

PHY 712 Electrodynamics

10-10:50 AM MWF in Olin 103

Class notes for Lecture 8:

Solution of Poisson/Laplace equation for special geometries –

- **Cylindrical – (Sec. 2.11, 3.7, 3.8, 3.11 in JDJ)**
- **Spherical -- (Sec. 3.1-3.6 in JDJ); next lecture**

Course schedule for Spring 2024

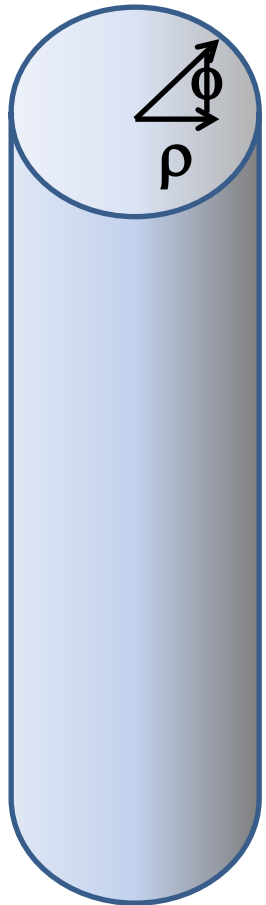
(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/17/2024	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/19/2024
2	Fri: 01/19/2024	Chap. 1	Electrostatic energy calculations	#2	01/29/2024
3	Mon: 01/22/2024	Chap. 1	Electrostatic energy calculations	#3	01/29/2024
4	Wed: 01/24/2024	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/29/2024
5	Fri: 01/26/2024	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/29/2024
6	Mon: 01/29/2024	Chap. 1 - 3	Brief introduction to numerical methods	#6	02/05/2024
7	Wed: 01/31/2024	Chap. 2 & 3	Image charge constructions	#7	02/05/2024
8	Fri: 02/2/2024	Chap. 2 & 3	Poisson equation in cylindrical geometries		
9	Mon: 02/05/2024	Chap. 3 & 4	Spherical geometry and multipole moments		

Comment on surface boundary conditions – As discussed in JDJ (Sec. 1.9), there are two named boundary conditions – named for famous mathematicians.

- Dirichlet boundary condition is named for specifying the potential function $\Phi(\mathbf{r})$ on the boundary.
- Neumann boundary condition is named for specifying the electric field function $\mathbf{E}(\mathbf{r})$ on the boundary.

Solution of the Poisson/Laplace equation in various geometries
→ cylindrical geometry with no z-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

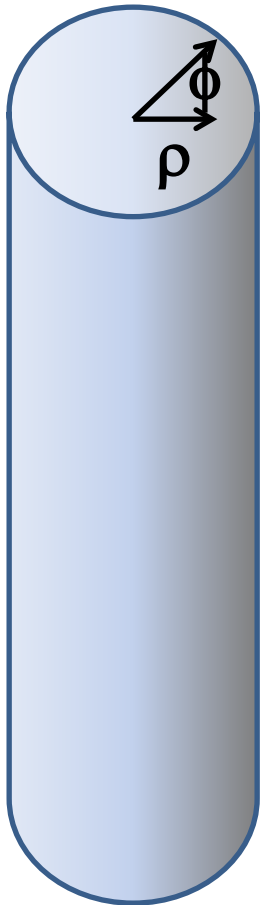
$$\Phi(\rho, \phi) = \Phi(\rho, \phi + m2\pi)$$

Assume: $\Phi(\rho, \phi) = f(\rho)g(\phi)$

Suppose $\frac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$

$g(\phi) = \cos(m\phi + \alpha_m) \Rightarrow m = \text{integer}, \alpha_m = \text{phase}$

Solution of the Poisson/Laplace equation in various geometries
→ cylindrical geometry with no z-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Assume: $\Phi(\rho, \phi) = f(\rho)g(\phi)$

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{df_m(\rho)}{d\rho} \right) - \frac{m^2}{\rho^2} f_m(\rho) = 0$$

$$f_0(\rho) = \begin{cases} 1 \\ \ln \rho \end{cases} \quad f_{m>0} = \rho^{\pm m}$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2 \Phi = 0$

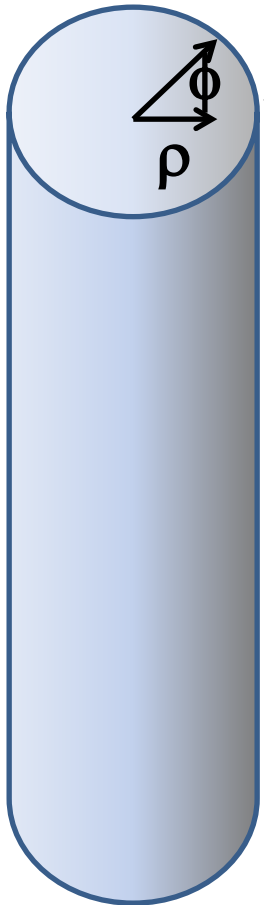
$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0$$

$$\Phi(\rho, \varphi) = \Phi(\rho, \varphi + m2\pi) \quad \rightarrow m = \text{integer}$$

\Rightarrow General solution of the Laplace equation
in these coordinates:

$$\Phi(\rho, \varphi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} \left(A_m \rho^m + B_m \rho^{-m} \right) \cos(m\varphi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):



Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) G(\rho, \rho', \phi, \phi') = -4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')$$

It can be shown that the following form can be used:

$$G(\rho, \rho', \phi, \phi') = -\ln(\rho_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_{<}}{\rho_{>}} \right)^m \cos(m(\phi - \phi'))$$

Note that the previous example is similar to the construction for the 2-d cartesian case --

For the 2-d cartesian case, for example, we can assume that the Green's function can be written in the form:

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') g_n(y, y') \quad \text{where} \quad \frac{d^2}{dx^2} u_n(x) = -\alpha_n u_n(x)$$

The y dependence of this equation will have the required

behavior, if we choose:
$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2} \right] g_n(y, y') = -4\pi\delta(y - y'),$$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n \right] v_{n_i}(y) = 0,$$

and the Wronskian constant:
$$K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$$

Cartesian example continued --

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2} \right] g_n(y, y') = -4\pi\delta(y - y'),$$

$$g_n(y, y') = \frac{4\pi}{K_n} v_{n_1}(y_{<}) v_{n_2}(y_{>})$$

where: $\left[\frac{d^2}{dy^2} - \alpha_n \right] v_{n_i}(y) = 0,$

and $K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n} y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n} (b - y))$

where $K_n = \sqrt{\alpha_n} \sinh(\sqrt{\alpha_n} b)$

using the identity: $\cosh(r) \sinh(s) + \sinh(r) \cosh(s) = \sinh(r + s)$

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') \frac{4\pi}{K_n} v_{n_1}(y_{<}) v_{n_2}(y_{>}).$$

In the cylindrical geometry case,

$$u_n(x) \rightarrow \{ \sin(m\varphi), \cos(m\varphi) \}$$

$$v_{n_{1,2}} \rightarrow \{ 1, \ln(\rho), \rho^m, \rho^{-m} \}$$

$$G(\rho, \rho', \varphi, \varphi') = -\ln(\rho_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_{<}}{\rho_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

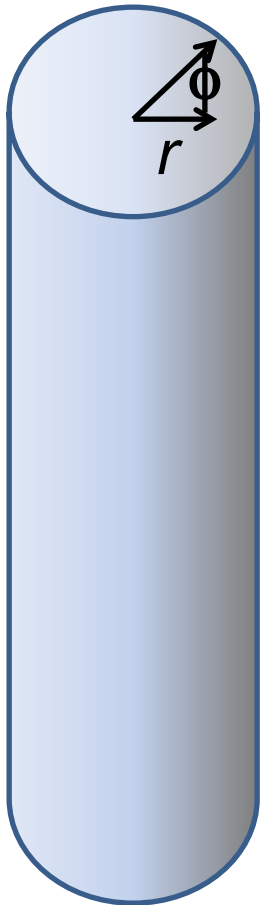
Note that, because we are using curvilinear coordinates, the Wronskian and the form of the delta function is modified.

More details given in **Jackson** Sec. 3.7 - 3.11.

Comments and details

Change notation

$$\rho \Rightarrow r$$

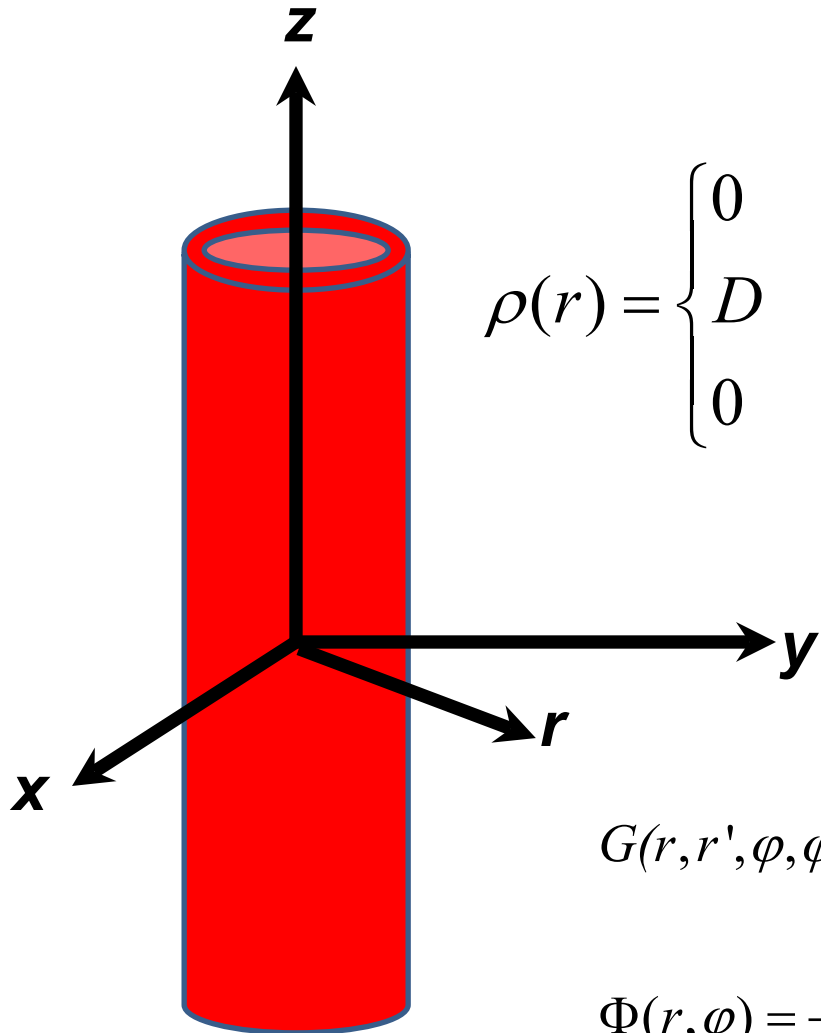


$$G(r, r', \varphi, \varphi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

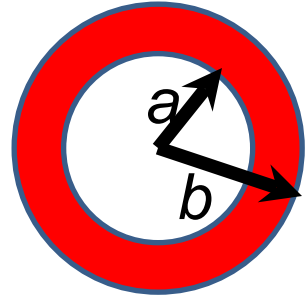
$$\Phi(r, \varphi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r, r', \varphi, \varphi') \rho(r', \varphi')$$

