Infinity

Math 165

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Infinity

Today, we explore the notion of infinity and infinite sets.

You've all encountered infinity before in calculus

$$\lim_{n\to\infty} \frac{n^2 - 2n + 5}{n - 4} = \infty \qquad\qquad \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$$

Today, we're more interested in the size of sets

How big is a set?

Consider the set of modern Romance Languages offered at Wake Forest

$$RL = \{\text{French, Italian, Portuguese, Spanish}\}$$

How many languages are in the set *RL*? 4

How are we sure it's 4?
By counting the languages, we are identifying each language with a number:

$$\begin{array}{lcl} \text{French} & \leftrightarrow & 1 \\ \text{Italian} & \leftrightarrow & 2 \\ \text{Portuguese} & \leftrightarrow & 3 \\ \text{Spanish} & \leftrightarrow & 4 \end{array}$$

So, the set *RL* is the same size (**cardinality**) as the set $\{1, 2, 3, 4\}$.

Bijections

Definition

A **bijection** *between sets A and B is one that pairs each element of A with one and only one element of B, and vice-versa.*

We could think of a bijection as a function $f : A \to B$.

Our example shows how we could construct a bijection
$f : \{1, 2, 3, 4\} \to RL$

$$\begin{aligned} f(1) = &\quad \text{French} \\ f(2) = &\quad \text{Italian} \\ f(3) = &\quad \text{Portuguese} \\ f(4) = &\quad \text{Spanish} \end{aligned}$$

Question: Is there a bijection between students in this room and RL? *No, too many students.*

Cardinality

Idea

Two sets A and B have the same cardinality if and only if there is a bijection between them

Finite sets are those in bijection with some set $\{1, \ldots, n\}$. We say that the cardinality of such a set is n.

The cardinality of RL is 4, which we write $\text{card}(RL) = 4$.

$$\text{card}\{165 \text{ students}\}) = 53$$

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Infinite sets

Infinite sets then are the sets that are not finite.

Examples:

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$	$\mathbb{Q}$, the rational numbers
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	$\mathbb{R}$, the real numbers
P, the prime natural numbers	$\mathbb{R}[[x]]$, all power series

Question

Do all infinite sets have the same cardinality?
Or, are there different sizes of infinity?

Countable sets

A **countable** set is, um, *one that you can count*, or more precisely, one that's in bijection with a subset of $\mathbb{N}$.
(Remember that every set is a subset of itself.)

Many familiar infinite sets turn out to be countable. For instance, consider the even natural numbers

$$E = \{2, 4, 6, 8, 10, \ldots\}$$

Let's find a bijection from $\mathbb{N}$ to E:

$\mathbb{N}$	1	2	3	4	5	$\cdots$
E	2	4	6	8	10	$\cdots$

We could write this bijection $f : \mathbb{N} \to E$ as $f(n) = 2n$. So ... a

subset of $\mathbb{N}$ can have the same size as $\mathbb{N}$.

The integers are countable

Can supersets of $\mathbb{N}$ be countable?

Sure. Consider the integers.

How could we find a bijection between $\mathbb{Z}$ and $\mathbb{N}$?

$\mathbb{N}$	1	2	3	4	5	6	7	$\cdots$
$\mathbb{Z}$	0	1	-1	2	-2	3	-3	$\cdots$

If we wanted to give a formula,

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

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Power sets

the **power set** $\mathcal{P}(A)$ of A is the set of all subsets of A

$A = \{1, 2\}$ means $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

If A has n elements, how many does $\mathcal{P}(A)$ have?

$$2^n$$

Cardinality rules

- If $A \subset B$, then $\text{card}(A) \leq \text{card}(B)$.
- a set A is infinite if and only if it has a bijection with a proper subset of itself
- $\text{card}(A) < \text{card}(\mathcal{P}(A))$

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Problems

Problem 1

Show that the following sets are countable:

1. $O = \{1, 3, 5, 7, 9, 11, \ldots\}$, the odd natural numbers
2. $EZ = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$, the even integers
3. $M = \{2, 3, 4, 5, 6, \ldots\}$, all the natural numbers bigger than 1

In each case, come up with an explicit formula for the bijection mapping $\mathbb{N}$ to the set.

Problems

Problem 2: Hilbert's Hotel

The Hilbert Hotel is a 5-star luxury hotel with one room for each natural number. