PHY 712 Electrodynamics
11-11:50 AM in Olin 103

Class notes for Lecture 4:

Reading: Chapter 1 in JDJ

1. Review of electrostatics with one-dimensional examples
2. Poisson and Laplace Equations
3. Green's Theorem and its use in electrostatics

Physics Colloquium

Thursday

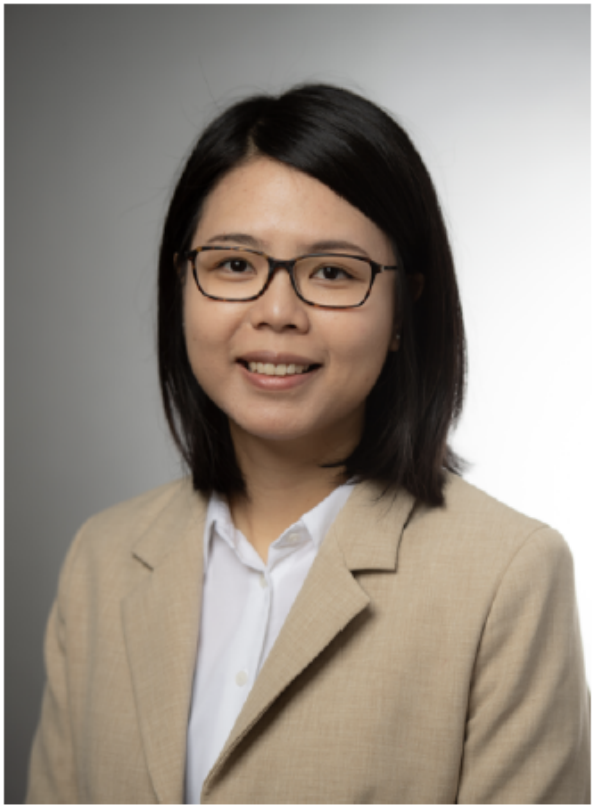
4 PM Olin 101

January 25th, 2024

Exploring emergent quantum phases in two-dimensional flat band systems

Quantum phases such as superconductivity and ferromagnetism are among the most important topics in condensed matter physics research. Recently, a family of two-dimensional flat band systems, including magic-angle twisted graphene, uncovered an abundance of symmetry breaking and novel quantum phases.

In this talk, I will introduce the recent advances in these materials and give two examples of how we engineered and revealed new quantum phases of matter in twisted graphene. These include an orbital ferromagnetic state induced by spin-orbit coupling and a zero-field superconducting diode effect. In the last part of the talk, I will present our discovery of a new type of Coulomb-driven rotational symmetry breaking state in the moiré-less bilayer graphene. These examples establish the two-dimensional flat band systems as a versatile platform with multiple tuning knobs, where new physics emerges from the interplay between various quantum phases.



Jiang-Xiazi Lin, Ph.D.
Brown University
Physics Department

PHY 712 Electrodynamics

MWF 10-10:50 AM	**Olin 103**	Webpage: **http://www.wfu.edu/~natalie/s24phy712/**

Instructor: Natalie Holzwarth	**Office:**300 OPL	**e-mail:**natalie@wfu.edu

Course schedule for Spring 2024

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/17/2024	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/19/2024
2	Fri: 01/19/2024	Chap. 1	Electrostatic energy calculations	#2	01/29/2024
3	Mon: 01/22/2024	Chap. 1	Electrostatic energy calculations	#3	01/29/2024
4	Wed: 01/24/2024	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/29/2024
5	Fri: 01/26/2024	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/29/2024

PHY 712 -- Assignment #4

Assigned: 1/24/2024 Due: 1/29/2024

Continue reading Chap. 1 in **Jackson**.

1. Using the Ewald summation methods developed in class, find the electrostatic interaction energy of a NaCl lattice having a cubic lattice constant a. Check that your result does not depend of the Ewald parameter η. You are welcome to copy (and modify) the maple file used in class. A FORTRAN code is also available upon request.

PHY 712 – Problem Set #5

Assigned: 01/26/2024 Due: 01/29/2024

Continue reading Chaper 1 & 2 in **Jackson**

1. Consider a one-dimensional charge distribution of the form:

$$\rho(x) = \begin{cases} 0 & \text{for} \quad x < -a \\ \rho_0 \sin(\pi x/a) & \text{for} \quad -a \leq x \leq a \\ 0 & \text{for} \quad x > a, \end{cases}$$

where ρ_0 and a are constants.