Note that in this case, we have assumed that the surface integral contributions are trivial.

Example – uniform cylindrical shell:



Top view:



$$\rho(r) = \begin{cases} 0 & r < a \\ D & a \leq r \leq b \\ 0 & r > b \end{cases}$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \Phi(r, \phi) = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi(r, \phi)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi(r, \phi)}{\partial^2 \phi}$$

$$G(r, r', \phi, \phi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\phi - \phi'))$$

$$\Phi(r, \phi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\infty} r' dr' G(r, r', \phi, \phi') \rho(r', \phi')$$

Boundary condition: $\lim_{r \rightarrow \infty} \left(\frac{\partial \Phi(r, \phi)}{\partial r} \right) = 0$

Question – Why only $m=0$ for this case?

$$G(r, r', \varphi, \varphi') = -\ln(r_>^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

$$\Phi(r, \varphi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r, r', \varphi, \varphi') \rho(r', \varphi')$$

Note that $\int_0^{2\pi} d\varphi' \cos(m(\varphi - \varphi')) = 0$ for $m > 0$

So that $\Phi(r, \varphi) = \frac{2\pi}{4\pi\epsilon_0} \int_0^{\infty} r' dr' (-\ln(r_>^2)) \rho(r')$

Some details

$$G(r, r', \varphi, \varphi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

$$\Phi(r, \varphi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r, r', \varphi, \varphi') \rho(r', \varphi')$$

$$\text{In our case: } \Phi(r, \varphi) = \frac{2\pi D}{4\pi\epsilon_0} \int_a^b r' dr' (-\ln(r_{>}^2)) = \frac{D}{\epsilon_0} \int_a^b r' dr' (-\ln(r_{>}))$$

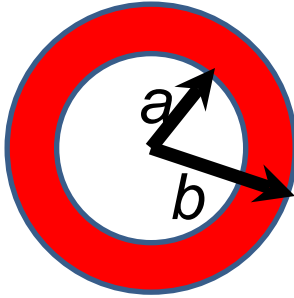
$$\text{For } 0 \leq r < a: \quad \Phi(r, \varphi) = \frac{D}{\epsilon_0} \int_a^b r' dr' (-\ln(r'))$$

$$\text{For } a \leq r < b: \quad \Phi(r, \varphi) = \frac{D}{\epsilon_0} \left(\int_a^r r' dr' (-\ln(r)) + \int_r^b r' dr' (-\ln(r')) \right)$$

$$\text{For } r > b: \quad \Phi(r, \varphi) = \frac{D}{\epsilon_0} \int_a^b r' dr' (-\ln(r))$$

Example continued -- $m=0$ only --

Top view:



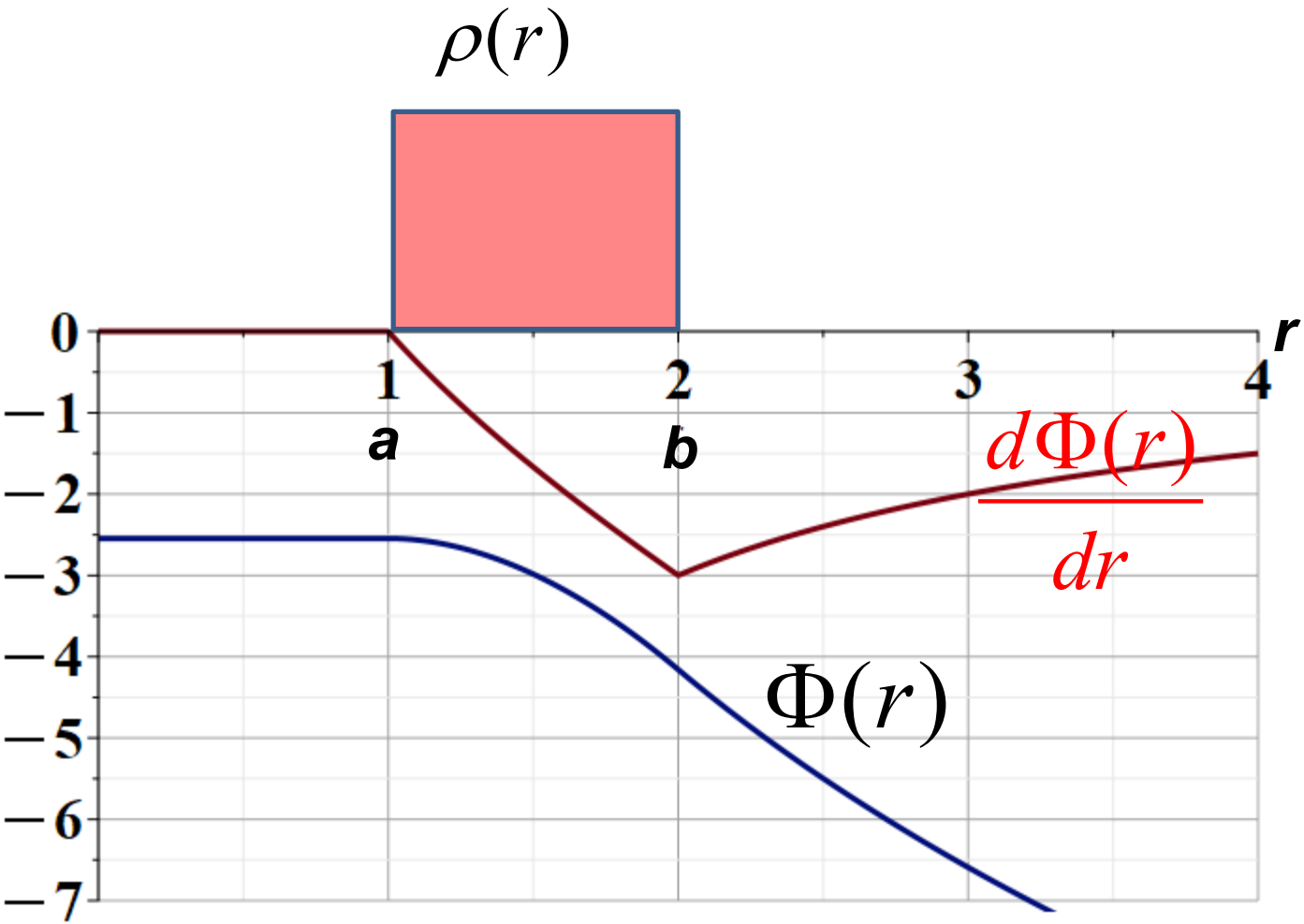
$$\rho(r) = \begin{cases} 0 & 0 < r < a \\ D & a \leq r \leq b \\ 0 & r > b \end{cases}$$

