PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 36:

Some quantum effects in electrodynamics

-- General quantum states of EM fields and related correlations functions

a. Review of eigenstates of EM Hamiltonian and of Glauber's coherent states

b. Comments on experimental situation

c. Squeezed states

33	Wed: 04/10/2024	Chap. 13 & 15	Other radiation -- Cherenkov & bremsstrahlung	#27	04/22/2024
34	Fri: 04/12/2024		Special topic: E & M aspects of superconductivity		
	Mon: 04/15/2024		Presentations I		
	Wed: 04/17/2024		Presentations II		
	Fri: 04/19/2024		Presentations III		
35	Mon: 04/22/2024		Special topic: Quantum Effects in E & M		
36	Wed: 04/24/2024		Special topic: Quantum Effects in E & M		
37	Fri: 04/26/2024		Special topic: Quantum Effects in E & M		
38	Mon: 04/29/2024		Review		
39	Wed: 05/01/2024		Review		

Final exam will be a take-home exam with similar form to mid-term due May 10??

The Physics Department Presents
the *2024*

Honors and Awards Ceremony

Honors Theses: Emily Foley & Emily Wang

Awards Ceremony: For the extraordinary dedication of our department members

New $\Sigma\Pi\Sigma$ Majors

Graduating Seniors

April 25 Olin Hall 3:30-5:00

References –

- Consultation with Professor Kandada
- Rodney Loudon, “The quantum theory of light” (1983)
- Leonard Mandel and Emil Wolf, “Optical Coherence and Quantum Optics” (2013)
- Yanhua Shih, “An Introduction to Quantum Optics” (2021) (some typos, but generally informative)
- Paul R Berman and Vladimir S. Malinovsky, “Principles of Laser Spectroscopy and Quantum Optics” (2011)

Review of what we learned from Lecture 35

For a single mode plane wave with wave vector **k**, frequency $\omega_{\mathbf{k}}$ and polarization σ:

EM Field Hamiltonian acting on eigenstate $\left|n_{\mathbf{k}\sigma}\right\rangle$:

where $\mathbf{k}$ denotes wavevector and σ denotes polarization direction --

$$H^{\text{fixed}}_{\text{field}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'}\left(\hbar\omega_{\mathbf{k}'}a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \hbar\omega_{\mathbf{k}}n_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle$$

Here $n_{\mathbf{k}\sigma} = 0,1,2,3,4.......$

$$a_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}}\left|n_{\mathbf{k}\sigma}-1\right\rangle$$

$$a^{\dagger}_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}+1}\left|n_{\mathbf{k}\sigma}+1\right\rangle$$

Commutation relations:

$$\left[a_{\mathbf{k}\sigma},a^{\dagger}_{\mathbf{k}'\sigma'}\right]=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \quad \left[a_{\mathbf{k}\sigma},a_{\mathbf{k}'\sigma'}\right]=0 \quad \left[a^{\dagger}_{\mathbf{k}\sigma},a^{\dagger}_{\mathbf{k}'\sigma'}\right]=0$$

In terms of the same operators and with polarization unit vectors $\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}$ – –

Vector potential:

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-\left(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t\right)}\right)$$

Electric field:

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}\quad\Rightarrow\mathbf{E}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-\left(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t\right)}\right)$$

Magnetic field:

$$\mathbf{B}=\nabla\times\mathbf{A}\quad\Rightarrow\mathbf{B}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-\left(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t\right)}\right)$$

While the photon eigenstates $\left|n_{\mathbf{k}'\sigma'}\right\rangle$ form a complete basis for describing quantum electromagnetic fields, they have some troublesome properties such as found in evaluating the field expectation values --

Vector potential:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{A}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle = \sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle = 0$$

Electric field:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{E}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle = i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle = 0$$

Magnetic field:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{B}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle = i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle = 0$$

A convenient superposition thanks to R. Glauber, PR 131, 2766 (1963)

$$\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}}\left|n\right\rangle \quad \text{based on a single mode } n \rightarrow n_{\mathbf{k}\sigma}$$

Electric field: $\left\langle c_\alpha \right| \mathbf{E}(\mathbf{r},t) \left| c_\alpha \right\rangle = i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(\alpha_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - \alpha^*_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$

Magnetic field: $\left\langle c_\alpha \right| \mathbf{B}(\mathbf{r},t) \left| c_\alpha \right\rangle = i\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(\alpha_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - \alpha^*_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$

Let $\alpha = \Lambda e^{i\psi}$ where both Λ and Ψ are unitless real values.

$$\left\langle c_\alpha \right| \mathbf{E}(\mathbf{r},t) \left| c_\alpha \right\rangle = -2\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\Lambda\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t+\psi\right)$$

$$\left\langle c_\alpha \right| \mathbf{B}(\mathbf{r},t) \left| c_\alpha \right\rangle = -2\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\Lambda\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t+\psi\right)$$