(a) Solve the Poisson equation for the electrostatic potential $\Phi(x)$ with the boundary conditions $\Phi(-a) = 0$ and $\frac{d\Phi}{dx}(a) = 0$.

(b) Find the corresponding electrostatic field $E(x)$.

(c) Plot $\Phi(x)$ and $E(x)$.

(d) Discuss your results in terms of possible electronic devices.

Comment on HW #2

Recall that we "proved" the following identity:

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r}) = -4\pi\ \delta(x)\delta(y)\delta(z)$$

Three dimensional δ function

Note that Jackson often writes $\mathbf{x}$ to denote a vector position, while we use $\mathbf{r}$ for this purpose.

In Cartesian coordinates: $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Your questions –

From Joe: I was working through Chapter 1 of Jackson and got stuck at the beginning of 1.6 (pg32) where he mentions the "dipole-layer distribution of strength D(x)". I don't think I've seen this quantity before, and the way he defines it seems a bit circular, so I'm guessing he assumes we know what it is. It'd be helpful if you could talk about what it is and how he uses it in that section.

Comment: From the previous slide, it is apparent that the transition from point charges q to continuous distributions $\rho(\mathbf{r})$ is non-trivial and singularities can arise. We will discuss dipolar fields in Chap. 4 where these ideas may be presented more clearly. (I personally find the notion of $D(\mathbf{r})$ rather confusing.)

Poisson and Laplace Equations

We are concerned with finding solutions to the Poisson equation:

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

and the Laplace equation:

$$\nabla^2 \Phi_L(\mathbf{r}) = 0$$

The Laplace equation is the “homogeneous” version of the Poisson equation. The Green's theorem allows us to determine the electrostatic potential from volume and surface integrals:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Poisson equation -- continued

Note that we have previously shown that the differential and integral forms of Coulomb's law is given by:

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \quad \text{and} \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Generalization of analysis for non-trivial boundary conditions:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

General comments on Green's theorem

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

This general form can be used in 1, 2, or 3 dimensions. In general, the Green's function must be constructed to satisfy the appropriate (Dirichlet or Neumann) boundary conditions. Alternatively, or in addition, boundary conditions can be adjusted using the fact that for any solution to the Poisson equation, $\Phi_P(\mathbf{r})$ other solutions may be generated by use of solutions of the Laplace equation

$$\Phi(\mathbf{r}) = \Phi_P(\mathbf{r}) + C\Phi_L(\mathbf{r}), \text{for any constant } C.$$

Physics Today **56**(12), 41 (2003); https://doi.org/10.1063/1.1650227

The Green of Green Functions

In 1828, an English miller from Nottingham published a mathematical essay that generated little response. George Green's analysis, however, has since found applications in areas ranging from classical electrostatics to modern quantum field theory.

Lawrie Challis and Fred Sheard

Nottingham, an attractive and thriving town in the English Midlands, is famous for its association with Robin Hood, whose statue stands in the shadow of the castle wall. The Sheriff of Nottingham still has a special role in the city government although happily no longer strikes terror into the hearts of the good citizens.

Recently a new attraction, a windmill, has appeared on the Nottingham skyline (see figure 1). The sails turn on windy days and the adjoining mill shop sells packets of stone ground flour but also, more surprisingly, tracts on mathematical physics. The connection between the flour and the physics is part of the mill's unique character and is explained by a plaque once attached to the side of the mill tower that said,

HERE LIVED AND LABOURED
GEORGE GREEN
MATHEMATICIAN
B.1793–D.1841.

his family built a house next to the mill, Green spent most of his days and many of his nights working and indeed living in the mill. When he was 31, Jane Smith bore him a daughter. They had seven children in all but never married. It was said that Green's father felt that Jane was not a suitable wife for the son of a prosperous tradesman and landowner and threatened to disinherit him.

Little is known about Green's life from 1802 until 1823. In particular, it is not known whether he received any help in his mathematical development or if he was entirely self-taught. He may have received help from John Toplis, a fellow of Queens' College in the University of Cambridge and headmaster of the Nottingham Grammar School. Toplis's translation of Pierre-Simon Laplace's book *Mécanique Céleste*, published in Nottingham in 1814, seems a likely source of Green's interest in potential theory. The work was unusual in Britain at that time inasmuch as Toplis used Gottfried Leibniz's more convenient notation for differentials rather than Isaac Newton's. Because Green adapted the Leibniz notation, it seems plausible that Green was influenced by Toplis, but there is no evidence that Toplis acted in any way as his tutor.