$$G(r, r', \varphi, \varphi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

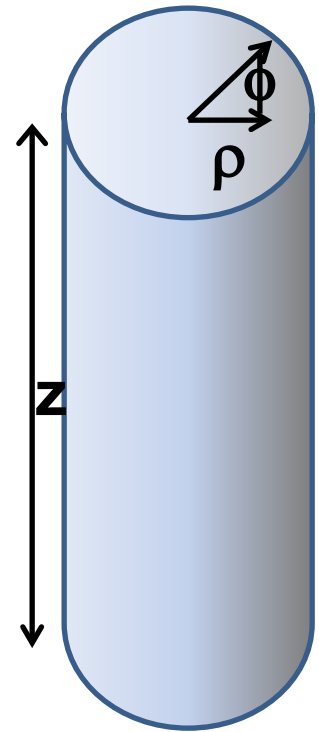
$$\Phi(r, \varphi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r, r', \varphi, \varphi') \rho(r', \varphi')$$

$$\Phi(r) = \begin{cases} \frac{D}{4\epsilon_0} (b^2 - a^2 - b^2 \ln(b^2) + a^2 \ln(a^2)) & 0 < r < a \\ \frac{D}{4\epsilon_0} (b^2 - r^2 - b^2 \ln(b^2) + a^2 \ln(r^2)) & a \leq r \leq b \\ \frac{D}{4\epsilon_0} (a^2 - b^2) \ln(r^2) & r > b \end{cases}$$

Example continued --



Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with z-dependence

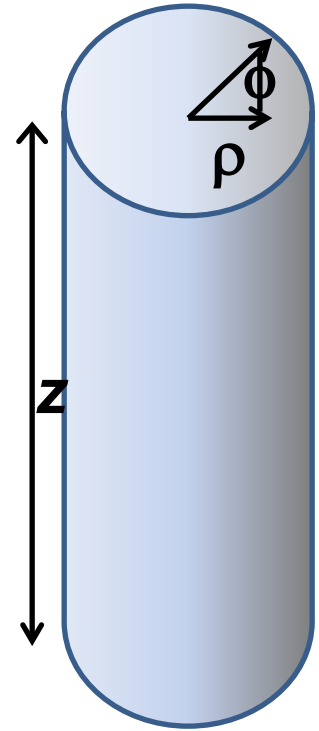


Laplace equation : $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2 \Phi = 0$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

One possibility :

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow J_m(k\rho), N_m(k\rho)$$

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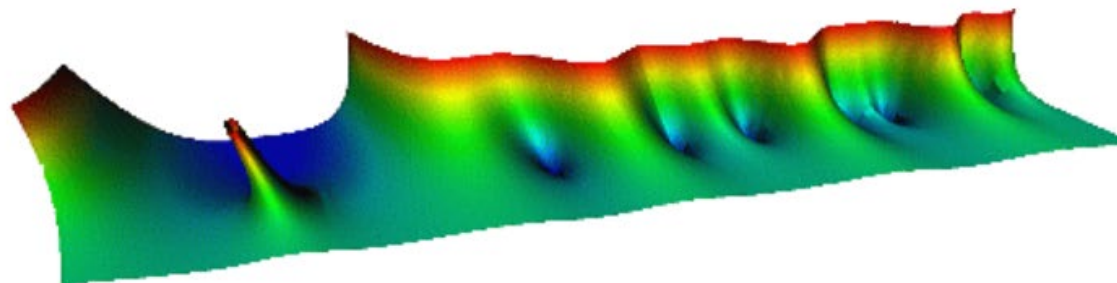
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About the Project



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Project News

2022-12-15 [DLMF Update; Version 1.1.8; MathML improvements](#)

2022-12-15 [Richard B. Paris, Associate Editor of the DLMF, dies at age 76](#)

2022-10-15 [DLMF Update; Version 1.1.7; Enhanced coverage of Lambert \$W\$](#)

2022-06-30 [DLMF Update; Version 1.1.6](#)

[More news](#)

§10.3 Graphics

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§10.3(i) Real Order and Variable

For the modulus and phase functions $M_\nu(x)$, $\theta_\nu(x)$, $N_\nu(x)$, and $\phi_\nu(x)$ see §10.18.