Single mode coherent state continued

It can also be shown that

$$\langle c_\alpha | |\mathbf{E}(\mathbf{r},t)|^2 | c_\alpha \rangle = \frac{\hbar\omega_\mathbf{k}}{2V\epsilon_0}\left(4\Lambda^2 \sin^2\left(\mathbf{k}\cdot\mathbf{r} - \omega_\mathbf{k} t + \psi\right) + 1\right)$$

Therefore

$$\langle c_\alpha | |\mathbf{E}(\mathbf{r},t)|^2 | c_\alpha \rangle - \left|\langle c_\alpha | \mathbf{E}(\mathbf{r},t) | c_\alpha \rangle\right|^2 = \frac{\hbar\omega_\mathbf{k}}{2V\epsilon_0}$$

This means that variance of the E field for the coherent state is independent of the amplitude Λ. Therefore, for large Λ the variance is small in comparison.

Gauber's coherent state: $\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle$

Here α represents a complex amplitude

It is possible to prove the following identies for the coherent states:

1. $\left\langle c_\alpha \middle| c_\alpha \right\rangle = 1$

2. $\left\langle c_\alpha \middle| a \middle| c_\alpha \right\rangle = \alpha$

3. $\left\langle c_\alpha \middle| a^\dagger \middle| c_\alpha \right\rangle = \alpha^*$

4. $\left|\left\langle c_\alpha \middle| c_\beta \right\rangle\right|^2 = e^{-|\alpha-\beta|^2}$

Visualization of coherent state electric fields for various amplitudes

Source:
R. Loudon, "The Quantum Theory of Light"

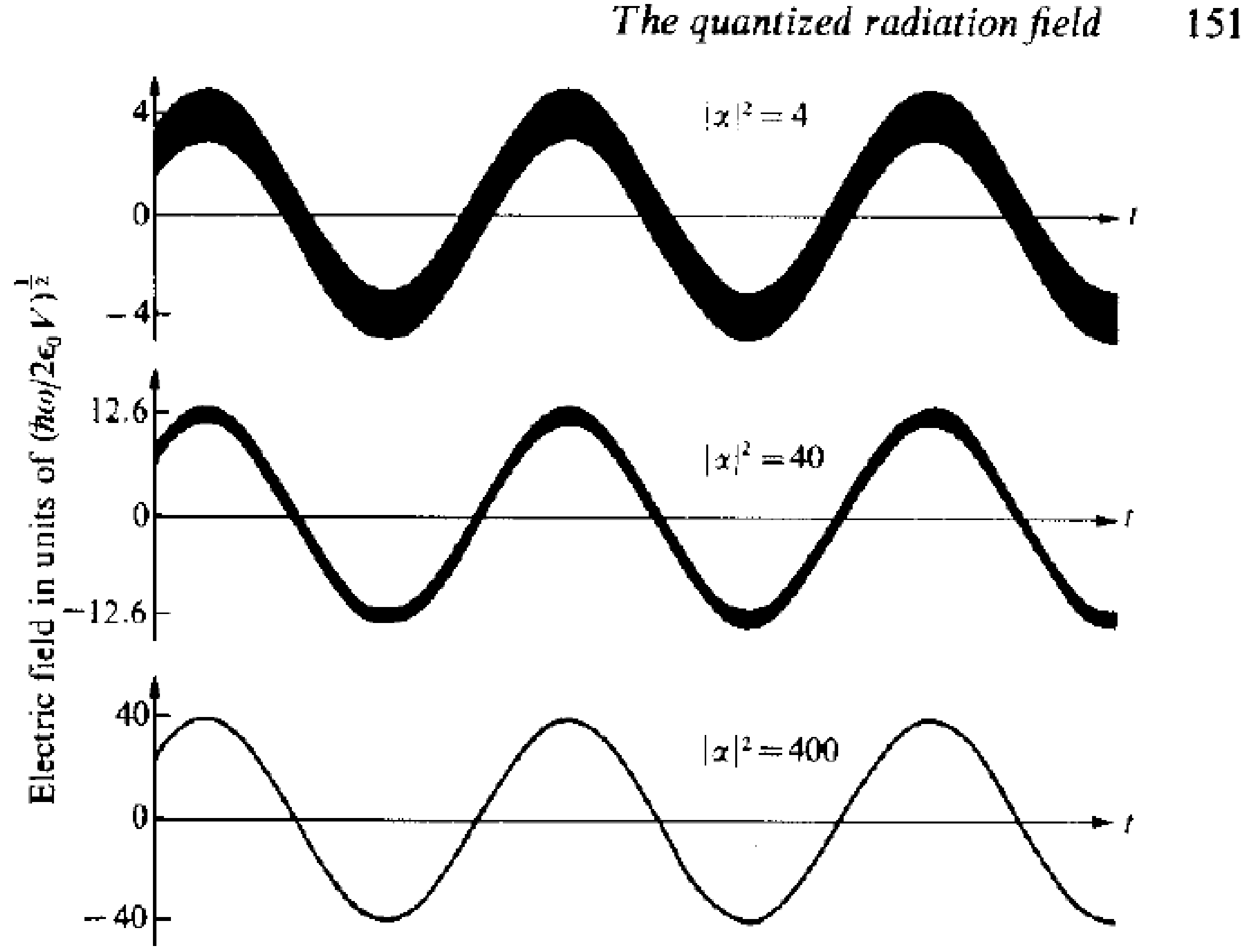


FIG. 4.3. Pictorial representation of the electric-field variation in a cavity mode excited to state $|\alpha\rangle$. Three different values of the mean photon number $|\alpha|^2$ are shown, the vertical scales being different for the three cases. The uncertainties in field values are indicated by the vertical widths $2\Delta E$ of the sine waves. These widths can also be regarded as combinations of the amplitude uncertainty associated with Δn and the phase uncertainty associated with $\Delta \cos\phi$.

Additional properties of single mode coherent state --

Consider the expectation values of the number operator and its square:

$$\mathbf{N}_{\mathbf{k}\sigma} \equiv a^{\dagger}_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$$

$$\left\langle c_\alpha \right| \mathbf{N}_{\mathbf{k}\sigma} \left| c_\alpha \right\rangle = \left|\alpha\right|^2 \qquad \left\langle c_\alpha \right| \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} \left| c_\alpha \right\rangle = \left|\alpha\right|^4 + \left|\alpha\right|^2$$

Square of the variance: $\left\langle c_\alpha \right| \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} \left| c_\alpha \right\rangle - \left| \left\langle c_\alpha \right| \mathbf{N}_{\mathbf{k}\sigma} \left| c_\alpha \right\rangle \right|^2 = \left|\alpha\right|^2$

Fractional uncertainty in the number of photons for the coherent state:

$$\frac{\sqrt{\left\langle c_\alpha \right| \mathbf{N}_{\mathbf{k}\sigma}\mathbf{N}_{\mathbf{k}\sigma} \left| c_\alpha \right\rangle - \left| \left\langle c_\alpha \right| \mathbf{N}_{\mathbf{k}\sigma} \left| c_\alpha \right\rangle \right|^2}}{\left\langle c_\alpha \right| \mathbf{N}_{\mathbf{k}\sigma} \left| c_\alpha \right\rangle} = \frac{\sqrt{\left|\alpha\right|^4 + \left|\alpha\right|^2 - \left|\alpha\right|^4}}{\left|\alpha\right|^2} = \frac{1}{\left|\alpha\right|} = \frac{1}{\Lambda}$$

when $\alpha = \Lambda \mathrm{e}^{i\psi}$

Interpretation of a single mode coherent state

$$\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle$$ based on a single mode $n \to n_{\mathbf{k}\sigma}$

The probability of finding n photons in this state is given by:

$$\left|\left\langle n | c_\alpha \right\rangle\right|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}$$

