# PHY 712 Electrodynamics
# 10-10:50 AM MWF Olin 103

## Notes for Lecture 39:

### Review –

1. Thanks for a great semester
2. Continued review of topics and problem solving strategies

| | | | | | |
|---|---|---|---|---|---|
| 29 | Fri: 03/29/2024 | Chap. 11 | Special Theory of Relativity | | |
| 30 | Mon: 04/01/2024 | Chap. 14 | Radiation from moving charges | #24 | 04/08/2024 |
| 31 | Wed: 04/03/2024 | Chap. 14 | Radiation from accelerating charged particles | #25 | 04/08/2024 |
| 32 | Fri: 04/05/2024 | Chap. 14 | Synchrotron radiation and Compton scattering | #26 | 04/08/2024 |
| | Mon: 04/08/2024 | No class | Eclipse related absences | | |
| 33 | Wed: 04/10/2024 | Chap. 13 & 15 | Other radiation -- Cherenkov & bremsstrahlung | #27 | 04/22/2024 |
| 34 | Fri: 04/12/2024 | | Special topic: E & M aspects of superconductivity | | |
| | Mon: 04/15/2024 | | Presentations I | | |
| | Wed: 04/17/2024 | | Presentations II | | |
| | Fri: 04/19/2024 | | Presentations III | | |
| 35 | Mon: 04/22/2024 | | Special topic: Quantum Effects in E & M | | |
| 36 | Wed: 04/24/2024 | | Special topic: Quantum Effects in E & M | | |
| 37 | Fri: 04/26/2024 | | Special topic: Quantum Effects in E & M | | |
| 38 | Mon: 04/29/2024 | | Review | | |
| 39 | Wed: 05/01/2024 | | Review | | |

Important dates: Final exams available ~May 2
Exams and outstanding HW due May 10

From Lecture 24 --
Electromagnetic waves from time harmonic sources

$$\text{Charge density:} \quad \rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$$

$$\text{Current density:} \quad \mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$$

Note that the continuity condition:

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r},t) = 0 \quad \Rightarrow -i\omega\tilde{\rho}(\mathbf{r},\omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$$

For dynamic problems where $\tilde{\rho}(\mathbf{r},\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r},\omega)$ are contained in a small region of space and $S \to \infty$,

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

# Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorenz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorenz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm}\widetilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\widetilde{\mathbf{J}}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

# Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

Power radiated:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2\hat{\mathbf{r}}}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\,(ik\mu_0)\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_0(kr_>)j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{J_0 k}{\varepsilon_0\omega R}\cos\theta\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_1(kr_>)j_1(kr_<)$$

Evaluation for $r >> R$ :

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}\;\frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=\frac{J_0 k}{\varepsilon_0\omega}\cos\theta\,\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\;\frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Note that this result is valid for all *k*.

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1(x) = -\left(1 + \frac{i}{x}\right)\frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3\cos(x)}{x^2} \qquad h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right)\frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x << 1 \qquad \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x >> 1 \qquad \Rightarrow h_l(x) \approx (-i)^{l+1}\frac{e^{ix}}{x}$$

# Some details

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \; (ik\mu_0) \int_0^\infty r'^2\, dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$= \hat{\mathbf{z}} J_0 \; (ik\mu_0) \left( h_0(kr) \int_0^r r'^2\, dr' e^{-r'/R} j_0(kr') + \cancel{j_0(kr) \int_r^\infty r'^2\, dr' e^{-r'/R} h_0(kr')} \right)$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{J_0 k}{\varepsilon_0 \omega R} \; \cos\theta \int_0^\infty r'^2\, dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

$$= -\frac{J_0 k}{\varepsilon_0 \omega R} \; \cos\theta \left( h_1(kr) \int_0^r r'^2\, dr' e^{-r'/R} j_1(kr') + \cancel{j_1(kr) \int_r^\infty r'^2\, dr' e^{-r'/R} h_1(kr')} \right)$$

For *r>>R*, these terms are negligible

# More details for *r>>R*

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \ (ik\mu_0) h_0(kr) \int_0^\infty r'^2 \, dr' e^{-r'/R} j_0(kr')$$

```
> assume(RR > 0); assume(k > 0)
> simplify(int(x^2·exp(-x/RR)·sin(k·x)/(k·x), x = 0 ..infinity))
```

$$\frac{2\,RR\sim^3}{\left(k\sim^2\,RR\sim^2+1\right)^2}$$

# More details for *r>>R*

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{J_0 k}{\varepsilon_0 \omega R}\cos\theta h_1(kr)\int_0^\infty r'^2\, dr' e^{-r'/R} j_1(kr')$$

```
> assume(RR > 0); assume(k > 0)
> simplify(int(x^2·exp(-x/RR)·(sin(k·x)/(k^2·x^2) - cos(k·x)/(k·x)), x = 0..infinity))
```

$$\frac{2\, RR{\sim}^4\, k{\sim}}{\left(k{\sim}^2\, RR{\sim}^2 + 1\right)^2}$$

Example of dipole radiation source -- continued

Evaluation for $r >> R$:

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \;\; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \, \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right) \;\; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Relationship to pure dipole approximation (exact when $kR \rightarrow 0$)

$$\mathbf{p}(\omega) \equiv \int d^3r\, \mathbf{r}\widetilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r\, \widetilde{\mathbf{J}}(\mathbf{r},\omega) = -\frac{8\pi R^3 J_0}{i\omega}\hat{\mathbf{z}}$$

Corresponding dipole fields: $\widetilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$

$$\widetilde{\Phi}(\mathbf{r},\omega) = -\frac{i}{4\pi\omega\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Electromagnetic waves from time harmonic sources – for dipole radiation --:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left( k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)\right)-\mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2\hat{\mathbf{r}}}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

Another useful approximation for analyzing radiation for time harmonic sources --

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{e^{ikr}}{r} e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \qquad k \equiv \frac{\omega}{c}$$

for $r >> r'$

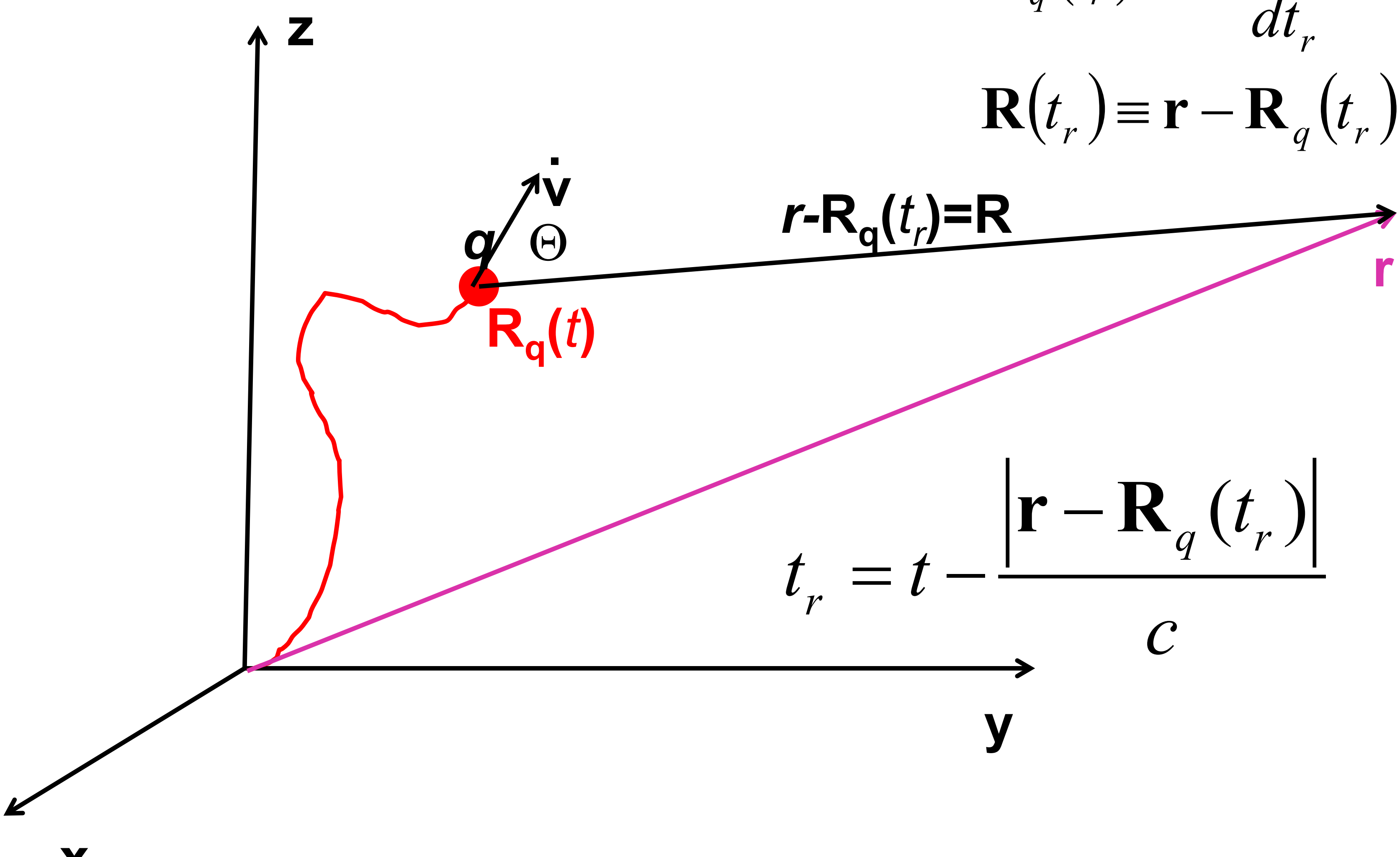
often used for antenna radiation

# Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$




$$t_r = t - \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c}$$

# Liènard-Wiechert potentials –(Gaussian units)

$$\dot{\mathbf{R}}_q\left(t_r\right) \equiv \frac{d\mathbf{R}_q\left(t_r\right)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}\left(t_r\right) \equiv \mathbf{r} - \mathbf{R}_q\left(t_r\right) \equiv \mathbf{R}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Electric and magnetic fields far from source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Let $\hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R}$ $\quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}$ $\quad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

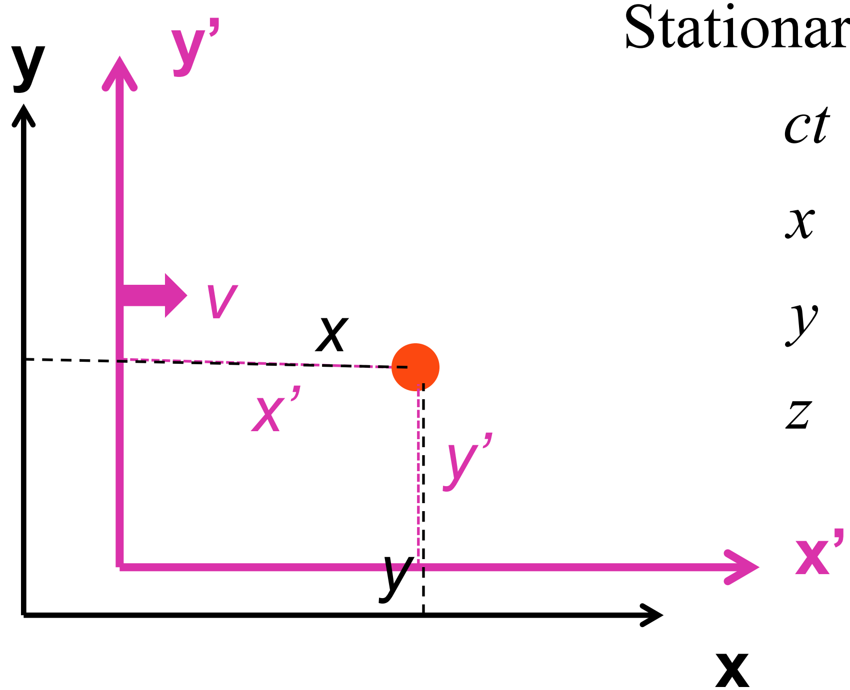
$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

## Lorentz transformations

Convenient notation :

$$\beta_v \equiv \frac{v}{c}$$

$$\gamma_v \equiv \frac{1}{\sqrt{1-\beta_v{}^2}}$$




| Stationary frame | | Moving frame |
|---|---|---|
| $ct$ | $=$ | $\gamma(ct'+\beta x')$ |
| $x$ | $=$ | $\gamma(x'+\beta ct')$ |
| $y$ | $=$ | $y'$ |
| $z$ | $=$ | $z'$ |

**Note, the concept of the Lorentz transformation is quite general, but the specific transformation form given in the following slides is special to the relative velocity along the x-axis.**

# Lorentz transformations -- continued

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$ :

$$\boldsymbol{\mathcal{L}}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{\mathcal{L}}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \boldsymbol{\mathcal{L}}_v \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{\mathcal{L}}_v^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice :

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

# Velocity relationships

Consider: $u_x = \dfrac{u'_x + v}{1 + vu'_x/c^2}$ $\quad u_y = \dfrac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)}$ $\quad u_z = \dfrac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$

Note that $\gamma_u \equiv \dfrac{1}{\sqrt{1-(u/c)^2}} = \dfrac{1 + vu'_x/c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v\gamma_{u'}\left(1 + vu'_x/c^2\right)$

$$\Rightarrow \gamma_u c = \gamma_v\left(\gamma_{u'}c + \beta_v\gamma_{u'}u'_x\right)$$

$$\Rightarrow \gamma_u u_x = \gamma_v\left(\gamma_{u'}u'_x + \gamma_{u'}v\right) = \gamma_v\left(\gamma_{u'}u'_x + \beta_v\gamma_{u'}c\right)$$

$$\Rightarrow \gamma_u u_y = \gamma_{u'}u'_y \qquad \gamma_u u_z = \gamma_{u'}u'_z$$

$$\rightarrow \quad \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$$

Field strength tensor $F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

For stationary frame

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

For moving frame

$$F'^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

➔ This analysis shows that the E and B fields must be treated as components of the field strength tensor and that in order to transform between inertial frames, we need to use the tensor transformation relationships:

Transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}_v{}^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v{}^{\delta\beta} \qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y + \beta_v B'_z\right) & -\gamma_v\left(E'_z - \beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z + \beta_v E'_y\right) & \gamma_v\left(B'_y - \beta_v E'_z\right) \\ \gamma_v\left(E'_y + \beta_v B'_z\right) & \gamma_v\left(B'_z + \beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z - \beta_v B'_y\right) & -\gamma_v\left(B'_y - \beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

# Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Summary of results:

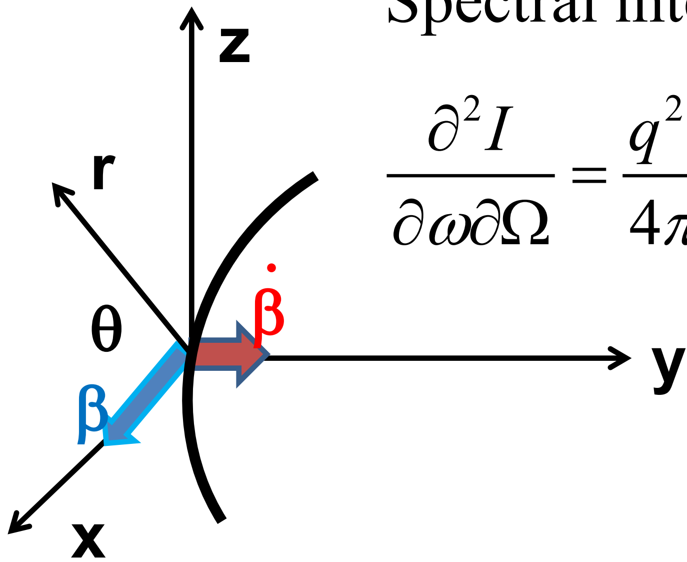
$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma_v\left(E_y - \beta_v B_z\right) \qquad B'_y = \gamma_v\left(B_y + \beta_v E_z\right)$$

$$E'_z = \gamma_v\left(E_z + \beta_v B_y\right) \qquad B'_z = \gamma_v\left(B_z - \beta_v E_y\right)$$

Some comments on synchrotron radiation spectra from large synchrotron facilities

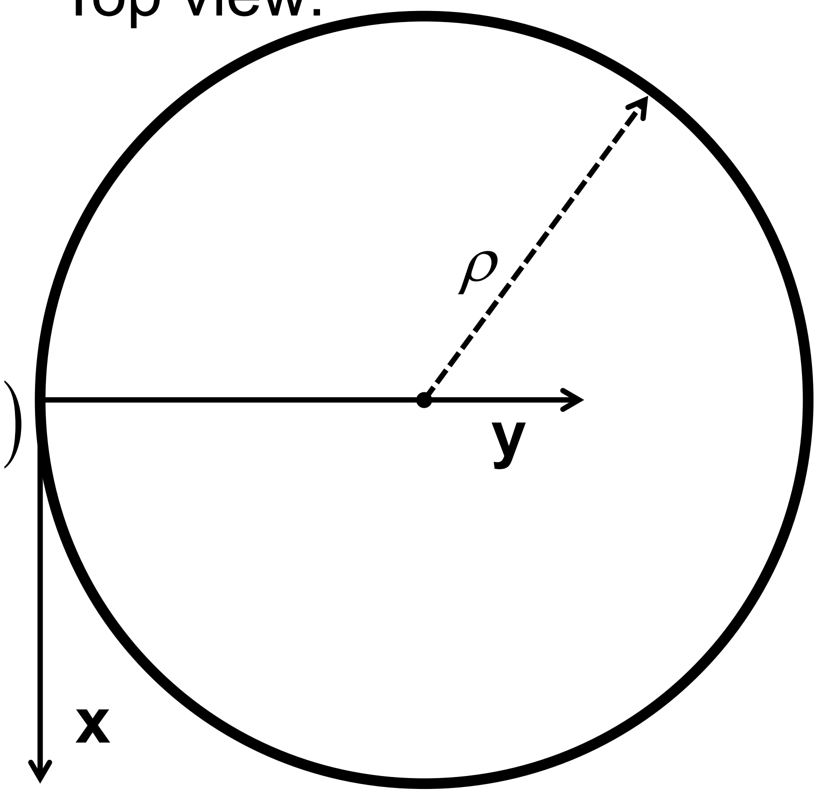
Slides from Lecture 31




# Spectral intensity relationship:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\ e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\ \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]\right|^2$$

Top view:

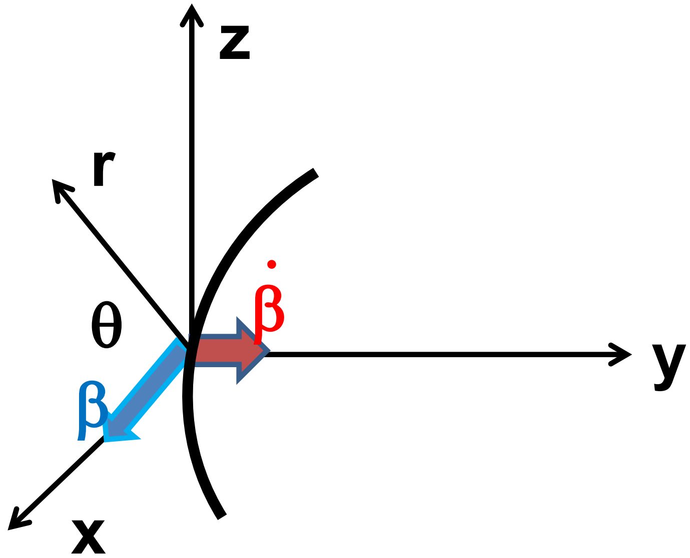



$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho)$$
$$+\ \ \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$




$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

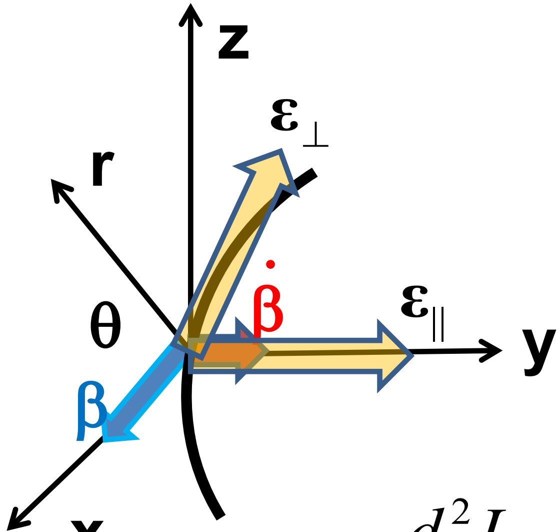
For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Note that we have previous shown that in the radiation zone, the Poynting vector is in the $\hat{\mathbf{r}}$ direction; we can then choose to analyze two orthogonal polarization directions:

$$\boldsymbol{\varepsilon}_{||} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{||}\sin(vt_r/\rho) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$




$$\boldsymbol{\varepsilon}_\parallel = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_\perp = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right) =$$

$$\beta\left(-\boldsymbol{\varepsilon}_\parallel \sin\left(vt_r/\rho\right) + \boldsymbol{\varepsilon}_\perp \sin\theta\cos\left(vt_r/\rho\right)\right)$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\beta)e^{i\omega(t-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)}dt\right|^2$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_\parallel(\omega)|^2 + |C_\perp(\omega)|^2\right\}$$

$$C_\parallel(\omega) = \int_{-\infty}^{\infty} dt\sin(vt/\rho)e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$C_\perp(\omega) = \int_{-\infty}^{\infty} dt\sin\theta\cos(vt/\rho)e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

We will analyze this expression for the case in which the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1-1/(2\gamma^2))$) passing a beam line port. In addition, because of the design of the radiation ports, $\theta \approx 0$, and the relevant integration times $t$ are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency $\omega_c \equiv \dfrac{3c\gamma^3}{2\rho}$.

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

Some details:

Modified Bessel functions

$$K_{1/3}(\xi) = \sqrt{3}\int_0^\infty dx\, \cos\left[\tfrac{3}{2}\,\xi\left(x + \tfrac{1}{3}\,x^3\right)\right] \qquad K_{2/3}(\xi) = \sqrt{3}\int_0^\infty dx\, x\sin\left[\tfrac{3}{2}\,\xi\left(x + \tfrac{1}{3}\,x^3\right)\right]$$

Exponential factor

$$\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)/c\right) = \omega\left(t_r - \frac{\rho}{c}\cos\theta\sin\left(vt_r/\rho\right)\right)$$

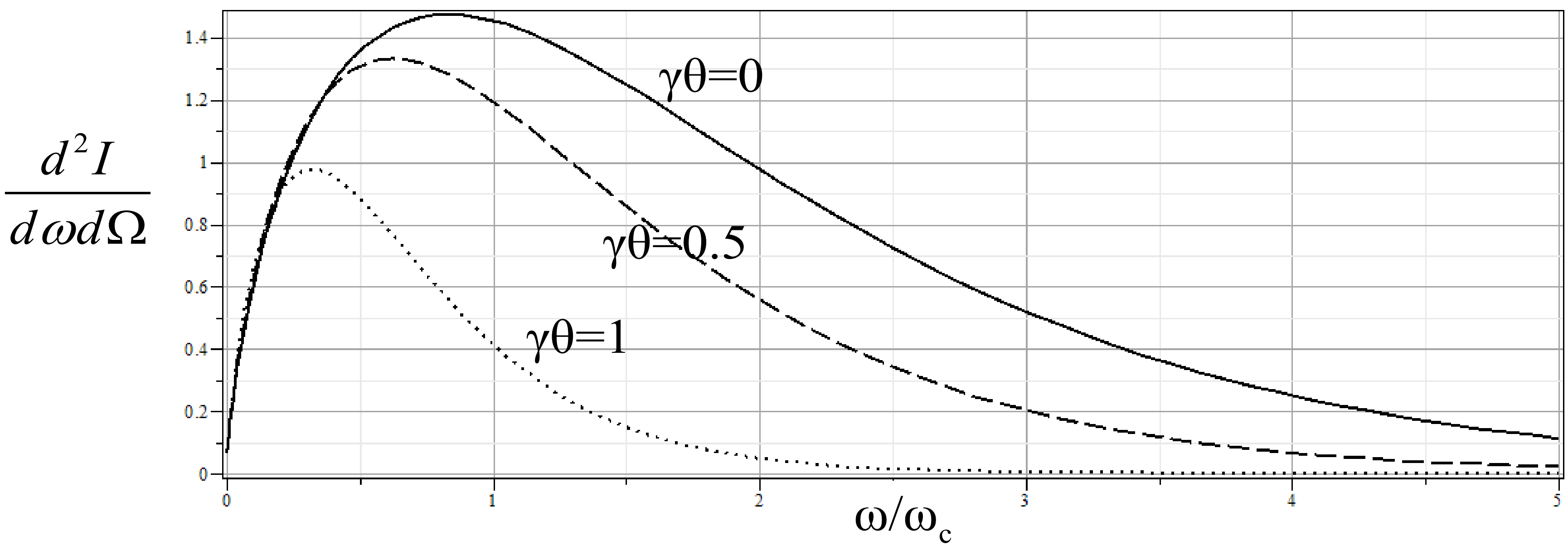
In the limit of $t_r \approx 0, \quad \theta \approx 0, \;\; v \approx c\left(1 - \dfrac{1}{2\gamma^2}\right)$

$$\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)/c\right) \approx \frac{\omega t_r}{2\gamma^2}\left(1+\gamma^2\theta^2\right) + \frac{\omega c^2 t_r^{\;3}}{6\rho^2} = \frac{3}{2}\xi\left(x + \frac{1}{3}x^3\right)$$

where $\displaystyle \xi = \frac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}$ and $\displaystyle x = \frac{c\gamma t_r}{\rho\left(1+\gamma^2\theta^2\right)^{1/2}}$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

By plotting the intensity as a function of $\omega$, we see that the intensity is largest near $\omega \approx \omega_c$. The plot below shows the intensity as a function of $\omega/\omega_c$ for $\gamma\theta$=0 , 0.5 and 1:

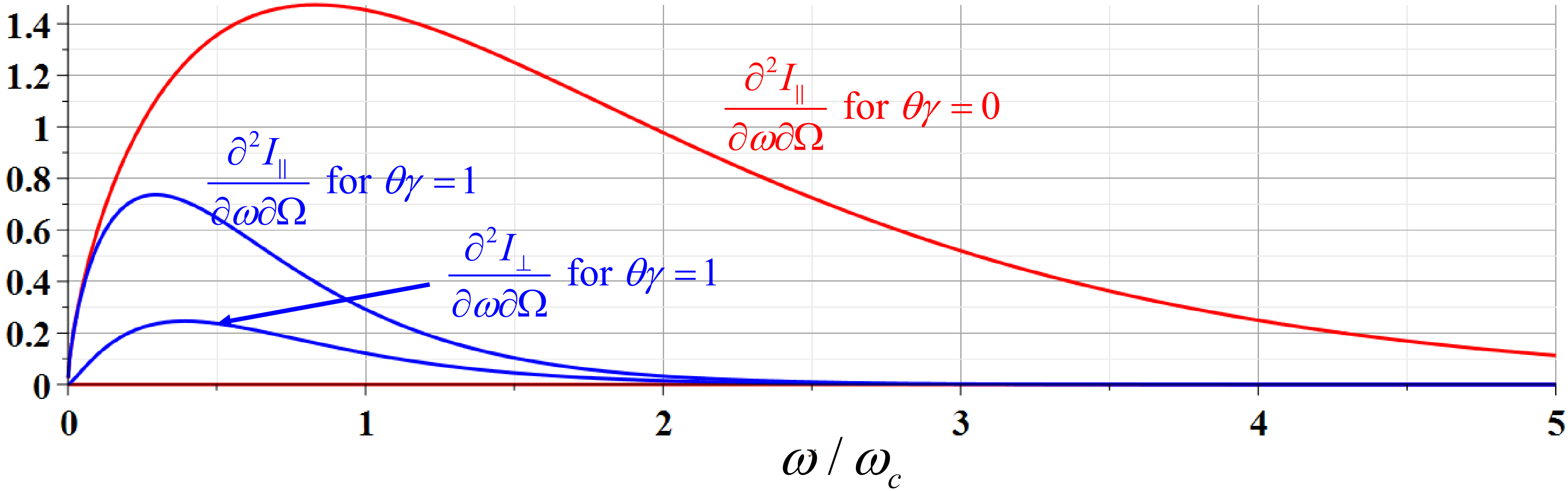


# More details

$$\frac{d^2I}{d\omega d\Omega} = \frac{d^2I_{\parallel}}{d\omega d\Omega} + \frac{d^2I_{\perp}}{d\omega d\Omega}$$

$$\frac{d^2I_{\parallel}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$

$$\frac{d^2I_{\perp}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$



Quantum effects in E & M
--Review of what we learned from Lecture 35

For a single mode plane wave with wave vector **k**, frequency $\omega_{\mathbf{k}}$ and polarization $\sigma$:

EM Field Hamiltonian acting on eigenstate $\left|n_{\mathbf{k}\sigma}\right\rangle$:

where $\mathbf{k}$ denotes wavevector and $\sigma$ denotes polarization direction --

$$H^{\text{fixed}}_{\text{field}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'}\left(\hbar\omega_{\mathbf{k}'}a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \hbar\omega_{\mathbf{k}}n_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle$$

Here $n_{\mathbf{k}\sigma} = 0,1,2,3,4.......$

$$a_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}}\left|n_{\mathbf{k}\sigma}-1\right\rangle$$

$$a^{\dagger}_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}+1}\left|n_{\mathbf{k}\sigma}+1\right\rangle$$

Commutation relations:

$$\left[a_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'}\right]=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \quad \left[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'}\right]=0 \quad \left[a^{\dagger}_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'}\right]=0$$

In terms of the same operators and with polarization unit vectors $\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}$ --

Vector potential:

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left( a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} + a^{\dagger}_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t)} \right)$$

Electric field:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad \Rightarrow \mathbf{E}(\mathbf{r},t) = i \sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left( a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} - a^{\dagger}_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t)} \right)$$

Magnetic field:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \Rightarrow \mathbf{B}(\mathbf{r},t) = i \sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left( a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} - a^{\dagger}_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t)} \right)$$

While the photon eigenstates $\left|n_{\mathbf{k}'\sigma'}\right\rangle$ form a complete basis for describing quantum electromagnetic fields, they have some troublesome properties such as found in evaluating the field expectation values --

Vector potential:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{A}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle = \sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} + a^\dagger_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle = 0$$

Electric field:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{E}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle = i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - a^\dagger_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle = 0$$

Magnetic field:

$$\left\langle n_{\mathbf{k}'\sigma'}\right|\mathbf{B}(\mathbf{r},t)\left|n_{\mathbf{k}'\sigma'}\right\rangle = i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left\langle n_{\mathbf{k}'\sigma'}\right|\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - a^\dagger_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)\left|n_{\mathbf{k}'\sigma'}\right\rangle = 0$$

## A convenient superposition thanks to R. Glauber, PR 131, 2766 (1963)

$$\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle \quad \text{based on a single mode } n \to n_{\mathbf{k}\sigma}$$

Electric field: $\left\langle c_\alpha \right| \mathbf{E}(\mathbf{r},t) \left| c_\alpha \right\rangle = i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(\alpha_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - \alpha^*_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$

Magnetic field: $\left\langle c_\alpha \right| \mathbf{B}(\mathbf{r},t) \left| c_\alpha \right\rangle = i\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(\alpha_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - \alpha^*_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$

Let $\alpha = \Lambda e^{i\psi}$ where both $\Lambda$ and $\Psi$ are unitless real values.

$$\left\langle c_\alpha \right| \mathbf{E}(\mathbf{r},t) \left| c_\alpha \right\rangle = -2\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\Lambda\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t+\psi\right)$$

$$\left\langle c_\alpha \right| \mathbf{B}(\mathbf{r},t) \left| c_\alpha \right\rangle = -2\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\Lambda\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t+\psi\right)$$

## Single mode coherent state continued

It can also be shown that

$$\left\langle c_\alpha \left| \left| \mathbf{E}(\mathbf{r},t) \right|^2 \right| c_\alpha \right\rangle = \frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}\left(4\Lambda^2 \sin^2\left(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t + \psi\right) + 1\right)$$

Therefore

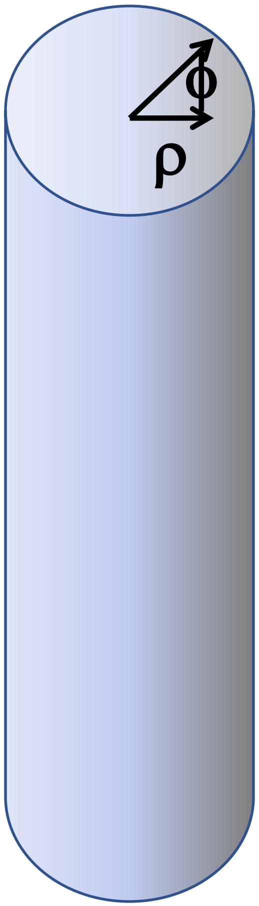
$$\left\langle c_\alpha \left| \left| \mathbf{E}(\mathbf{r},t) \right|^2 \right| c_\alpha \right\rangle - \left| \left\langle c_\alpha \left| \mathbf{E}(\mathbf{r},t) \right| c_\alpha \right\rangle \right|^2 = \frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}$$

**This means that variance of the E field for the coherent state is independent of the amplitude $\Lambda$. Therefore, for large $\Lambda$ the variance is small in comparison.**

Review of cylindrical coordinates

Slides from Lecture 8 --

Solution of the Poisson/Laplace equation in various geometries
➔cylindrical geometry with no *z*-dependence (infinitely long wire, for example):




Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

$$\Phi(\rho,\varphi) = \Phi(\rho,\varphi + m2\pi)$$

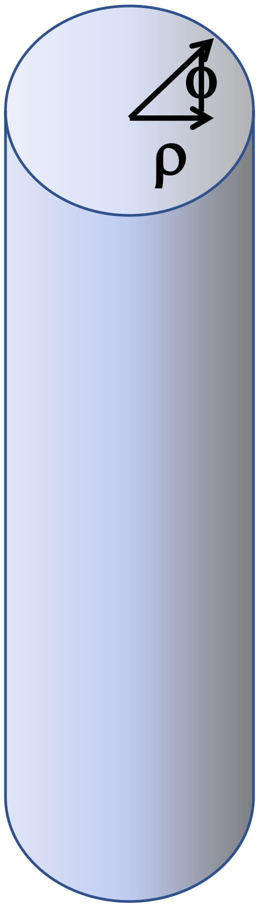
Assume: $\Phi(\rho,\phi) = f(\rho)g(\phi)$

Suppose $\frac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$

$g(\phi) = \cos(m\phi + \alpha_m) \quad \Rightarrow m = \text{integer}, \alpha_m = \text{phase}$

Solution of the Poisson/Laplace equation in various geometries

➔cylindrical geometry with no *z*-dependence (infinitely long wire, for example):




Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

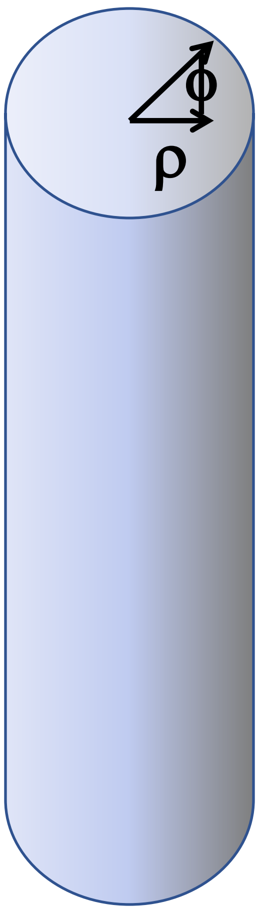
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}=0$$

Assume: $\Phi(\rho,\phi)=f(\rho)g(\phi)$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df_m(\rho)}{d\rho}\right)-\frac{m^2}{\rho^2}f_m(\rho)=0$$

$$f_0(\rho)=\begin{cases}1\\ \ln\rho\end{cases}\qquad f_{m>0}=\rho^{\pm m}$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):




Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

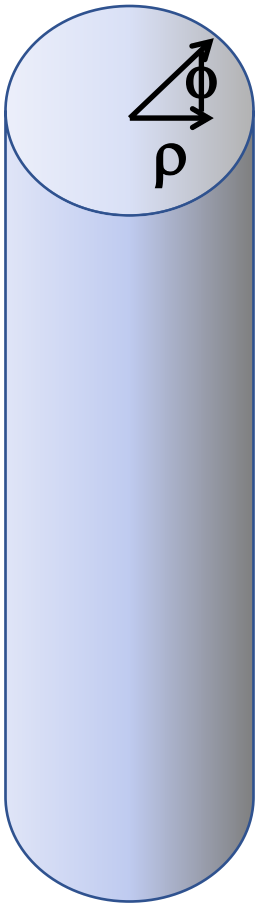
$\Phi(\rho,\varphi) = \Phi(\rho,\varphi + m2\pi)$ ➔ *m*=integer

$\Rightarrow$ General solution of the Laplace equation

in these coordinates:

$$\Phi(\rho,\varphi) = \mathrm{A}_0 + \mathrm{B}_0\ln(\rho) + \sum_{m=1}^{\infty}\left(A_m\rho^m + B_m\rho^{-m}\right)\cos(m\varphi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):




Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)G(\rho,\rho',\varphi,\varphi') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\varphi-\varphi')$$

It can be shown that the following form can be used:

$$G(\rho,\rho',\varphi,\varphi') = -\ln\left(\rho_>^{\,2}\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left(m(\varphi-\varphi')\right)$$

Note that this example is similar to the construction for the 2-d cartesian case --

For the 2-d cartesian case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y') \quad \text{where} \quad \frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

The $y$ dependence of this equation will have the required behavior, if we choose:

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y,y') = -4\pi\delta(y-y'),$$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0,$$

and the Wronskian constant: $K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$

## Cartesian example continued --

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y,y') = -4\pi\delta(y-y'),$$

$$g_n(y,y') = \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>)$$

where: $$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0,$$

and $$K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n}\, y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n}(b-y))$

where $K_n = \sqrt{\alpha_n}\sinh(\sqrt{\alpha_n}\, b)$

using the identity: $\cosh(r)\sinh(s) + \sinh(r)\cosh(s) = \sinh(r+s)$

$$G(x,x',y,y') = \sum_n u_n(x) u_n(x') \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>).$$

# In the cylindrical geometry case,

$$u_n(x) \rightarrow \{\sin(m\varphi), \cos(m\varphi)\}$$

$$v_{n_{1,2}} \rightarrow \{1, \ln(\rho), \rho^m, \rho^{-m}\}$$

$$G(\rho, \rho', \varphi, \varphi') = -\ln\left(\rho_>{}^2\right) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos\left(m\left(\varphi - \varphi'\right)\right)$$

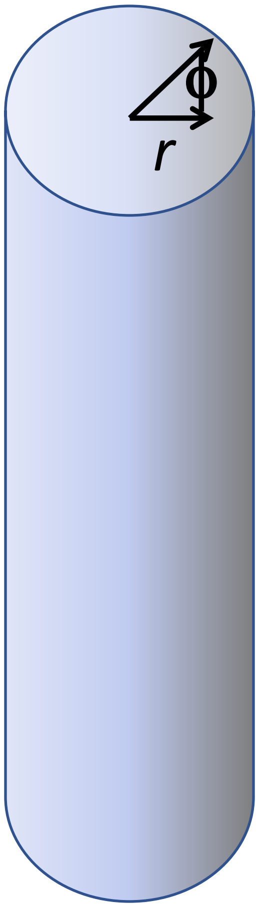
Note that, because we are using curvilinear coordinates, the Wronskian and the form of the delta function is modified.

More details given in **Jackson** Sec. 3.7 - 3.11.

Comments and details

Change notation

$\rho \Rightarrow r$

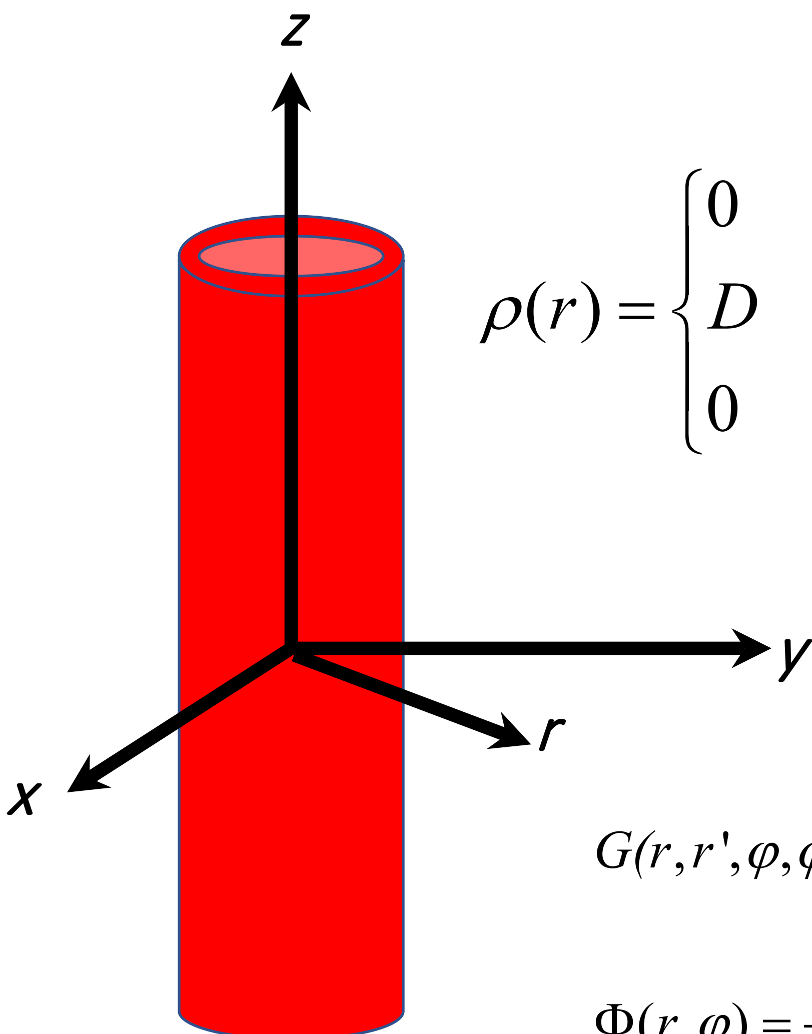



$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\;2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

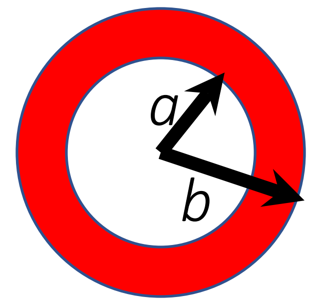
$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that in this case, we have assumed that the surface integral contributions are trivial.

# Example – uniform cylindrical shell:




Top view:




$$\rho(r) = \begin{cases} 0 & r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \Phi(r,\phi) = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi(r,\phi)}{\partial^2 \phi}$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>{}^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Boundary condition: $\lim_{r\to\infty}\left(\frac{\partial \Phi(r,\varphi)}{\partial r}\right) = 0$

# Question – Why only m=0 for this case?

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that $\int_0^{2\pi} d\varphi' \cos(m(\varphi-\varphi')) = 0$ for $m > 0$

So that $\Phi(r,\varphi) = \frac{2\pi}{4\pi\epsilon_0}\int_0^{\infty} r'dr'\left(-\ln\left(r_>^{\,2}\right)\right)\rho(r')$

# Some details

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

In our case: $\Phi(r,\varphi) = \frac{2\pi D}{4\pi\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>^2)\right) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>)\right)$

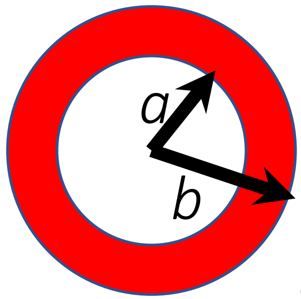
For $0 \le r < a$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r')\right)$

For $a \le r < b$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\left(\int_a^r r'dr'\left(-\ln(r)\right) + \int_r^b r'dr'\left(-\ln(r')\right)\right)$

For $r > b$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r)\right)$

Example continued -- *m=0* only --

Top view:



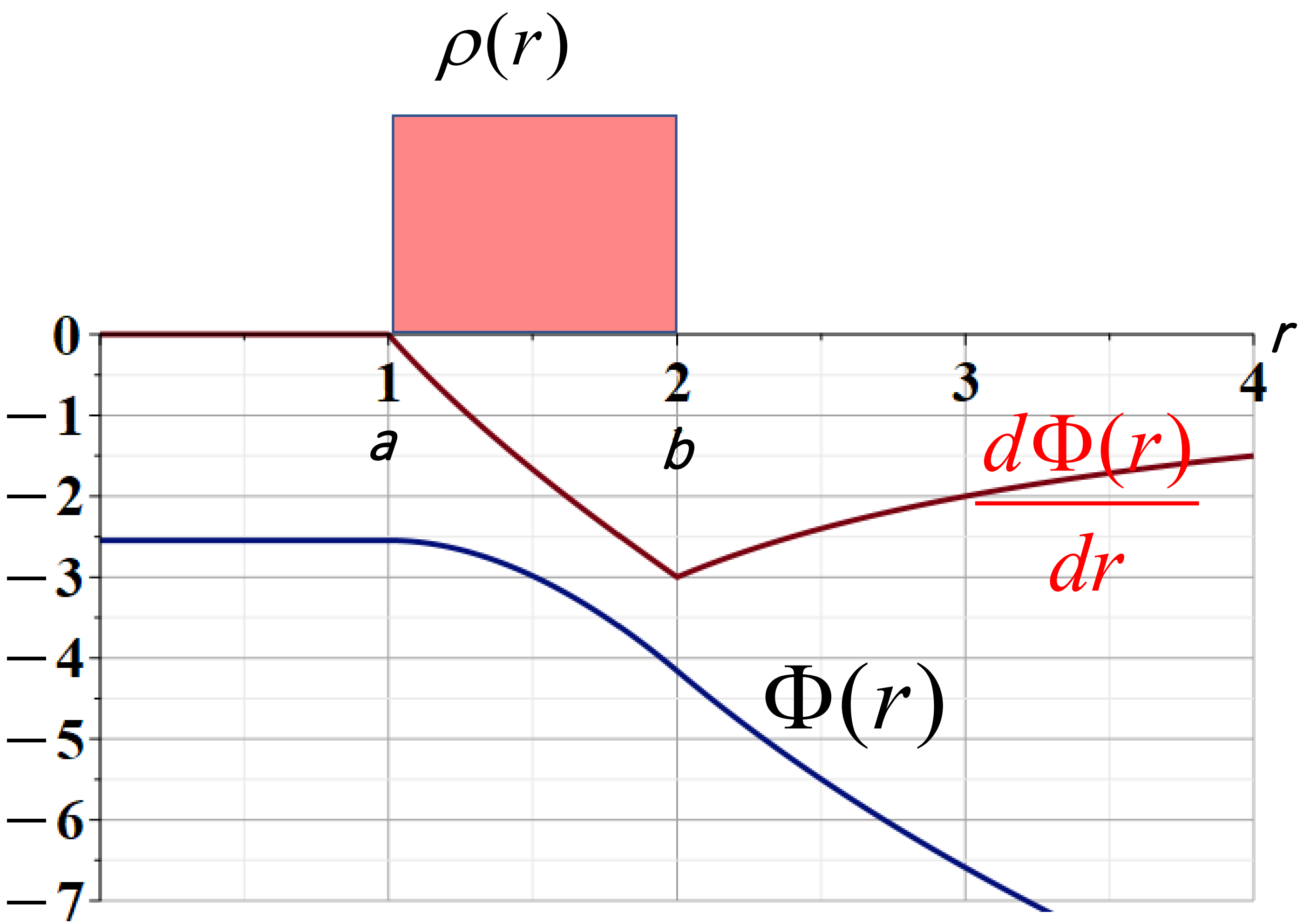

$$\rho(r)=\begin{cases}0 & 0<r<a\\ D & a\le r\le b\\ 0 & r>b\end{cases}$$

$$G(r,r',\varphi,\varphi')=-\ln\left(r_>^{\;2}\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m\cos\left(m\left(\varphi-\varphi'\right)\right)$$

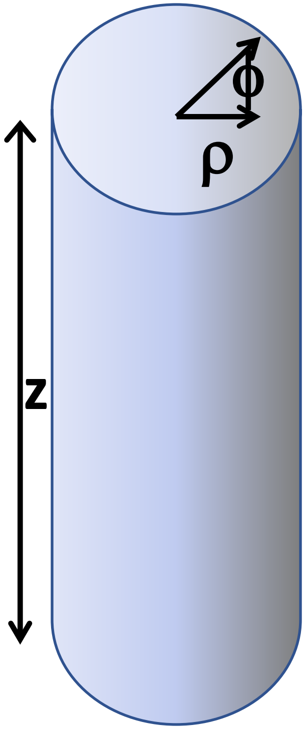
$$\Phi(r,\varphi)=\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}d\varphi'\int_0^{\infty}r'dr'G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

$$\Phi(r)=\begin{cases}\dfrac{D}{4\epsilon_0}\left(b^2-a^2-b^2\ln(b^2)+a^2\ln(a^2)\right) & 0<r<a\\[2ex] \dfrac{D}{4\epsilon_0}\left(b^2-r^2-b^2\ln(b^2)+a^2\ln(r^2)\right) & a\le r\le b\\[2ex] \dfrac{D}{4\epsilon_0}\left(a^2-b^2\right)\ln(r^2) & r>b\end{cases}$$

# Example continued --



# Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence

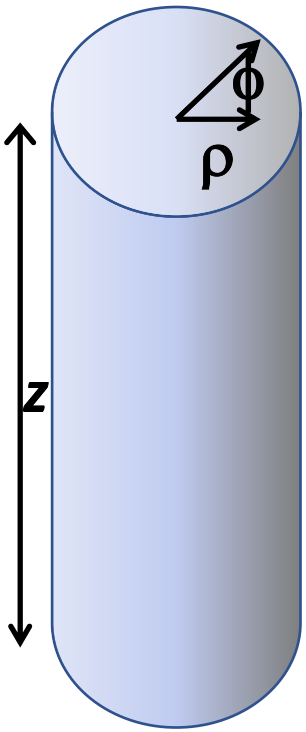



$$\text{Laplace equation:} \qquad \nabla^2\Phi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$

# Cylindrical geometry continued:




$$\text{Laplace equation}: \qquad \nabla^2\Phi = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

One possibility :

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

https://dlmf.nist.gov/

Digital Library of Mathematical Functions

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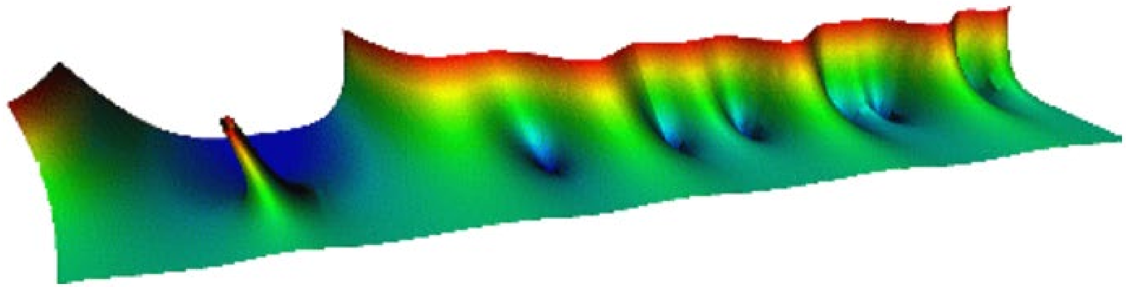


# NIST Digital Library of Mathematical Functions

***Project News***

2022-12-15 *DLMF Update; Version 1.1.8; MathML improvements*
2022-12-15 *Richard B. Paris, Associate Editor of the DLMF, dies at age 76*
2022-10-15 *DLMF Update; Version 1.1.7; Enhanced coverage of Lambert* $W$
2022-06-30 *DLMF Update; Version 1.1.6*
*More news*

# §10.3 Graphics

*Contents*



## §10.3(i) Real Order and Variable

For the modulus and phase functions $M_\nu(x)$, $\theta_\nu(x)$, $N_\nu(x)$, and $\phi_\nu(x)$ see §10.18.

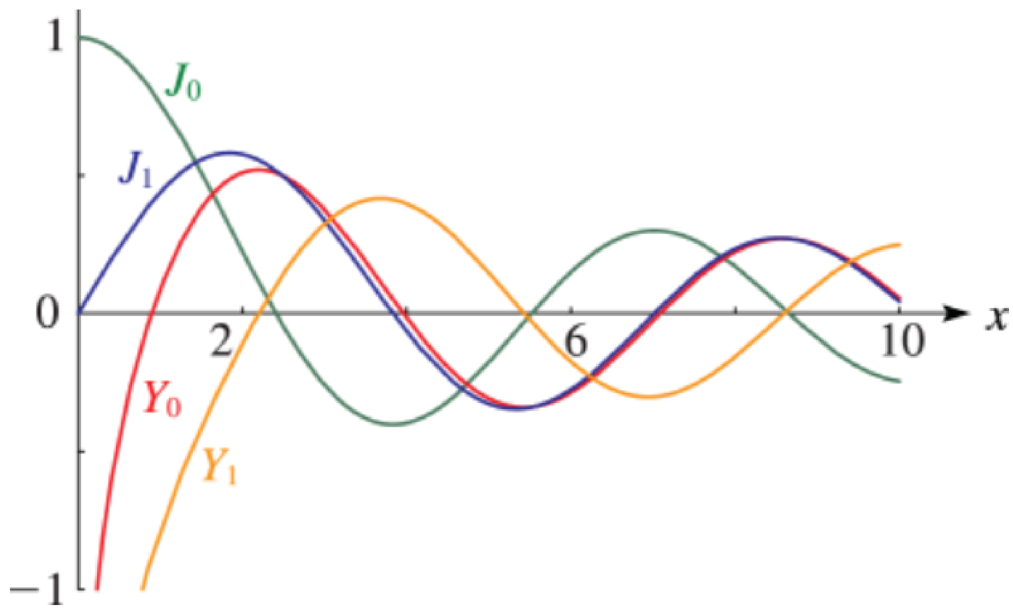
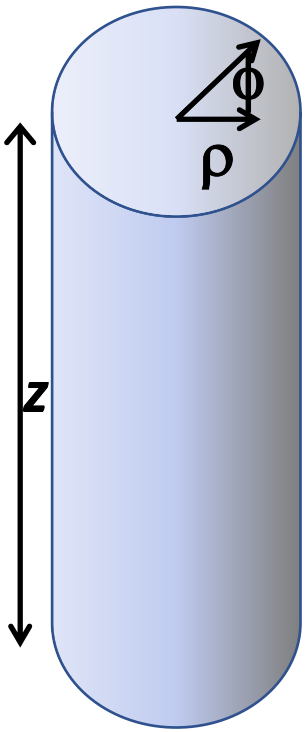



Figure 10.3.1: $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$,

# Cylindrical geometry continued:




Laplace equation: $\nabla^2\Phi = 0$

$$\Phi(\rho,\varphi,z) = R(\rho)Q(\varphi)Z(z)$$

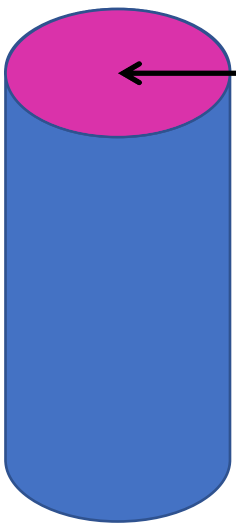
Another possibility:

$$\frac{d^2Z}{dz^2} + k^2 Z = 0 \qquad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow I_m(k\rho), K_m(k\rho)$$

# Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*



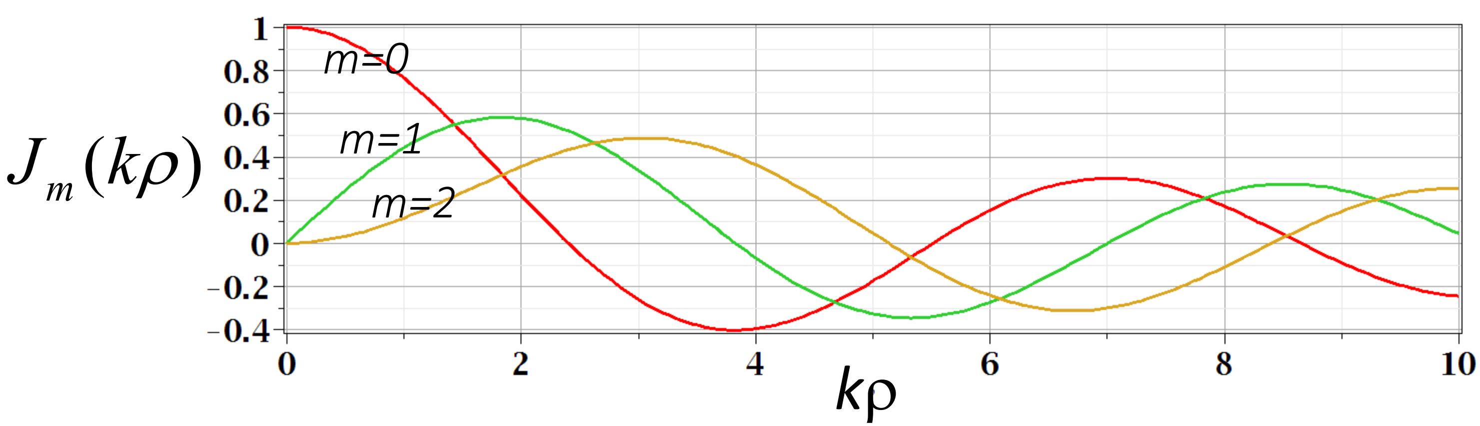
$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

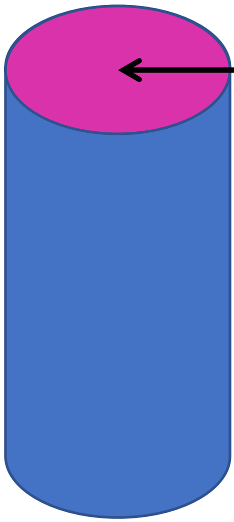
Well behaved at ***ρ=0***

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\phi+\alpha_{mn}\right)$$

$$\text{where} \quad J_m(k_{mn}a)=0$$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at $z=L$



$$\Phi(\rho,\varphi,z=L)=V(\rho,\varphi)$$

$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\varphi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(\rho,\varphi)$ is an even function of $\varphi$ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\varphi\cos\left(m\varphi\right)\int_0^a \rho d\rho J_m(k_{mn}\rho)V(\rho,\varphi)}{\sinh(k_{mn}L)\int_0^{2\pi} d\varphi\cos^2\left(m\varphi\right)\int_0^a \rho d\rho J_m^{\,2}(k_{mn}\rho)}$$

# §10.26 Graphics

***Contents***



## §10.26(i) Real Order and Variable

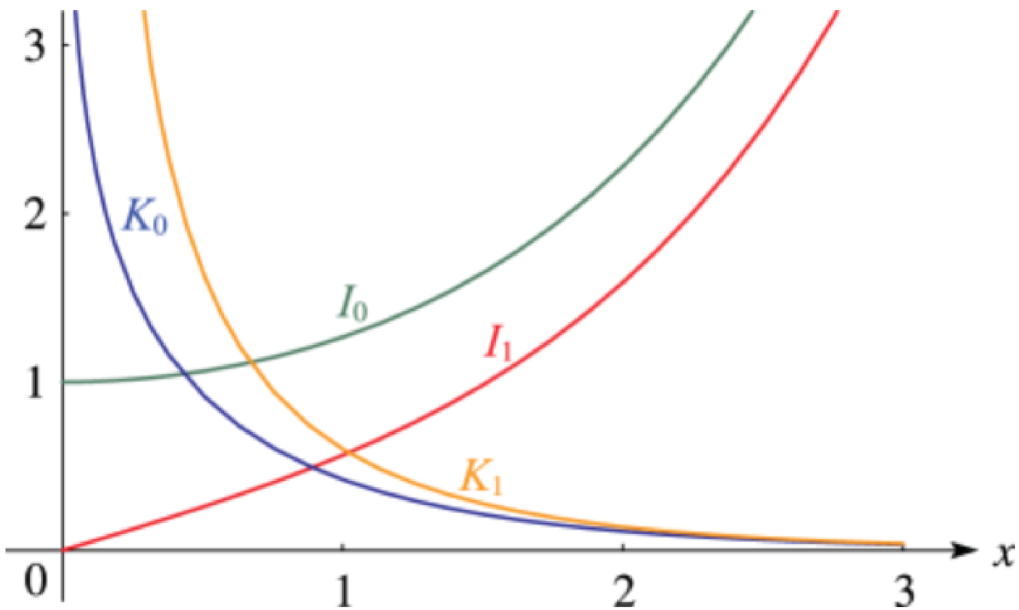
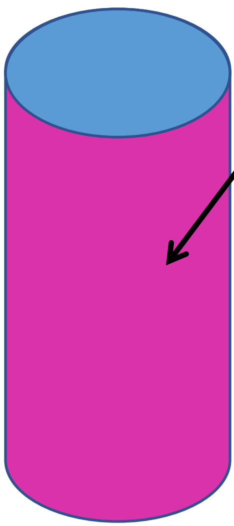



Figure 10.26.1: $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$,

# Solutions of Laplace equation inside cylindrical shape
## Example with non-trivial boundary value at *ρ=a*

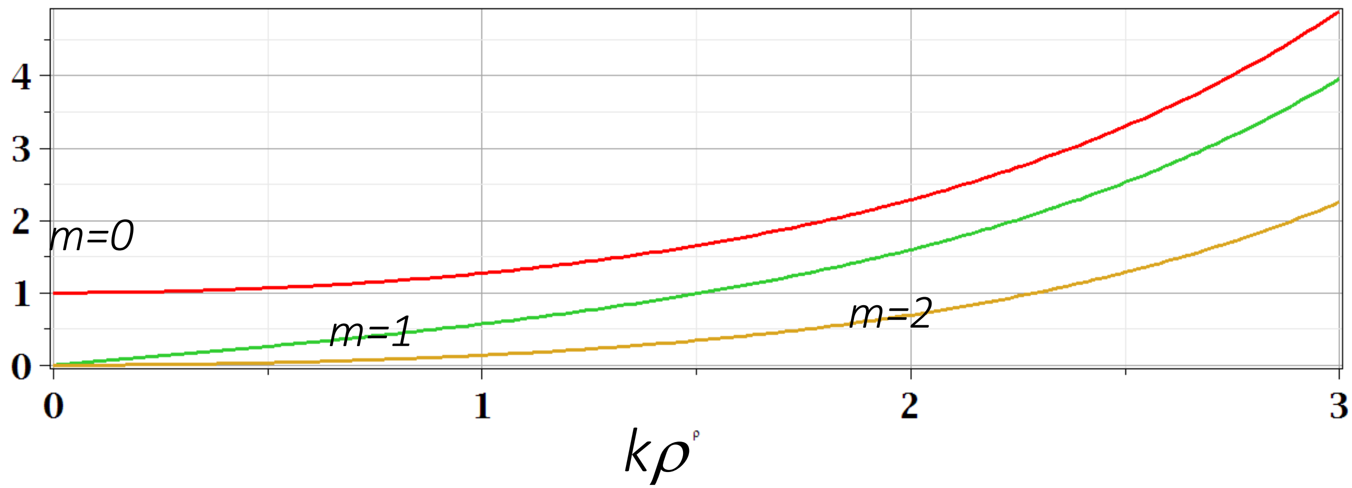


$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$

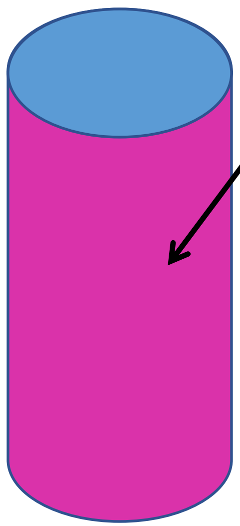
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi+\alpha_{mn}\right)$$