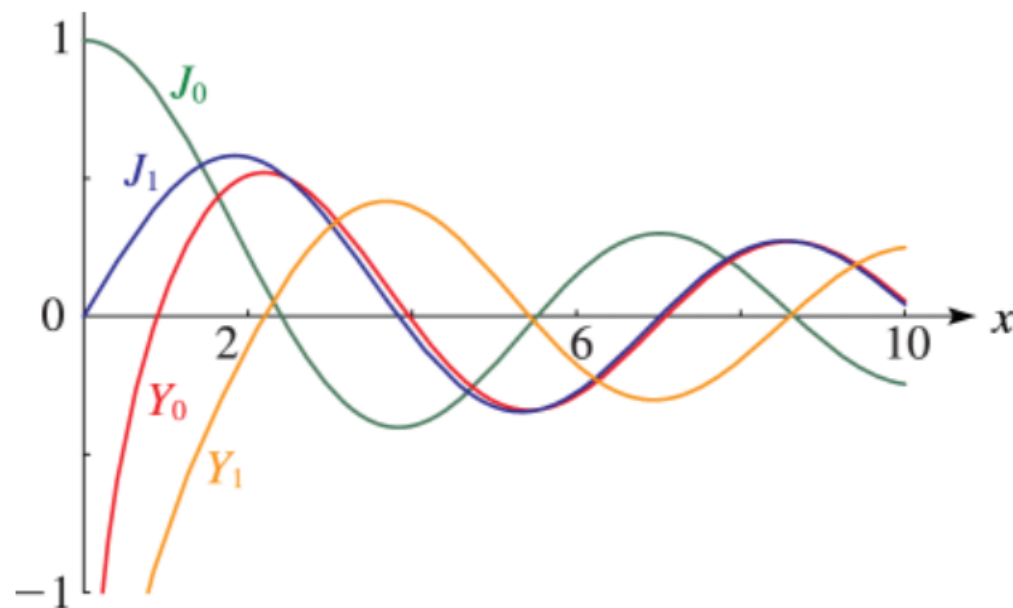


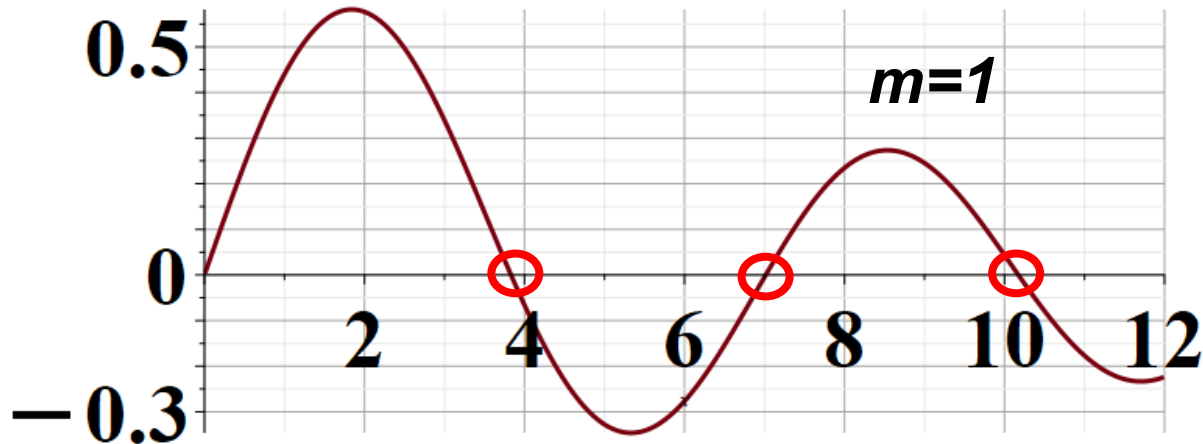
Figure 10.3.1: $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$,

Some useful identities involving cylindrical Bessel functions from **Jackson** Sec. 3.7

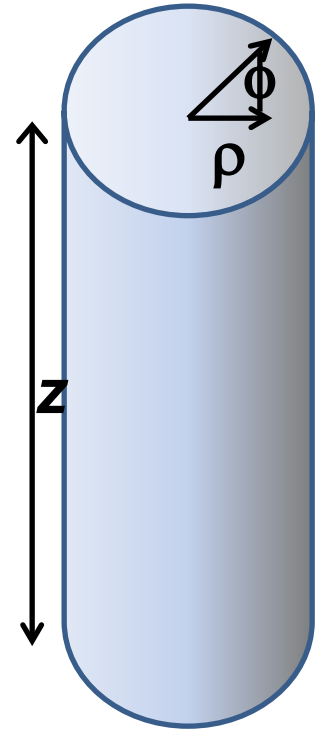
$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(1 - \frac{m^2}{u^2} \right) \right) J_m(u) = 0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn} ; $J_m(x_{mn}) = 0$

$$\int_0^a \rho d\rho J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{m'n'}\rho}{a}\right) = \frac{a^2}{2} (J_{m+1}(x_{mn}))^2 \delta_{nn'}$$