In 1823, Green joined the Nottingham Subscription Library, the center of intellectual activity in the town. The library was situated in Bromley House (see figure 2). Library membership provided Green with encouragement

"Derivation" of Green's Theorem

Poisson equation: $\nabla^2 \Phi(\mathbf{r}) = -\dfrac{\rho(\mathbf{r})}{\varepsilon_0}$

Green's relation: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}').$

Divergence theorm: $\int_V d^3r\ \nabla\cdot\mathbf{A} = \oint_S d^2r\ \mathbf{A}\cdot\hat{\mathbf{r}}$

Let $\mathbf{A} = f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})$

$$\int_V d^3r\ \nabla\cdot\left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right) = \oint_S d^2r\ \left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right)\cdot\hat{\mathbf{r}}$$

$$\Downarrow$$

$$\int_V d^3r\ \left(f(\mathbf{r})\nabla^2 g(\mathbf{r}) - g(\mathbf{r})\nabla^2 f(\mathbf{r})\right)$$

"Derivation" of Green's Theorem

Poisson equation: $\nabla^2\Phi(\mathbf{r}) = -\dfrac{\rho(\mathbf{r})}{\varepsilon_0}$

Green's relation: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}').$

$$\int_V d^3r\,\left(f(\mathbf{r})\nabla^2 g(\mathbf{r}) - g(\mathbf{r})\nabla^2 f(\mathbf{r})\right) = \oint_S d^2r\,\left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right)\cdot\hat{\mathbf{r}}$$

$$f(\mathbf{r}) \leftrightarrow \Phi(\mathbf{r}) \qquad\qquad g(\mathbf{r}) = G(\mathbf{r},\mathbf{r}')$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Example of charge density and potential varying in one dimension

Consider the following one dimensional charge distribution:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0},$$

with the boundary condition $\Phi(-\infty) = 0$. and $\frac{d\Phi}{dx}(\infty) = 0$

Electrostatic field solution

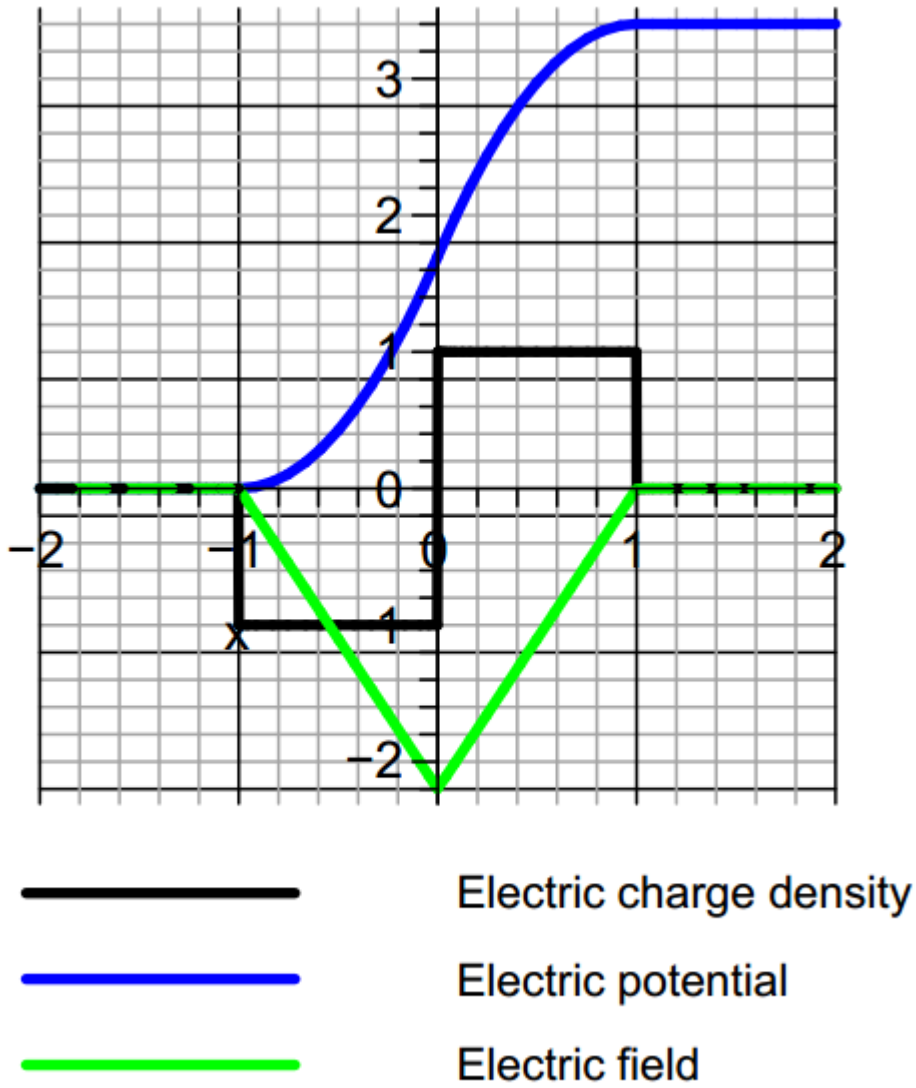
The solution to the Poisson equation is given by:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases}.$$

