PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Plan for Lecture 7:

Continue reading Chapters 2 & 3

1. **Methods of images -- planes, spheres**
2. **Solution of Poisson equation in for other geometries -- cylindrical**

Physics Colloquium

Thursday

4 PM in Olin 101

February 1st, 2024

Principles for Modeling Physically-Relevant Quantum Systems of Many Particles with Computers

Systems of many strongly-interacting particles are key to explaining many phenomena: from the magnets in our everyday experience to more exotic phenomena such as superconductivity, quantum hall physics, and emergent gauge symmetries. However, the necessary quantum mechanical treatments of these systems involve Hilbert spaces that grow exponentially with the system volume, putting naive calculations out of reach. In this talk, I will motivate three useful principles for building models that are both relevant to nature and amenable to computer simulation in polynomial time: locality, symmetry, and small ultralocal Hilbert spaces. With classical computers we will see how locality as a guiding principle allows us to study antiferromagnetism and superconductivity with relativity, and how symmetry as a guiding principle allows us to detect conformal field theories using the quantum hall effect. Finally for quantum computers we make use of small ultralocal Hilbert spaces as a guiding principle, and then design and study resource-efficient qubit-friendly models that realize continuous gauge symmetries found in fundamental physics.

Emilie Huffman, Ph.D.
Perimeter Institute for Theoretical Physics

Course schedule for Spring 2024

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/17/2024	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/19/2024
2	Fri: 01/19/2024	Chap. 1	Electrostatic energy calculations	#2	01/29/2024
3	Mon: 01/22/2024	Chap. 1 & 3	Electrostatic energy calculations	#3	01/29/2024
4	Wed: 01/24/2024	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/29/2024
5	Fri: 01/26/2024	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/29/2024
6	Mon: 01/29/2024	Chap. 1 - 3	Brief introduction to numerical methods	#6	02/05/2024
7	Wed: 01/31/2024	Chap. 2 & 3	Image charge constructions	#7	02/05/2024

PHY 712 -- Assignment #7

Assigned: 1/31/2024   Due: 2/05/2024

Continue reading Chap. 2 in **Jackson**.

1. Eq. 2.5 on page 59 of **Jackson** was derived as the surface change density on a sphere of radius *a* due to a charge *q* placed at a radius $y > a$ outside the sphere. Determine the total surface charge on the sphere's outer surface.
2. Now consider the same system except assume $y < a$ representing the charge q being placed inside the sphere. What is the surface charge density and the total charge on the inner sphere surface in this case?

Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

1. Direct solution of differential equation
2. Solution by means of an integral equation; Green's function techniques
3. Orthogonal function expansions
4. Numerical methods (finite differences and finite element methods)
5. Method of images ← **today**

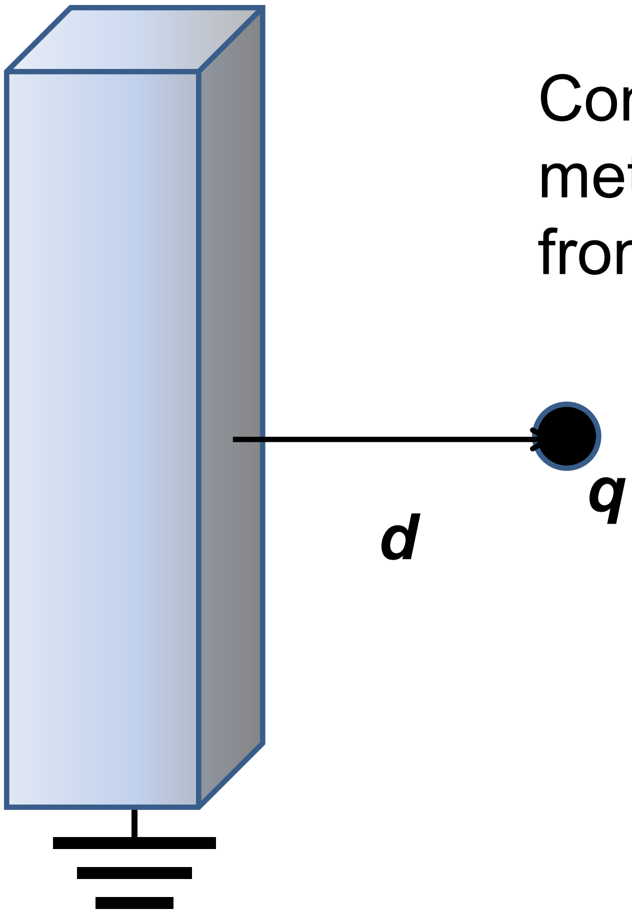
Depends on geometry; Cartesian, spherical, and cylindrical cases considered in textbook