Cylindrical geometry continued:



Laplace equation: $\nabla^2 \Phi = 0$

$$\Phi(\rho, \varphi, z) = R(\rho)Q(\varphi)Z(z)$$

Another possibility:

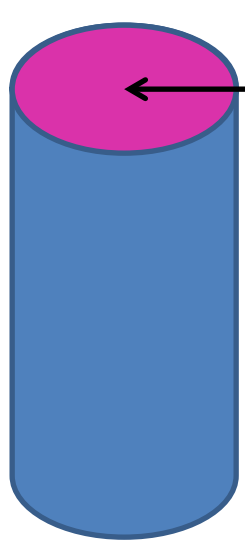
$$\frac{d^2 Z}{dz^2} + k^2 Z = 0 \quad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow I_m(k\rho), K_m(k\rho)$$

Solutions of Laplace equation inside cylindrical shape

Example with non-trivial boundary value at $z=L$



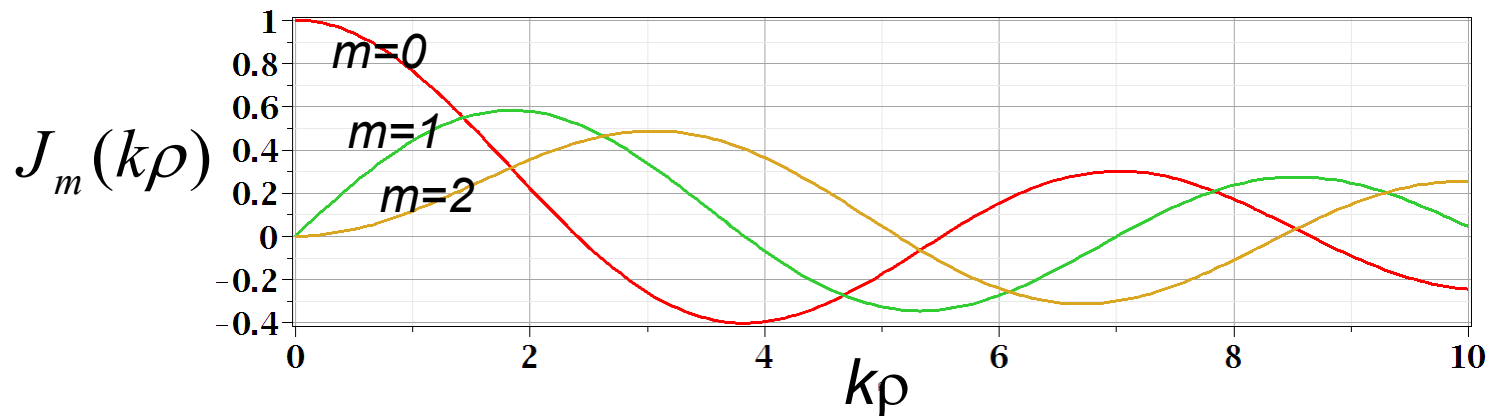
$$\Phi(\rho, \phi, z = L) = V(\rho, \phi)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho) \sinh(k_{mn}z) \sin(m\phi + \alpha_{mn})$$

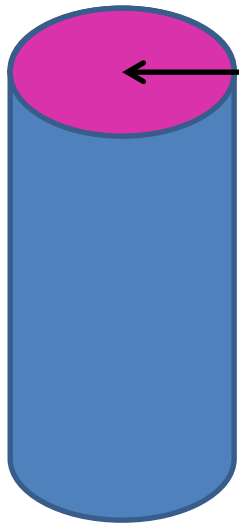
Well behaved at $\rho=0$

$$\text{where } J_m(k_{mn}a) = 0$$



Solutions of Laplace equation inside cylindrical shape

Example with non-trivial boundary value at $z=L$



$$\leftarrow \Phi(\rho, \varphi, z = L) = V(\rho, \varphi)$$

$$\Phi(\rho, \varphi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho) \sinh(k_{mn}z) \sin(m\varphi + \alpha_{mn})$$

If $V(\rho, \varphi)$ is an even function of φ so that $\alpha_{mn} = \pi / 2$:

$$A_{mn} = \frac{\int_0^{2\pi} d\varphi \cos(m\varphi) \int_0^a \rho d\rho J_m(k_{mn}\rho) V(\rho, \varphi)}{\sinh(k_{mn}L) \int_0^{2\pi} d\varphi \cos^2(m\varphi) \int_0^a \rho d\rho J_m^2(k_{mn}\rho)}$$

§10.26 Graphics

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§10.26(i) Real Order and Variable