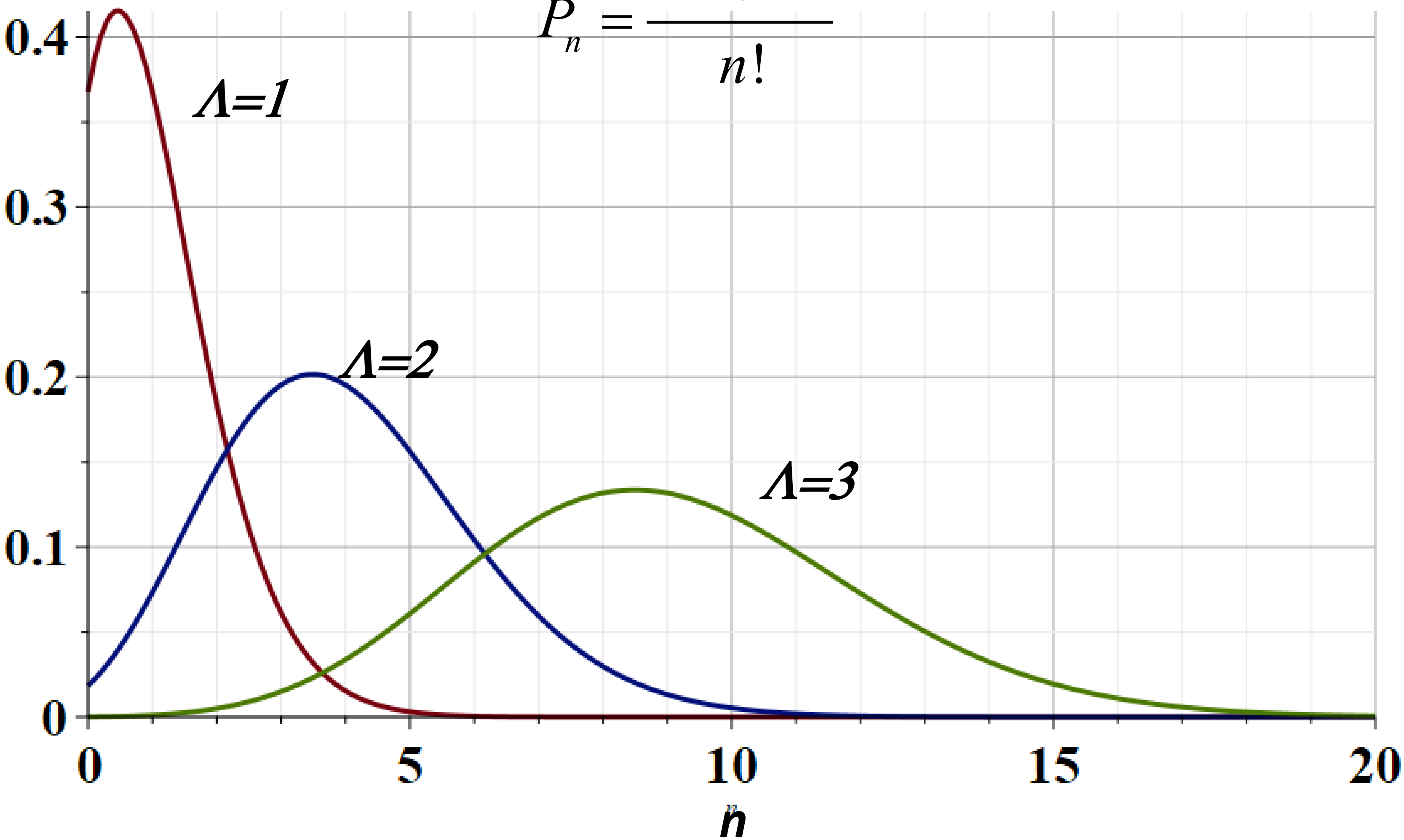
This is the form of a Poisson distribution for a mean value of $|\alpha|^2$.

For $\alpha = \Lambda e^{i\psi}$, the probability of finding the eigenstate with eigenstate $\left|n\right\rangle$ is given by

$$P_n = \left|\left\langle n | c_\alpha \right\rangle\right|^2 = \frac{|\Lambda|^{2n} e^{-|\Lambda|^2}}{n!}$$

Poisson distributions

$$P_n = \frac{\Lambda^{2n} e^{-\Lambda^2}}{n!}$$



Focusing on a particular pure EM mode with wavenumber $\mathbf{k}$ and frequency $\omega_{\mathbf{k}}$:

For a coherent state c_α with $\alpha = \Lambda e^{i\Psi}$, the probability of finding the eigenstate with photon number $\left|n\right\rangle$ is given by

$$P_n^{\text{Coherent}} = \left|\left\langle n \middle| c_\alpha \right\rangle\right|^2 = \frac{\left|\Lambda\right|^{2n} e^{-\left|\Lambda\right|^2}}{n!}$$

