \left\langle \left(A - \left\langle A \right\rangle\right)\psi \middle| \left(A - \left\langle A \right\rangle\right)\psi \right\rangle$$

Similarly,

$$\left(\Delta B\right)^2 \equiv \left\langle \psi \left| \left(B - \left\langle B \right\rangle\right)^2 \right| \psi \right\rangle$$

$$= \left\langle \left(B - \left\langle B \right\rangle\right)\psi \middle| \left(B - \left\langle B \right\rangle\right)\psi \right\rangle$$

Define $\left|\psi_A\right\rangle \equiv \left|\left(A - \left\langle A \right\rangle\right)\psi\right\rangle$

$$\left|\psi_B\right\rangle \equiv \left|\left(B - \left\langle B \right\rangle\right)\psi\right\rangle$$

Schwarz inequality:

$$\left\langle \psi_A \middle| \psi_A \right\rangle \left\langle \psi_B \middle| \psi_B \right\rangle \geq \left| \left\langle \psi_A \middle| \psi_B \right\rangle \right|^2$$

Define $\left|\psi_A\right\rangle \equiv \left|\left(A-\langle A\rangle\right)\psi\right\rangle$ and $\left|\psi_B\right\rangle \equiv \left|\left(B-\langle B\rangle\right)\psi\right\rangle$

Schwarz inequality:

$$\left\langle\psi_A\middle|\psi_A\right\rangle\left\langle\psi_B\middle|\psi_B\right\rangle \geq \left|\left\langle\psi_A\middle|\psi_B\right\rangle\right|^2$$

$$\left\langle\psi_A\middle|\psi_B\right\rangle = \left\langle\psi\right|\left(A-\langle A\rangle\right)\left(B-\langle B\rangle\right)\left|\psi\right\rangle$$

$$\left(A-\langle A\rangle\right)\left(B-\langle B\rangle\right) = \frac{1}{2}\Big(\left(A-\langle A\rangle\right)\left(B-\langle B\rangle\right)+\left(B-\langle B\rangle\right)\left(A-\langle A\rangle\right)\Big)$$

$$+\frac{1}{2}\Big(\left(A-\langle A\rangle\right)\left(B-\langle B\rangle\right)-\left(B-\langle B\rangle\right)\left(A-\langle A\rangle\right)\Big)$$

$$= F + \frac{i}{2}C$$

$$\langle\psi_A|\psi_B\rangle = \langle\psi|\left(A-\langle A\rangle\right)\left(B-\langle B\rangle\right)|\psi\rangle \qquad = \langle\psi|F|\psi\rangle + \frac{i}{2}\langle\psi|C|\psi\rangle$$

$$\left|\langle\psi_A|\psi_B\rangle\right|^2 = \left|\langle\psi|F|\psi\rangle\right|^2 + \frac{1}{4}\left|\langle\psi|C|\psi\rangle\right|^2 \geq \frac{1}{4}\left|\langle\psi|C|\psi\rangle\right|^2$$

Putting it all together:

$$\langle\psi_A|\psi_A\rangle\langle\psi_B|\psi_B\rangle \geq \left|\langle\psi_A|\psi_B\rangle\right|^2 \geq \frac{1}{4}\left|\langle\psi|C|\psi\rangle\right|^2$$

$$\Rightarrow (\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4}\left|\langle C\rangle\right|^2$$

Therefore: $[A,B]=iC$ implies $\Delta A\Delta B \geq \frac{1}{2}\left|\langle C\rangle\right|$

Example: $A = X, \quad B = P$

$[X,P]=i\hbar$ implies $\Delta X\Delta P \geq \frac{\hbar}{2}$

What does this have to do with quantum EM fields?

In fact, Carlson's textbook shows that although

$$\langle n_{\mathbf{k}'\sigma'} | \mathbf{E}(\mathbf{r},t) | n_{\mathbf{k}'\sigma'} \rangle = 0 \quad \text{and} \quad \langle n_{\mathbf{k}'\sigma'} | \mathbf{B}(\mathbf{r},t) | n_{\mathbf{k}'\sigma'} \rangle = 0,$$

the variances of the fields are both infinite for a pure eigenstate --

$$\langle 0 | \mathbf{E}^2(\mathbf{r}) | 0 \rangle = \left| \mathbf{E}(\mathbf{r}) | 0 \rangle \right|^2 = \frac{\hbar}{2\varepsilon_0 V} \sum_{\mathbf{k}\sigma} \sum_{\mathbf{k}'\sigma'} \sqrt{\omega_k \omega_{k'}} \left(\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \boldsymbol{\varepsilon}^*_{\mathbf{k}'\sigma'} \right) e^{i\mathbf{k}\cdot\mathbf{r} - i\mathbf{k}'\cdot\mathbf{r}} \langle 1, \mathbf{k}, \sigma | 1, \mathbf{k}', \sigma' \rangle$$