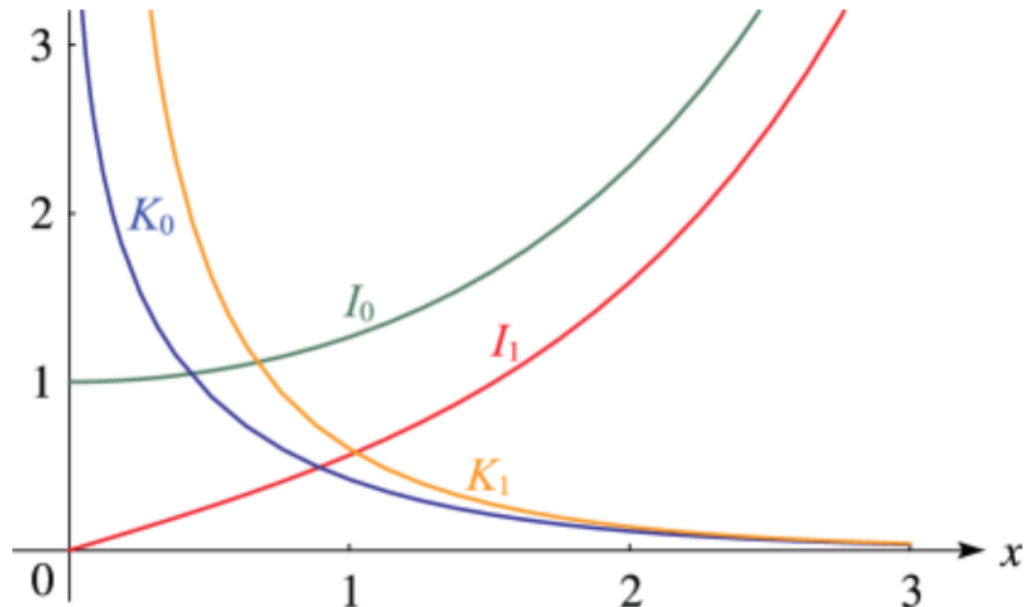
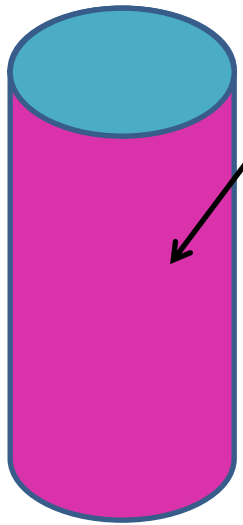


Figure 10.26.1: $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$,

Solutions of Laplace equation inside cylindrical shape

Example with non-trivial boundary value at $\rho=a$

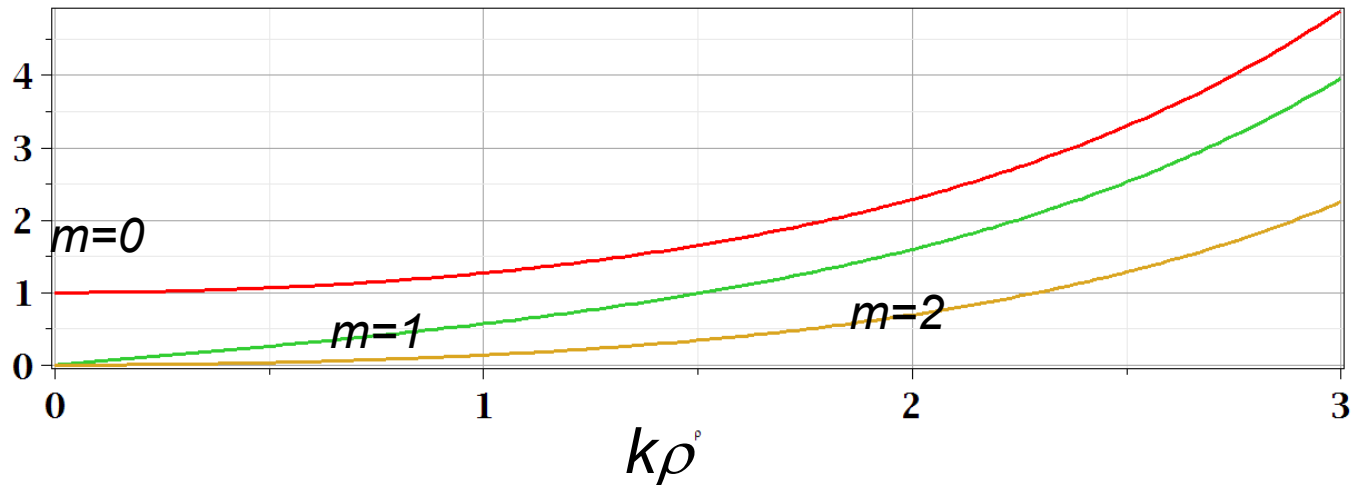


$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

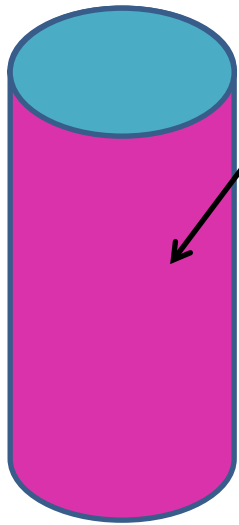
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m \left(\frac{n\pi\rho}{L} \right) \sin \left(\frac{n\pi z}{L} \right) \sin(m\phi + \alpha_{mn})$$