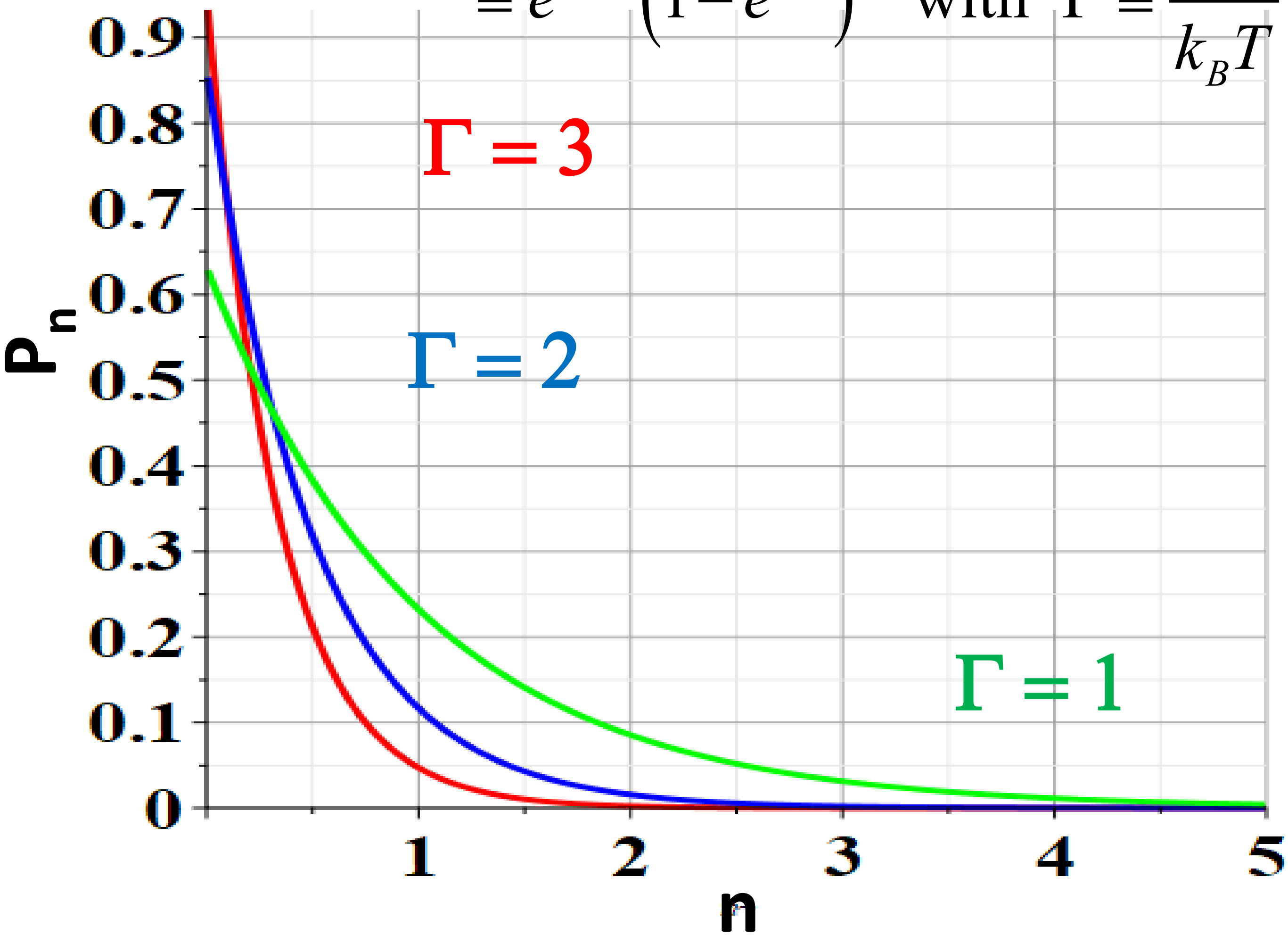
For "a black body system" at temperature T, the probability of finding the eigenstate with photon number $\left|n\right\rangle$ is given by

$$P_n^{\text{Thermal}}(T) = e^{-n\hbar\omega/k_B T}\left(1 - e^{-\hbar\omega/k_B T}\right)$$

Thermal distributions:

$$P_n^{\text{Thermal}}(T) = e^{-n\hbar\omega/k_BT}\left(1-e^{-\hbar\omega/k_BT}\right)$$

$$\equiv e^{-n\Gamma}\left(1-e^{-\Gamma}\right) \quad \text{with} \quad \Gamma \equiv \frac{\hbar\omega}{k_BT}$$



Other thoughts about the coherent photon state from Professor Kandada and from the Mandel and Wolf textbook –

1. **It turns out that the coherent state basis can be quite well realized using laser technology**
2. **There are problematic issues with the coherent state basis stemming from the fact that it is mathematically "over complete".**
3. **Despite these mathematical difficulties, because of #1 and related experimental processes, the coherent state formalism remains useful.**

A derivation of the coherent state from Mandel and Wolf:

Here we will focus on a single photon mode $\mathbf{k}, \sigma$ dropping those indicesλ in describing the creation $a^\dagger$, anhilation a, and number $a^\dagger a$ operators.

While $a^\dagger$ and a are not hermition operators, we can attempt to find their eigenvalues λ and functions $|\lambda\rangle$, expecting the eigenvalues to be complex.

$$a|\lambda\rangle = \lambda|\lambda\rangle \quad \text{and} \quad \langle\lambda|a^\dagger = \langle\lambda|\lambda^*$$

Assume that the eigenfunctions can be expanded in the number operator basis:

$$|\lambda\rangle = \sum_n c_n |n\rangle \text{ where the coefficients } c_n \text{ can be determined.}$$

$$a|\lambda\rangle = \lambda|\lambda\rangle \quad \Rightarrow \sum_n c_n \sqrt{n}|n-1\rangle = \lambda \sum_n c_n |n\rangle \quad \Rightarrow \quad c_{n+1} = \frac{\lambda}{\sqrt{n+1}} c_n$$

After several steps the normalized coherent eigenfunction is given by

$$|\lambda\rangle = e^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

The result $\left|\lambda\right\rangle = e^{-|\lambda|^2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left|n\right\rangle$ is equivalent to the previous result from Glauber. Writing $\lambda = \Lambda e^{i\psi}$ we found that the expecion value of the electric and magnetic fields take the form

$$\left\langle\lambda\right|\mathbf{E}(\mathbf{r},t)\left|\lambda\right\rangle = -2\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\Lambda\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t+\psi\right)$$

$$\left\langle\lambda\right|\mathbf{B}(\mathbf{r},t)\left|\lambda\right\rangle = -2\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\Lambda\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t+\psi\right)$$

Where Λ and ψ determine the amplitudes and phases of the fields.