$I_m(k\rho)$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at $\rho$=$a$



$$\Phi(\rho=a,\varphi,z)=V(\varphi,z)$$

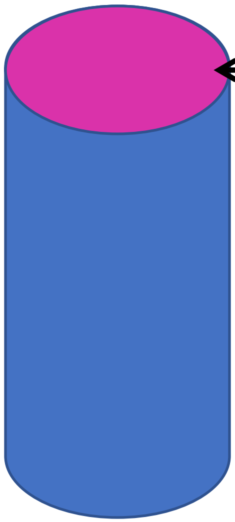
$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\varphi,z)=\sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(z,\varphi)$ is an even function of $\varphi$ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\varphi\cos\left(m\varphi\right)\int_0^L dz\sin\left(\frac{n\pi z}{L}\right)V(z,\varphi)}{I_m\left(\frac{n\pi a}{L}\right)\int_0^{2\pi} d\varphi\cos^2\left(m\varphi\right)\int_0^L dz\sin^2\left(\frac{n\pi z}{L}\right)}$$

Green's function for Dirchelet boundary value inside cylinder:



$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho=a,\phi,z)=0,\ \ \Phi(\rho,\phi,z=0)=0$$

Expansion in terms of Bessel function zeros : $J_m(k_{mn}a)=0$

$$G(\rho,\rho',\phi,\phi',z,z')=$$

$$\frac{8\pi}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}J_m(k_{mn}\rho)J_m(k_{mn}\rho')\sinh(k_{mn}z_<)\sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2\sinh(k_{mn}L)}$$

$$\Phi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_0}\int_V d\phi'\,\rho'\,d\rho'\,dz'\,G(\rho,\rho',\phi,\phi',z,z')\rho(\rho',\phi',z')$$

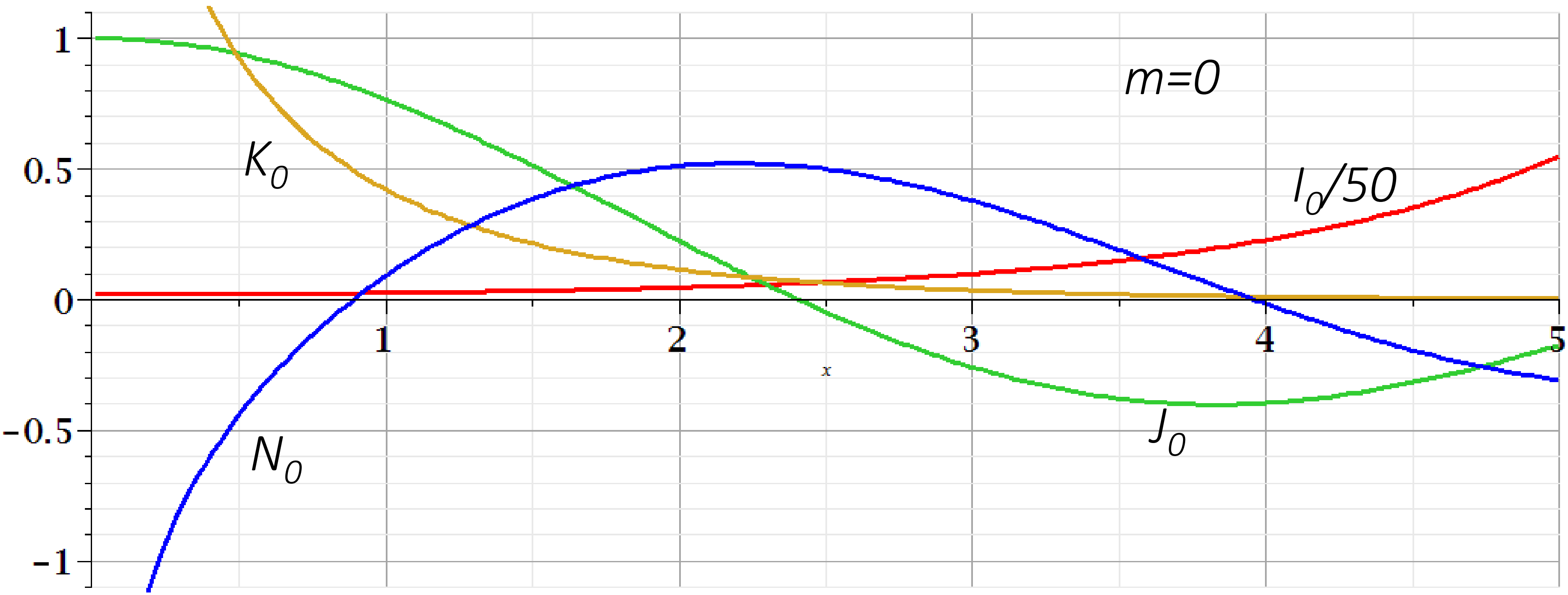
$$+\frac{1}{4\pi}\int_{S;z'=L} d\phi'\,\rho'\,d\rho'\left.\frac{\partial G(\rho,\rho',\phi,\phi',z,z')}{\partial z'}\right|_{z'=L}V(\rho',\phi')$$

# Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u), N_m(u), H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u), K_m(u)$$

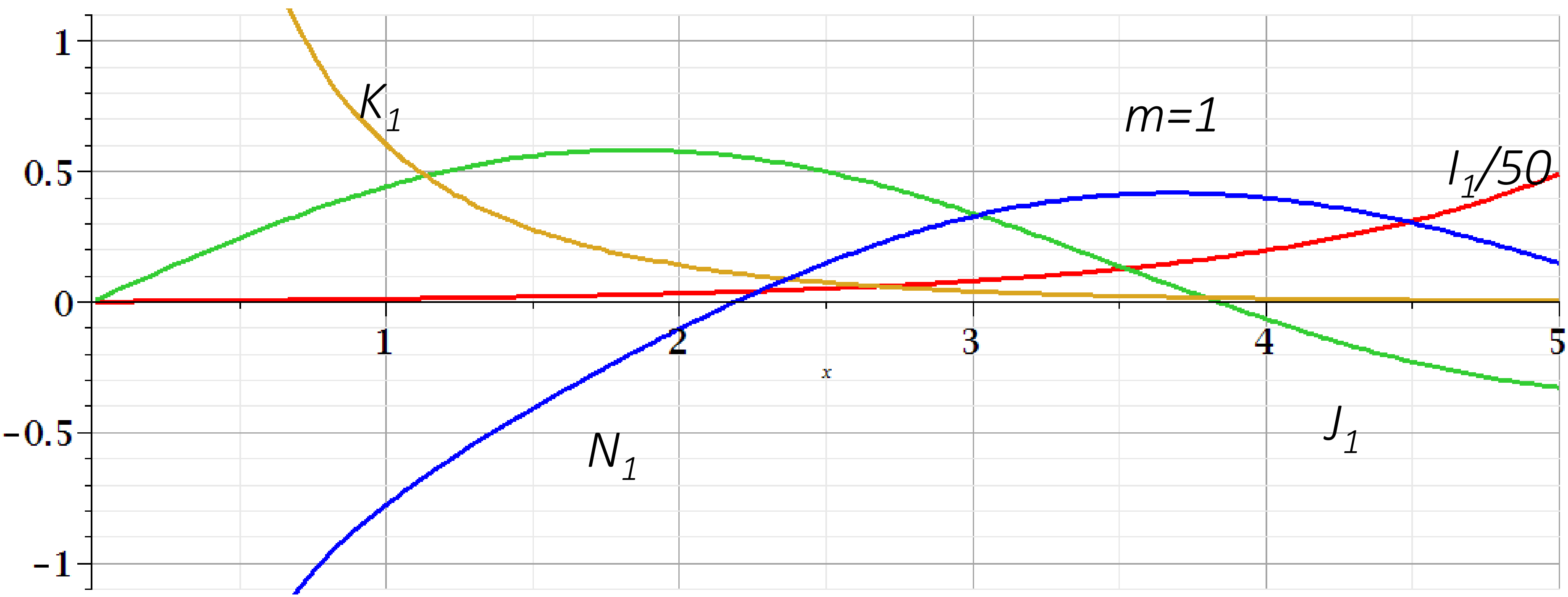


# Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$



Some useful identities involving cylindrical Bessel functions from **Jackson** Sec. 3.7

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(1-\frac{m^2}{u^2}\right)\right)J_m(u)=0 \qquad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: $x_{mn}$; $\quad J_m(x_{mn})=0$

$$\int_0^a \rho\, d\rho\, J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn'}\rho}{a}\right)=\frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2\delta_{nn'}$$