$I_m(k\rho)$



Solutions of Laplace equation inside cylindrical shape

Example with non-trivial boundary value at $\rho=a$



$$\Phi(\rho = a, \varphi, z) = V(\varphi, z)$$

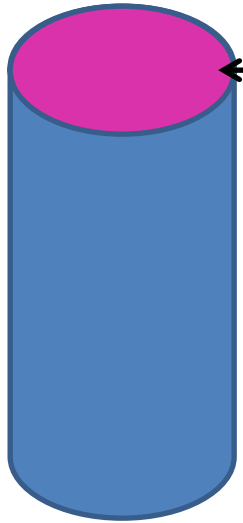
$$\Phi(\rho, \varphi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} I_m \left(\frac{n\pi\rho}{L} \right) \sin \left(\frac{n\pi z}{L} \right) \sin(m\varphi + \alpha_{mn})$$

If $V(z, \varphi)$ is an even function of φ so that $\alpha_{mn} = \pi/2$:

$$A_{mn} = \frac{\int_0^{2\pi} d\varphi \cos(m\varphi) \int_0^L dz \sin \left(\frac{n\pi z}{L} \right) V(z, \varphi)}{I_m \left(\frac{n\pi a}{L} \right) \int_0^{2\pi} d\varphi \cos^2(m\varphi) \int_0^L dz \sin^2 \left(\frac{n\pi z}{L} \right)}$$

Green's function for Dirchelet boundary value inside cylinder:



$$\Phi(\rho, \phi, z = L) = V(\rho, \phi)$$

$$\Phi(\rho = a, \phi, z) = 0, \quad \Phi(\rho, \phi, z = 0) = 0$$

Expansion in terms of Bessel function zeros : $J_m(k_{mn}a) = 0$

$$G(\rho, \rho', \phi, \phi', z, z') =$$

$$\frac{8\pi}{\pi a^2} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')} J_m(k_{mn}\rho) J_m(k_{mn}\rho') \sinh(k_{mn}z_{<}) \sinh(k_{mn}(L-z_{>}))}{k_{mn} (J_{m+1}(k_{mn}a))^2 \sinh(k_{mn}L)}$$

$$\Phi(\rho, \phi, z) = \frac{1}{4\pi\epsilon_0} \int_V d\phi' \rho' d\rho' dz' G(\rho, \rho', \phi, \phi', z, z') \rho(\rho', \phi', z')$$

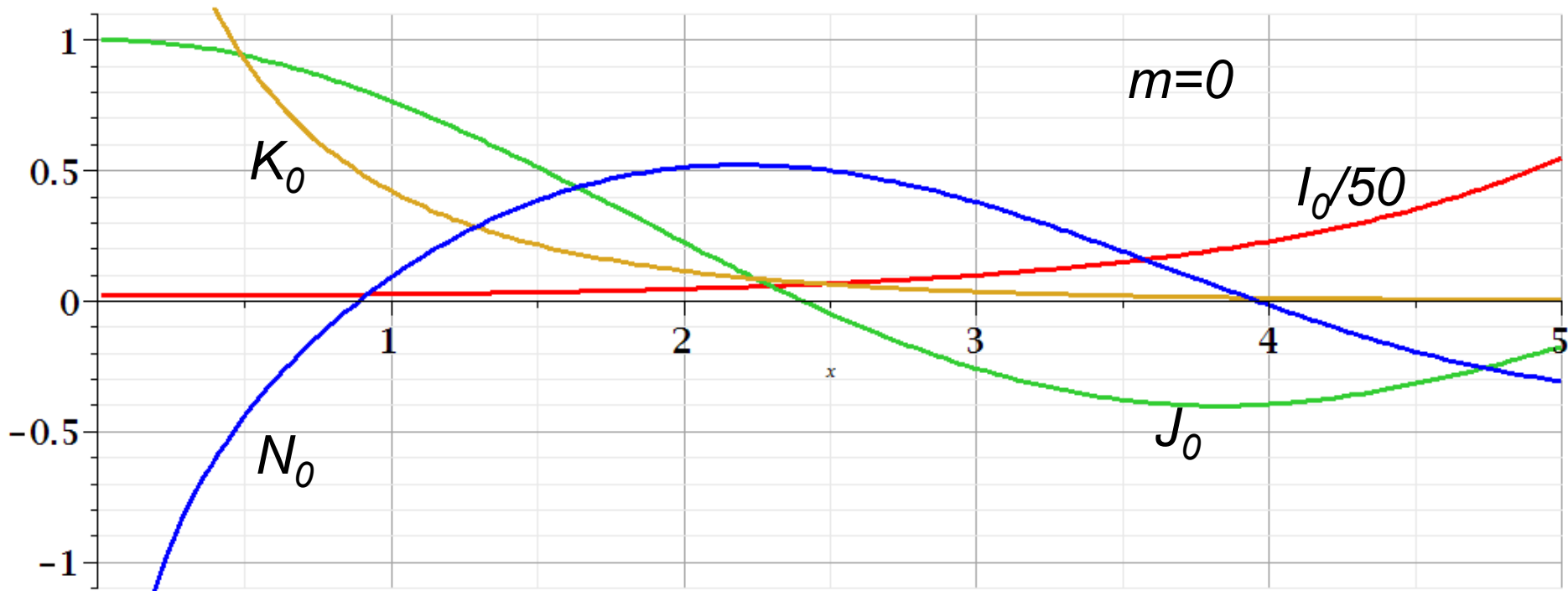
$$+ \frac{1}{4\pi} \int_{S; z'=L} d\phi' \rho' d\rho' \left. \frac{\partial G(\rho, \rho', \phi, \phi', z, z')}{\partial z'} \right|_{z'=L} V(\rho', \phi')$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2} \right) \right) F_m^\pm(u) = 0$$

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