It turns out that the amplitude and phase of the fields have complicated non-commuting relationships resulting in a Heisenberg uncertainty relationship. Unfortunately, the direct representation of the phase operator is complicated, and it is convenient to express the phenomenon through related operators.

Further analysis and modifications of the "coherent state"

Recall that we can write the EM Hamiltonian for a single mode $\omega_{\mathbf{k}} \equiv \omega$ --

$$H = \frac{1}{2}\hbar\omega\left(a^\dagger a + a a^\dagger\right) \quad \text{where } \left[a, a^\dagger\right] = 1$$

Define convenient unitless Hermitian operators

$$\hat{Q} \equiv \left(a^\dagger + a\right) \quad \text{and} \quad \hat{P} \equiv i\left(a^\dagger - a\right) \quad \Rightarrow \left[\hat{Q}, \hat{P}\right] = 2i$$

$$H = \frac{\hbar\omega}{2}\left(\hat{Q}^2 + \hat{P}^2\right)$$

From the Heisenberg uncertainty ideas applied to the standard deviations:

$$\Delta\hat{Q}\Delta\hat{P} \geq 1$$

In terms of the eigenstates of the EM Hamiltonian:

$$H\left|n\right\rangle = \hbar\omega\left(n+\frac{1}{2}\right)\left|n\right\rangle$$

$$\Delta\hat{Q}_n = \sqrt{\left\langle n\left|\hat{Q}^2\right|n\right\rangle - \left|\left\langle n\left|\hat{Q}\right|n\right\rangle\right|^2} = \sqrt{2n+1} = \Delta\hat{P}_n$$

$$\Rightarrow \Delta\hat{Q}_n\Delta\hat{P}_n = 2n+1 \geq 1$$

In terms of coherent states: --

For the coherent state:

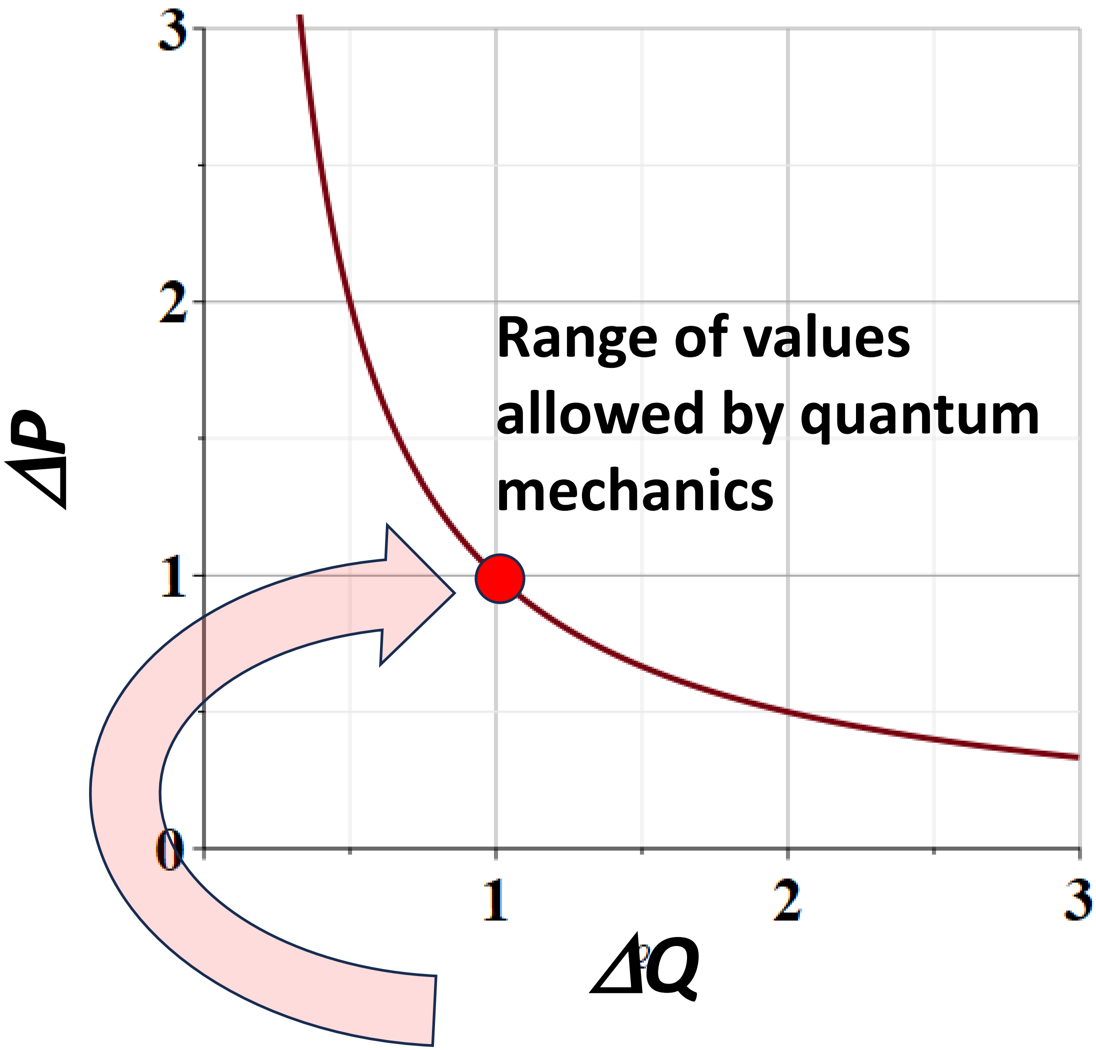
$$\left|\lambda\right\rangle = e^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \left|n\right\rangle$$