$$= \frac{\hbar}{2\varepsilon_0 V} \sum_{\mathbf{k}\sigma} \omega_k = \frac{\hbar c}{\varepsilon_0 V} \sum_{\mathbf{k}} k = \frac{\hbar c}{\varepsilon_0} \int \frac{d^3\mathbf{k}}{(2\pi)^3} k, \qquad (17.19a)$$

infinite

$$\langle 0 | \mathbf{B}^2(\mathbf{r}) | 0 \rangle = \left| \mathbf{B}(\mathbf{r}) | 0 \rangle \right|^2 = \frac{\hbar}{2\varepsilon_0 V} \sum_{\mathbf{k},\sigma} \sum_{\mathbf{k}',\sigma'} \frac{e^{i\mathbf{k}\cdot\mathbf{r} - i\mathbf{k}'\cdot\mathbf{r}}}{\sqrt{\omega_k \omega_{k'}}} \left(\mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \right) \cdot \left(\mathbf{k}' \times \boldsymbol{\varepsilon}^*_{\mathbf{k}'\sigma'} \right) \langle 1, \mathbf{k}, \sigma | 1, \mathbf{k}', \sigma' \rangle$$

$$= \frac{\hbar}{2\varepsilon_0 V} \sum_{\mathbf{k},\sigma} \frac{\left| \mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \right|^2}{\omega_k} = \frac{\hbar}{2\varepsilon_0 V} \sum_{\mathbf{k},\sigma} \frac{k^2}{\omega_k} = \frac{\hbar}{\varepsilon_0 V c} \sum_{\mathbf{k}} k = \frac{\hbar}{\varepsilon_0 c} \int \frac{d^3\mathbf{k}}{(2\pi)^3} k, \qquad (17.19b)$$

infinite

A more careful treatment shows relations such as

$$[\mathrm{E}_x(\mathbf{r},t), B_y(\mathbf{r}',t)] = ic\hbar \frac{\partial \delta(\mathbf{r} - \mathbf{r}')}{\partial z}$$

It is also possible to show that components of the E and B field have nontrivial commutation relations, indicating that in general it is not possible to simultaneously determine E and B at the same point in space to arbitrary accuracy.

Effects of the phase of each mode.

In deriving these equations, we neglected the phase of each mode. A more careful treatment of photon number and phase show that these also have nontrivial commutation relations.

How is this quantum treatment of the electromagnetic fields consistent with the classical picture?

1. **There is no need for consistency.?**
2. **There should be consistency in certain ranges of the parameters.?**

$$\text{Glauber's coherent state: } \left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle \quad \text{based on a single mode } n \to n_{\mathbf{k}\sigma}$$

$$\text{Electric field: } \left\langle c_\alpha \left| \mathbf{E}(\mathbf{r},t) \right| c_\alpha \right\rangle = i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(\alpha_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - \alpha^*_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)} \right)$$

$$\text{Magnetic field: } \left\langle c_\alpha \left| \mathbf{B}(\mathbf{r},t) \right| c_\alpha \right\rangle = i\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(\alpha_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - \alpha^*_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)} \right)$$

Note that α is a complex number which can be written in terms of a real amplitude and phase: E_0 and ψ :

$$\left\langle c_\alpha \left| \mathbf{E}(\mathbf{r},t) \right| c_\alpha \right\rangle = -2\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} E_0 \sin\left(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t + \psi\right)$$

$$\left\langle c_\alpha \left| \mathbf{B}(\mathbf{r},t) \right| c_\alpha \right\rangle = -2\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} E_0 \sin\left(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t + \psi\right)$$

$$\text{Let} \quad \alpha = E_0 e^{i\Psi}$$

Single mode coherent state continued

It can also be shown that

$$\langle c_\alpha \left| \left| \mathbf{E}(\mathbf{r},t) \right|^2 \right| c_\alpha \rangle = \frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}\left(4E_0^2 \sin^2\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t+\psi\right)+1\right)$$

Therefore

$$\langle c_\alpha \left| \left| \mathbf{E}(\mathbf{r},t) \right|^2 \right| c_\alpha \rangle - \left| \langle c_\alpha \left| \mathbf{E}(\mathbf{r},t) \right| c_\alpha \rangle \right|^2 = \frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}$$

This means that variance of the E field for the coherent state is independent of the amplitude E_0. Therefore, for large E_0 the variance is small in comparison.