Method of images

Clever trick for specialized geometries:

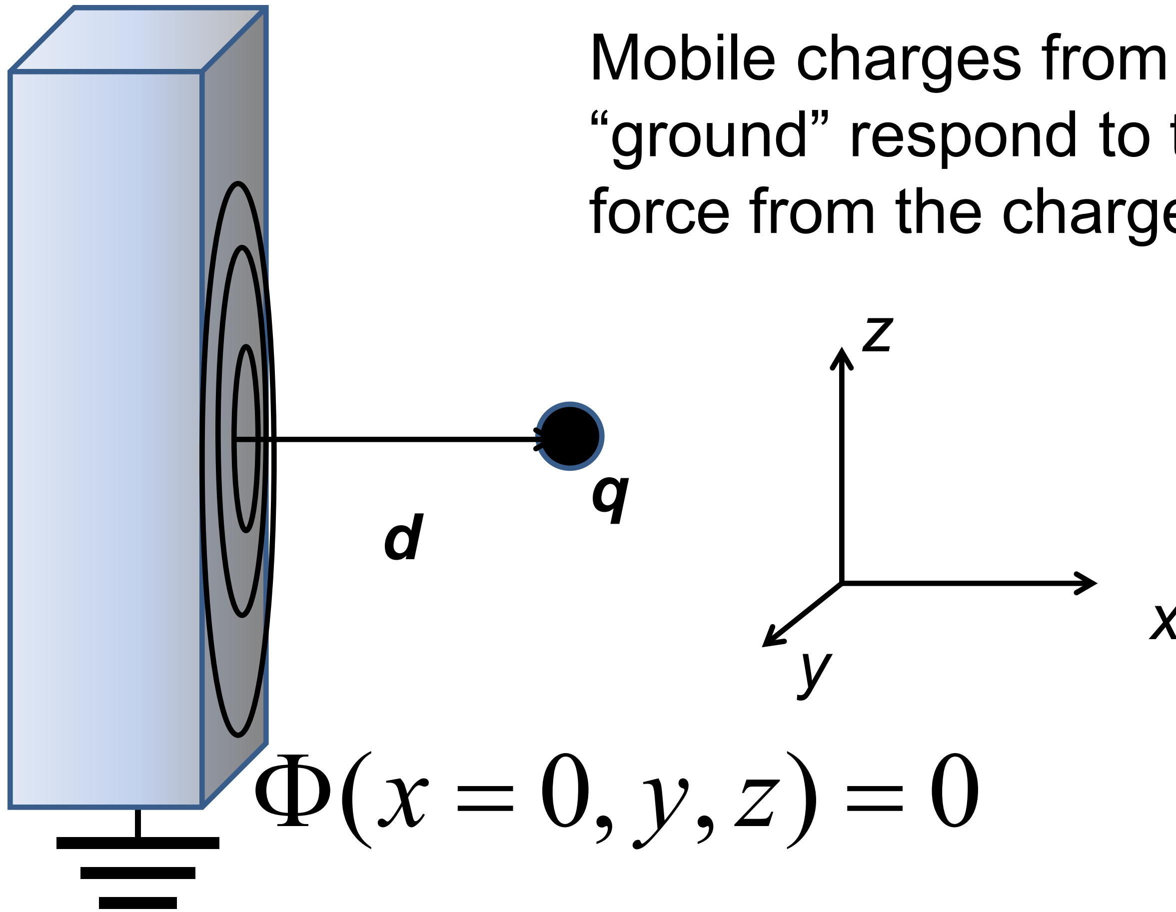
- Flat plane (surface)
- Sphere

Planar case:



Consider a grounded metal sheet, a distance ***d*** from a point charge ***q***.

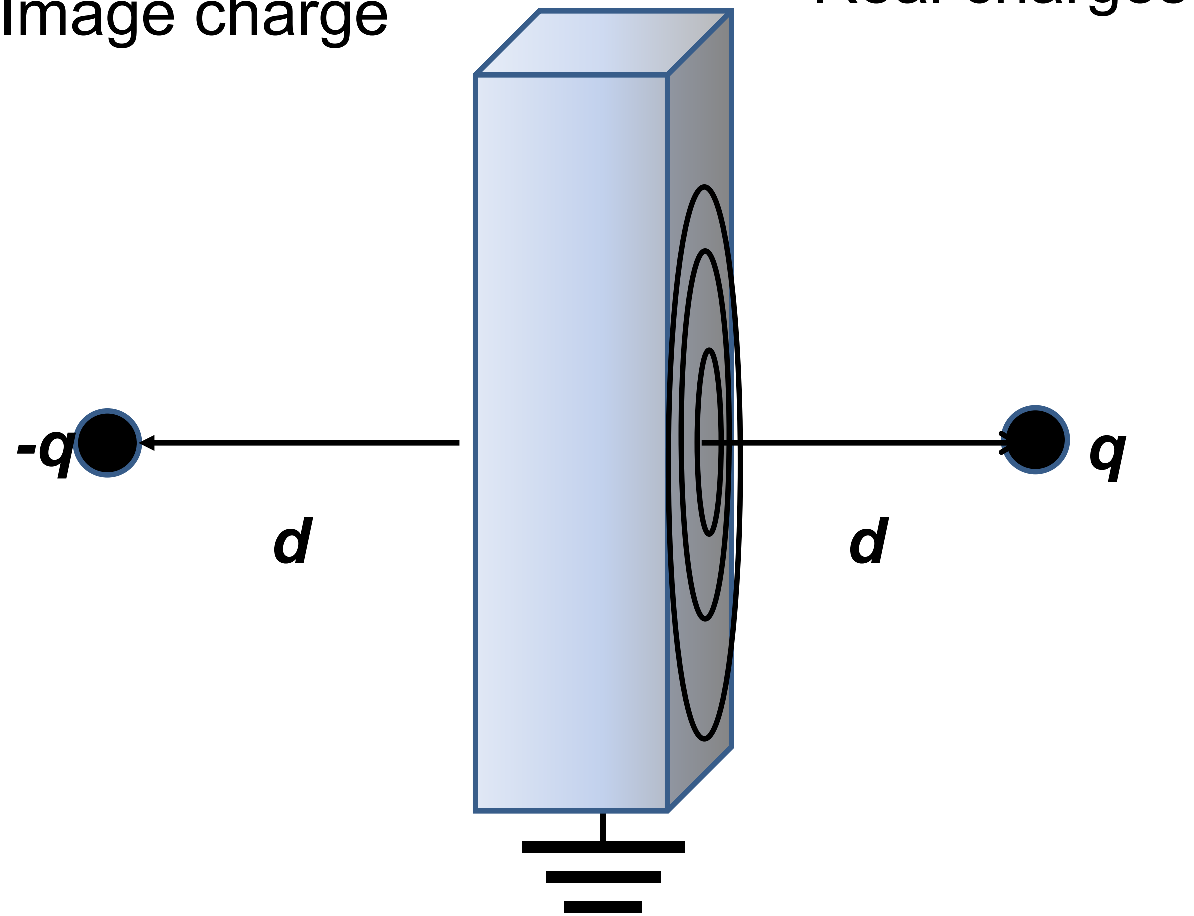
A grounded metal sheet, a distance ***d*** from a point charge ***q***.



Mobile charges from the "ground" respond to the force from the charge ***q***.

$$\Phi(x=0, y, z) = 0$$

Fiction | Truth



A grounded metal sheet, a distance ***d*** from a point charge ***q***.

$$\nabla^2\Phi = -\frac{q}{\varepsilon_0}\delta^3(\mathbf{r} - d\hat{\mathbf{x}})$$

$$\Phi(x=0, y, z) = 0$$

Trick for $x \geq 0$:

$$\Phi(x \geq 0, y, z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r} - d\hat{\mathbf{x}}|} - \frac{q}{|\mathbf{r} + d\hat{\mathbf{x}}|}\right)$$

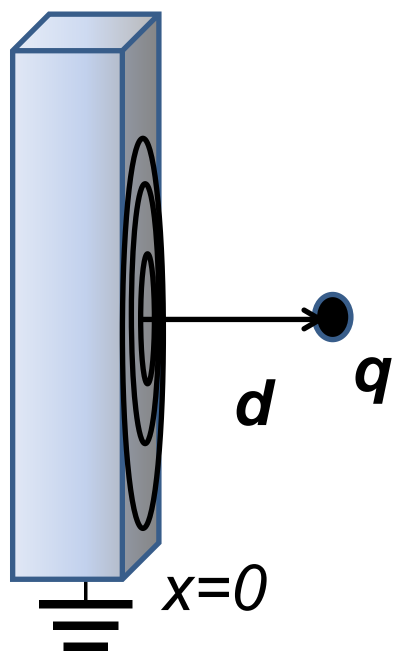
Surface charge density:

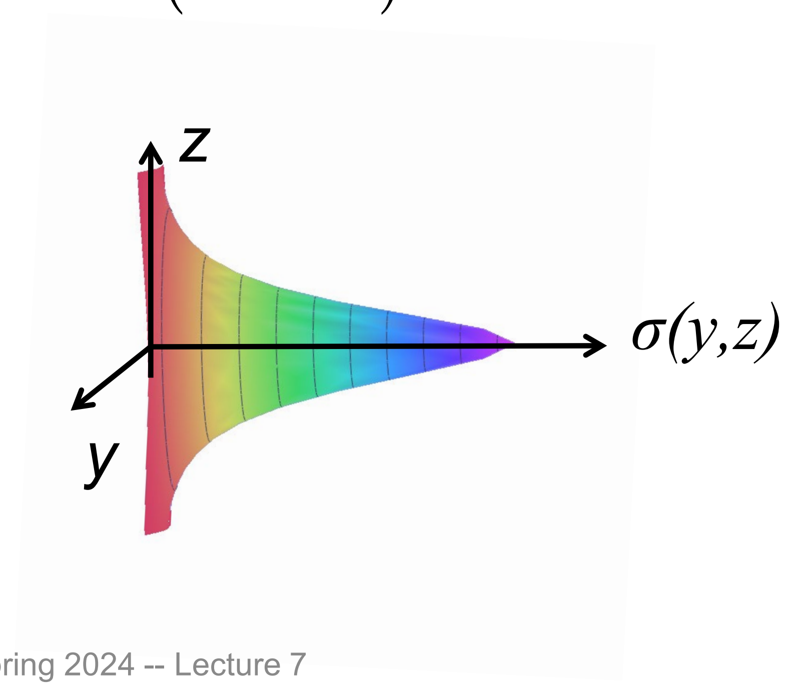
$$\sigma(y,z) = \varepsilon_0 E(0, y, z) = -\varepsilon_0 \frac{d\Phi(0, y, z)}{dx} = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2 + y^2 + z^2\right)^{3/2}}\right)$$

A grounded metal sheet, a distance *d* from a point charge *q*.

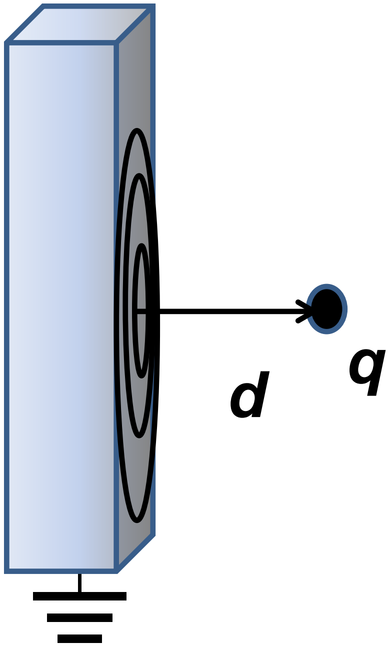
$$\text{Surface charge density:} \quad \sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

$$\text{Note:} \iint dydz \;\; \sigma(y,z) = -\frac{q2d}{4\pi} 2\pi \int_0^\infty \frac{udu}{\left(d^2+u^2\right)^{3/2}} = -q$$





A grounded metal sheet, a distance ***d*** from a point charge ***q***.



Surface charge density :

$$\sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

Force between charge and sheet :

$$\mathbf{F} = \frac{-q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0 (2d)^2}$$

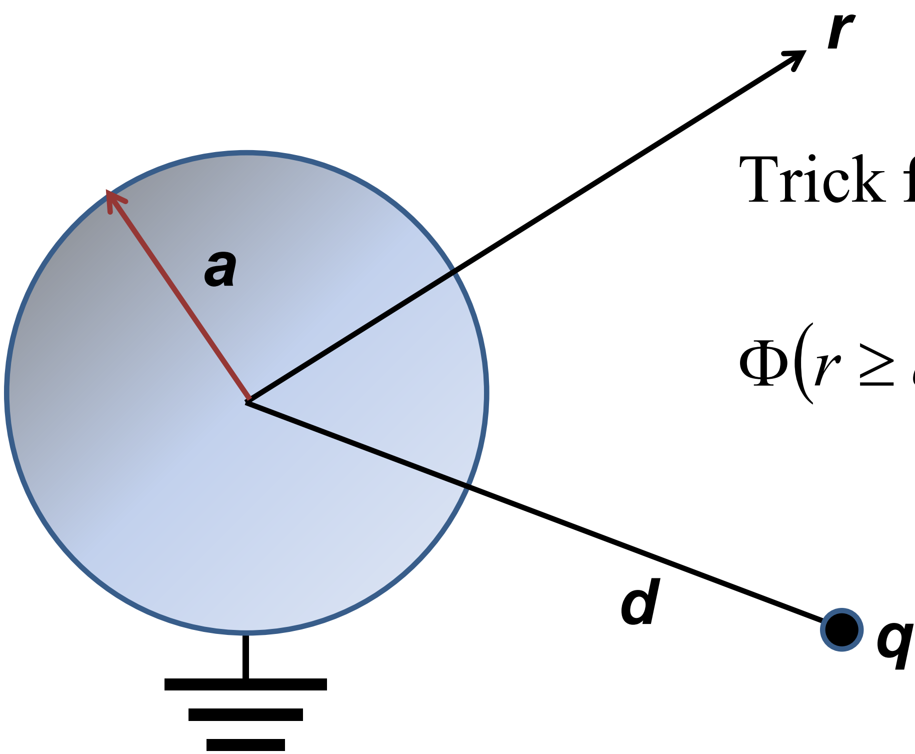
Image potential between charge and sheet at distance x :

$$V(x) = \frac{-q^2}{4\pi\varepsilon_0 (4x)}$$

Note: this effect can be observed in photoemission experiments.

Image charge methods can be used in some other geometries --

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Trick for $r \geq a$:

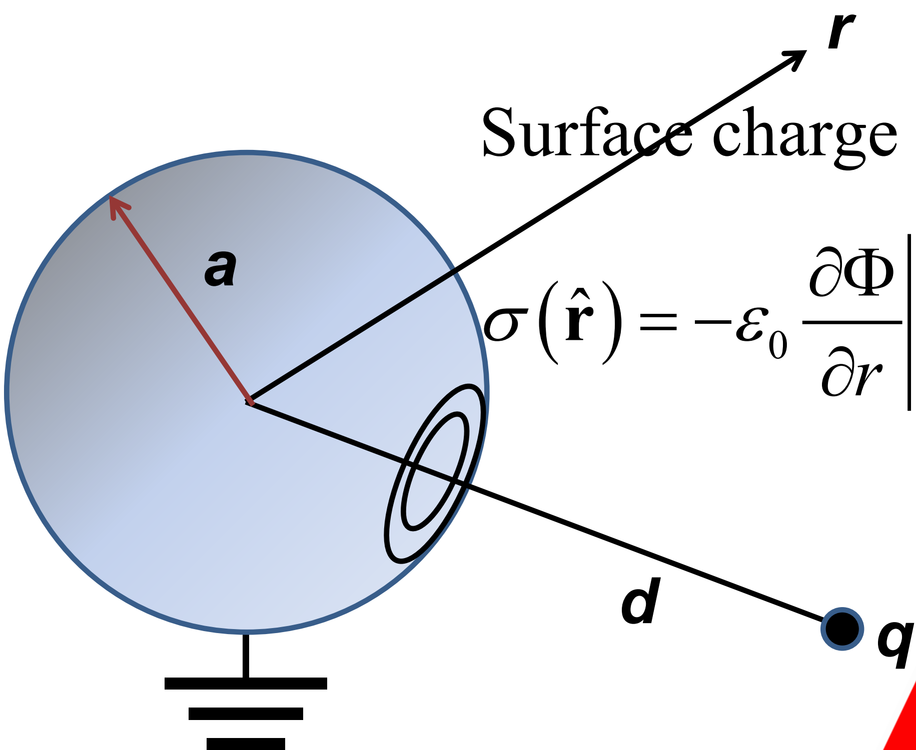
$$\Phi(r \geq a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{\frac{d}{a}\left|\mathbf{r}-\mathbf{d}\frac{a^2}{d^2}\right|}\right)$$

Interpreted as

Image charge of $q' = -q\frac{a}{d}$

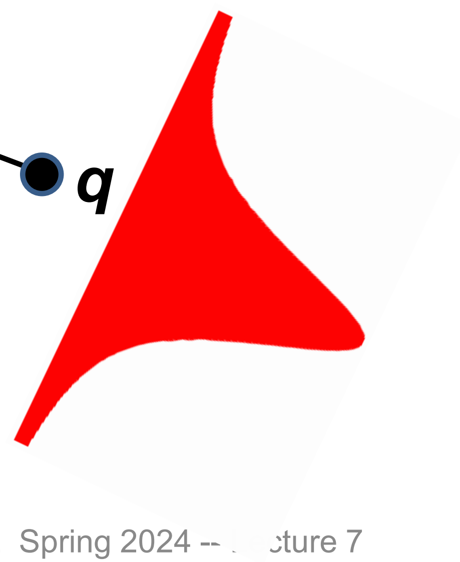
Located along $\hat{\mathbf{d}}$ at $\hat{\mathbf{d}}a\frac{a}{d}$

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.

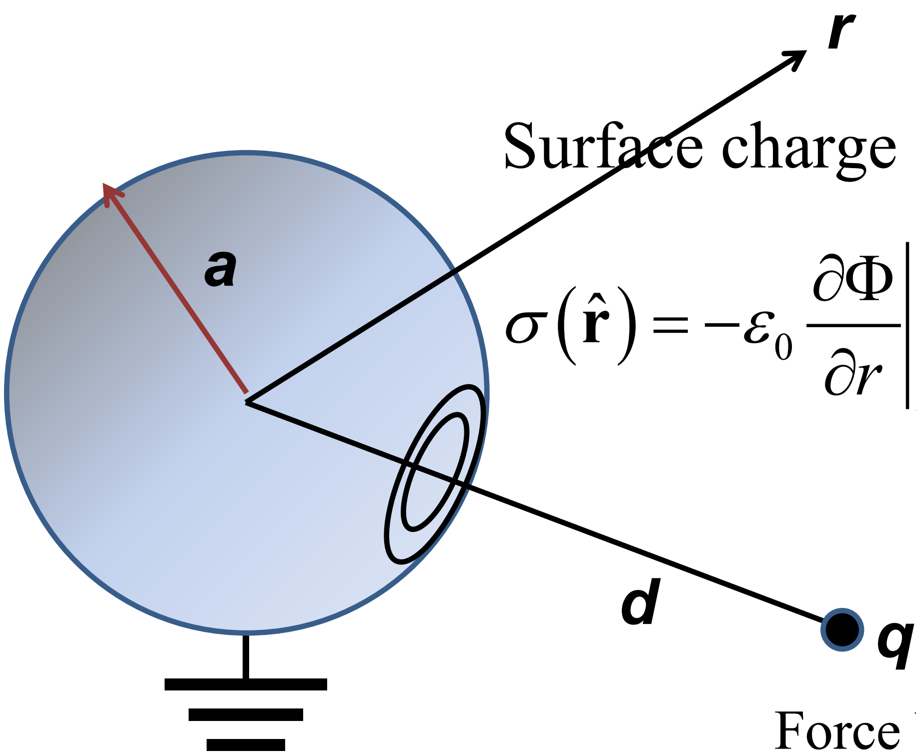


Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$



A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.

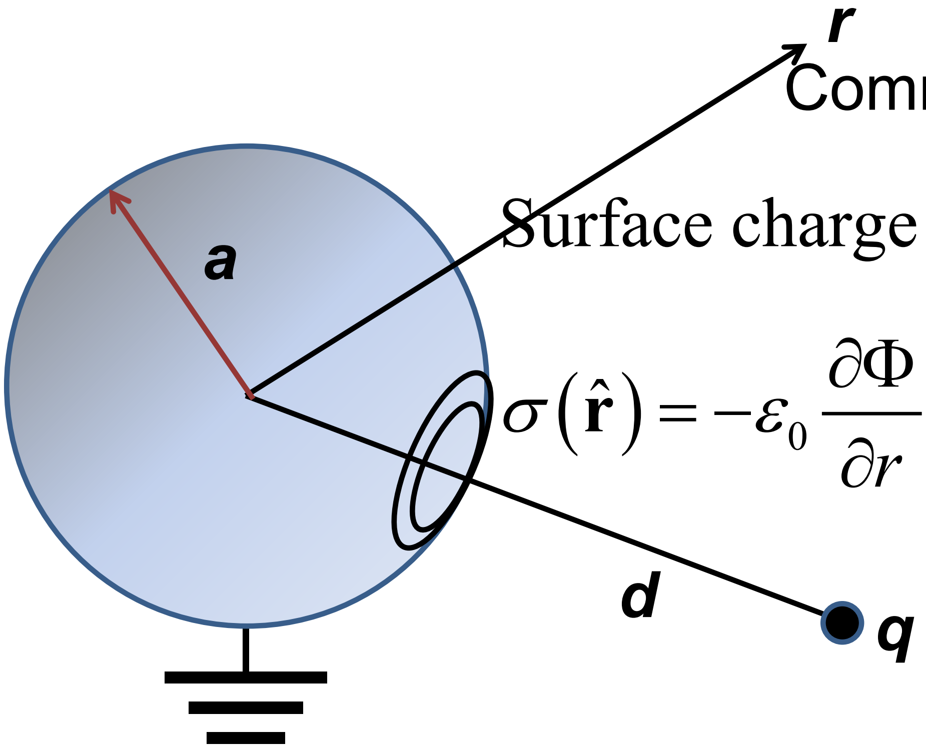


Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

Force between q and sphere

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2(a/d)}{(d-a^2/d)^2} = \frac{q^2}{4\pi\epsilon_0}\frac{ad}{(d^2-a^2)^2}$$



Comment on HW problem #7

Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

For #1, integrate the charge induced on the outer surface of the sphere due to the point charge q at the point $d > a$.

$$\int \sigma(\hat{\mathbf{r}})dS = -\int \frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}dS = -\frac{q}{4\pi a^2}\frac{a}{d}\left(1-\frac{a^2}{d^2}\right)2\pi a^2\int \frac{d\cos\theta}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}}$$

For #2, the point charge q is located at a point $d < a$ and a similar analysis follows. Integrate the charge induced on the inner surface of the sphere.

(Answer to #2 should be different from that of #1.)

Use of image charge formalism to construct Green's function

Example:

Suppose we have a Dirichlet boundary value problem

on a sphere of radius a:

$$\nabla^2\Phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \Phi(r=a)=0$$

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow \text{For} \quad r,r' > a: \qquad G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\frac{r'}{a}\left|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'\right|}$$

Analysis of Poisson/Laplace equation in various regular geometries

1. Rectangular geometries ➔ previous lectures
2. Cylindrical geometries ➔ now
3. Spherical geometries ➔ later

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}=0$$

$$\Phi(\rho,\varphi)=\Phi(\rho,\varphi+m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates:

$$\Phi(\rho,\varphi)=\mathrm{A}_0+\mathrm{B}_0\ln(\rho)+\sum_{m=1}^{\infty}\left(A_m\rho^m+B_m\rho^{-m}\right)\sin(m\varphi+\alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

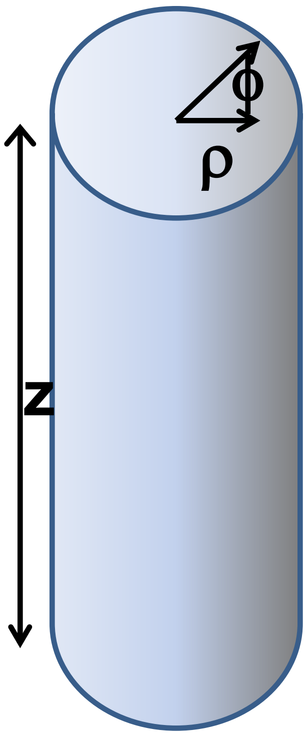
Note that here ρ means radial coordinate

Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>{}^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left(m(\phi-\phi')\right)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi(\rho,\varphi,z)=\Phi(\rho,\varphi+m2\pi,z)$$

$$\Phi(\rho,\varphi,z)=R(\rho)Q(\varphi)Z(z)$$

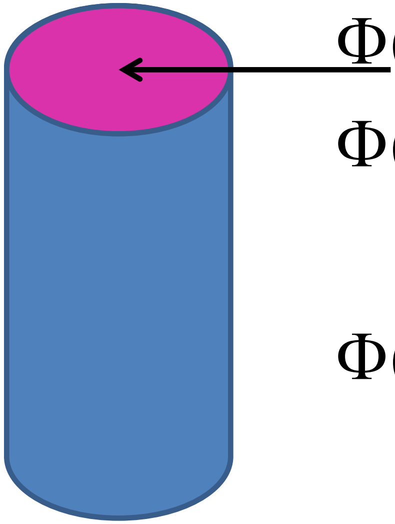
Cylindrical geometry continued:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry example:

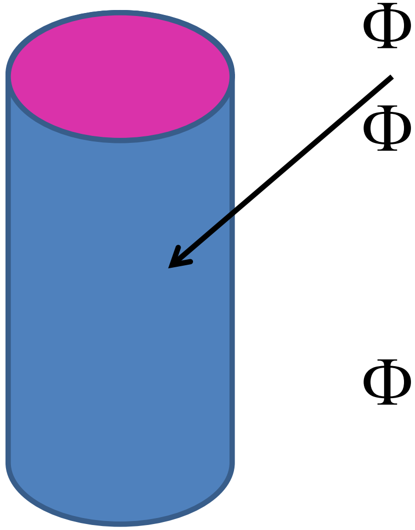


$$\Phi(\rho,\phi,z=L) = V(\rho,\phi)$$

$$\Phi(\rho,\phi,z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\phi + \alpha_{mn}\right)$$

Cylindrical geometry example:



$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

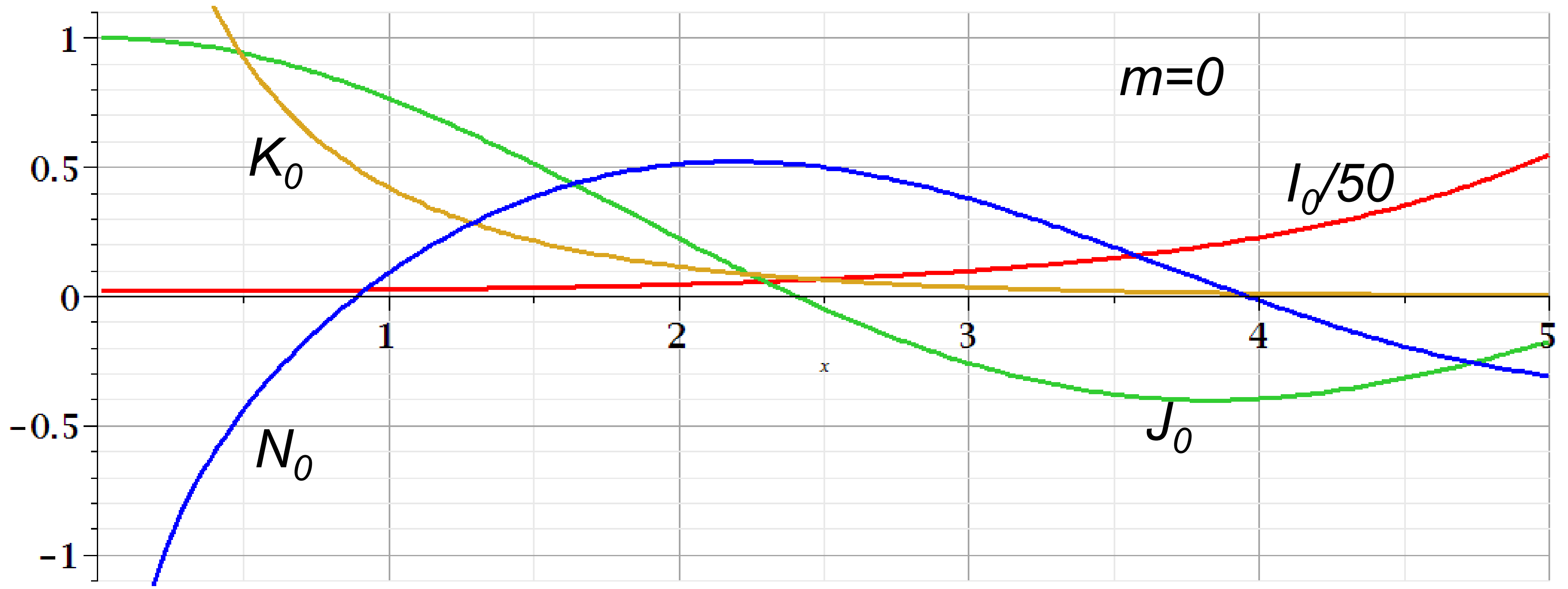
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi + \alpha_{mn}\right)$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$



Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$