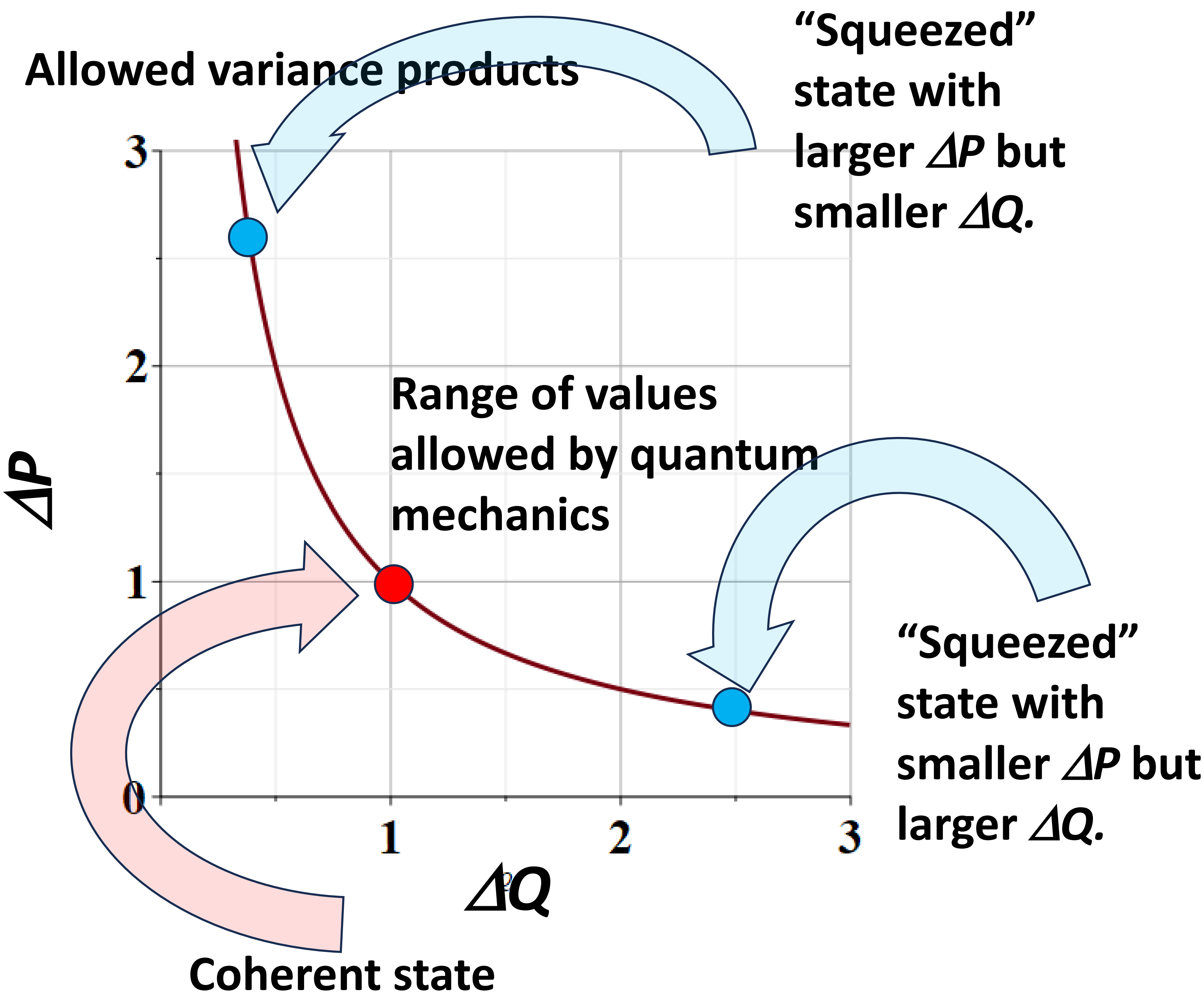
$$\Delta\hat{Q}_\lambda = \sqrt{\left\langle\lambda\middle|\hat{Q}^2\middle|\lambda\right\rangle - \left|\left\langle\lambda\middle|\hat{Q}\middle|\lambda\right\rangle\right|^2} = 1 = \Delta\hat{P}_\lambda$$