Visualization of coherent state electric fields for various amplitudes

Source: Rodney Loudon, "The Quantum Theory of Light"

The quantized radiation field 151

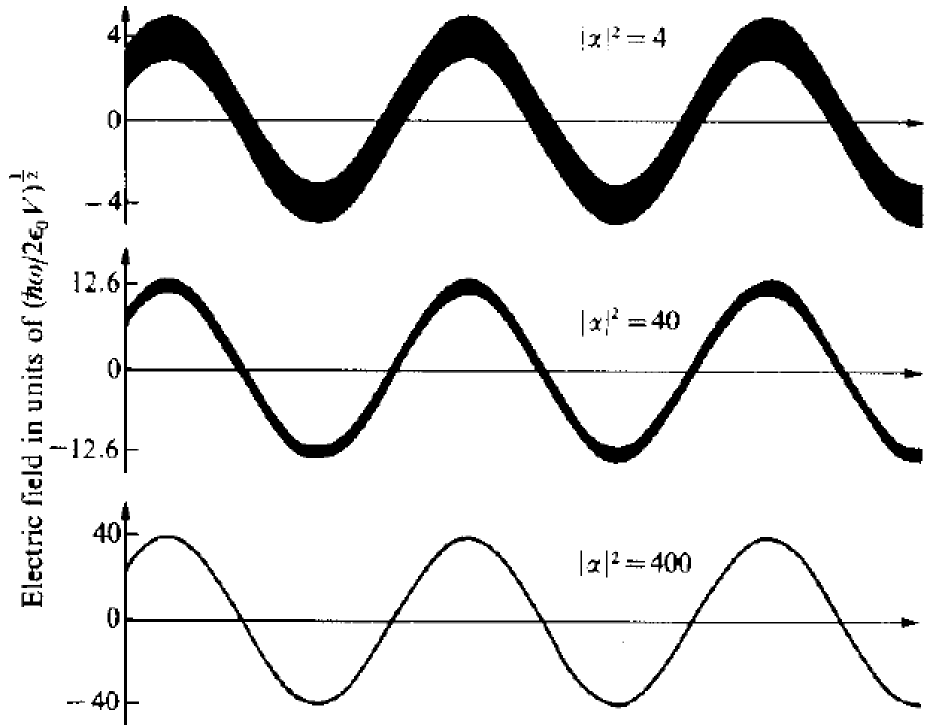

FIG. 4.3. Pictorial representation of the electric-field variation in a cavity mode excited to state $|\alpha\rangle$. Three different values of the mean photon number $|\alpha|^2$ are shown, the vertical scales being different for the three cases. The uncertainties in field values are indicated by the vertical widths $2\Delta E$ of the sine waves. These widths can also be regarded as combinations of the amplitude uncertainty associated with Δn and the phase uncertainty associated with $\Delta \cos \phi$.

Single mode coherent state continued

Now consider the expectation values of the number operator and its square:

$$\mathbf{N}_{\mathbf{k}\sigma} \equiv a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$$

$$\langle c_\alpha | \mathbf{N}_{\mathbf{k}\sigma} | c_\alpha \rangle = |\alpha|^2 \qquad \langle c_\alpha | \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} | c_\alpha \rangle = |\alpha|^4 + |\alpha|^2$$

Square of the variance: $\langle c_\alpha | \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} | c_\alpha \rangle - \left|\langle c_\alpha | \mathbf{N}_{\mathbf{k}\sigma} | c_\alpha \rangle\right|^2 = |\alpha|^2$

Fractional uncertainty in the number of photons for the coherent state:

$$\frac{\sqrt{\langle c_\alpha | \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} | c_\alpha \rangle - \left|\langle c_\alpha | \mathbf{N}_{\mathbf{k}\sigma} | c_\alpha \rangle\right|^2}}{\langle c_\alpha | \mathbf{N}_{\mathbf{k}\sigma} | c_\alpha \rangle} = \frac{1}{|\alpha|}$$

Interpretation of a single mode coherent state

$$\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle$$ based on a single mode $n \to n_{\mathbf{k}\sigma}$

The probability of finding n photons in this state is given by:

$$\left|\left\langle n \middle| c_\alpha \right\rangle\right|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}$$

This is the form of a Poisson distribution

for a mean value of $|\alpha|^2$.

More reading -- **page 231**

REVIEWS OF

MODERN PHYSICS

VOLUME 37, NUMBER 2 APRIL 1965

Coherence Properties of Optical Fields*

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This article presents a review of coherence properties of electromagnetic fields and their measurements, with special emphasis on the optical region of the spectrum. Analyses based on both the classical and quantum theories are described. After a brief historical introduction, the elementary concepts which are frequently employed in the discussion of interference phenomena are summarized. The measure of second-order coherence is then introduced in connection with the analysis of a simple interference experiment and some of the more important second-order coherence effects are studied. Their uses in stellar interferometry and interference spectroscopy are described. Analysis of partial polarization from the standpoint of correlation theory is also outlined. The general statistical description of the field is discussed in some detail. The recently discovered universal "diagonal" representation of the density operator for free fields is also considered and it is shown how, with the help of the associated generalized phase-space distribution function, the quantum-mechanical correlation functions may be expressed in the same form as the classical ones. The sections which follow deal with the statistical properties of thermal and nonthermal light, and with the temporal and spatial coherence of black-body radiation. Later sections, dealing with fourth- and higher-order coherence effects include a discussion of the photoelectric detection process. Among the fourth-order effects described in detail are bunching phenomena, the Hanbury Brown–Twiss effect and its application to astronomy. The article concludes with a discussion of various transient superposition effects, such as light beats and interference fringes produced by independent light beams.