Laplace
Poisson
Poisson
Laplace

The electrostatic field is given by:

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -\frac{\rho_0}{\varepsilon_0}(x+a) & \text{for } -a < x < 0 \\ \frac{\rho_0}{\varepsilon_0}(x-a) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}.$$


3
2
1
0
−2
−1
0
1
2
x
−1
−2
Electric charge density
Electric potential
Electric field

Comment about the example and solution

This particular example is one that is used to model semiconductor junctions where the charge density is controlled by introducing charged impurities near the junction.

The solution of the Poisson equation for this case can be determined by piecewise solution within each of the four regions. Alternatively, from Green's theorem in one-dimension, one can use the Green's function

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx' \quad \text{where} \quad G(x,x') = 4\pi x_<$$

$x_<$ should be take as the smaller of x and x'.

Notes on the one-dimensional Green's function

The Green's function for the one-dimensional Poisson equation can be defined as a solution to the equation: $\nabla^2 G(x,x') \equiv \frac{\partial^2}{\partial x^2} G(x,x') = -4\pi\delta(x-x')$

Here the factor of 4π is not really necessary, but ensures consistency with your text's treatment of the 3-dimensional case. The meaning of this expression is that x' is held fixed while taking the derivative with respect to x.

Construction of a Green's function in one dimension

Consider two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0$$

where $i = 1$ or 2. Let

$$G(x, x') = \frac{4\pi}{W} \phi_1(x_<) \phi_2(x_>).$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

W is defined as the "Wronskian":

$$W \equiv \frac{d\phi_1(x)}{dx} \phi_2(x) - \phi_1(x) \frac{d\phi_2(x)}{dx}.$$

Summary

$$\nabla^2 G(x,x') = -4\pi\delta(x-x')$$

$$G(x,x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>)$$

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}$$

$$\lim_{\epsilon\to 0}\left(\left.\frac{dG(x,x')}{dx}\right|_{x=x'+\epsilon} - \left.\frac{dG(x,x')}{dx}\right|_{x=x'-\epsilon}\right) = -4\pi$$

One dimensional Green's function in practice

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\int_{-\infty}^{x} G(x,x')\rho(x')dx' + \int_{x}^{\infty} G(x,x')\rho(x')dx'\right\}$$

For the one-dimensional Poisson equation, we can construct the Green's function by choosing $\phi_1(x) = x$ and $\phi_2(x) = 1; W = 1$:

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty} \rho(x')dx'\right\}.$$

$$G(x,x') = 4\pi x_<$$

This expression gives the same result as previously obtained for the example $\rho(x)$ and more generally is appropriate for any neutral charge distribution.

Question -- How do we know which one of x and x' is the $x_<$ term?

$$G(x,x') = 4\pi x_<$$

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty}\rho(x')dx'\right\}.$$

x'<x x'>x

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}.$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x').$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x').$$

Orthogonal function expansions –continued

Therefore, if

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x),$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write t
Green's functions as

$$G(x, x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}.$$

Example

For example, consider the example discussed earlier in the interval $-a \leq x \leq a$ with

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{24}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. For this purpose, we may choose

$$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right). \tag{25}$$

The Green's function for this case as:

$$G(x, x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right) \sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}. \tag{26}$$

Example – continued

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right).$$

Constant shift to allow $\Phi(0) = 0$.