$$\Rightarrow \Delta\hat{Q}_\lambda \Delta\hat{P}_\lambda = 1$$

In this sense, the coherent state represents the minimum uncertainty process.

Allowed variance products




Allowed variance products
"Squeezed" state with larger ΔP but smaller ΔQ.
Range of values allowed by quantum mechanics
"Squeezed" state with smaller ΔP but larger ΔQ.
Coherent state
ΔP
ΔQ
0
1
2
3
1
2
3

Altered operators --

$$\hat{Q}_\beta \equiv \left(a^\dagger e^{i\beta} + a e^{-i\beta}\right) \quad \text{and} \quad \hat{P}_\beta \equiv i\left(a^\dagger e^{i\beta} - a e^{-i\beta}\right)$$

Note that $[\hat{Q}_\beta,\ \hat{P}_\beta] = 2i$ which implies $\Delta\hat{Q}_\beta \Delta\hat{P}_\beta \geq 1$

Also note that $\hat{Q}_{\beta=0} \equiv \hat{Q}$ and $\hat{P}_{\beta=0} \equiv \hat{P}$

It also can be shown that

$$\hat{Q}_\beta = \cos\beta\ \hat{Q}_0 + \sin\beta \hat{P}_0$$

$$\hat{P}_\beta = -\sin\beta \hat{Q}_0 + \cos\beta \hat{P}_0$$

This means that by changing β, it may be possible to make $\Delta\hat{Q}_\beta < 1$ and $\Delta\hat{P}_\beta > 1$ or $\Delta\hat{Q}_\beta > 1$ and $\Delta\hat{P}_\beta < 1$ which leads to the notion of "squeezing".

Estimation of $\left\langle \left(\Delta\hat{Q}_\beta \right)^2 \right\rangle$

Suppose that the expectation value can be evaluated with the help of a density matrix expressed in terms of coherent states $|\lambda\rangle$ and positive weight factor $\phi(\lambda)$

$$\hat{\rho} \equiv \int \phi(\lambda) |\lambda\rangle\langle\lambda| d(\Re\lambda) d(\Im\lambda) \equiv \int \phi(\lambda)|\lambda\rangle\langle\lambda| d^2\lambda$$

$$\left\langle \hat{Q}_\beta \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{Q}_\beta\right) = \int \phi(\lambda) \langle\lambda|\hat{Q}_\beta|\lambda\rangle d^2\lambda = \int \phi(\lambda)\langle\lambda| a^\dagger e^{i\beta} + a e^{-i\beta} |\lambda\rangle d^2\lambda$$

Letting $\lambda \equiv \Lambda e^{i\psi}$ $\quad \left\langle \hat{Q}_\beta \right\rangle = 2\int \phi(\lambda)\Lambda\cos(\psi-\beta) d^2\lambda$

$$\left\langle \left(\hat{Q}_\beta\right)^2 \right\rangle = \int \phi(\lambda)\left[4\Lambda^2\cos^2(\psi-\beta)+1\right]d^2\lambda$$

$$\left\langle \left(\Delta\hat{Q}_\beta\right)^2 \right\rangle = \left\langle \left(\hat{Q}_\beta\right)^2 \right\rangle - \left\langle \hat{Q}_\beta \right\rangle^2$$

Next time, following Mandel and Wolf, we introduce the squeeze operator

$$\hat{S}(z) \equiv \exp\left(\frac{1}{2}\left(z^* \hat{a}^2 - z\hat{a}^{\dagger 2}\right)\right) \quad \text{where } z = re^{i\theta}$$

With this transformation, we see that

$$\left\langle \left(\Delta \hat{Q}_{\text{Squeezed}}\right)^2 \right\rangle = e^{-2r} \left\langle \left(\Delta \hat{Q}\right)^2 \right\rangle$$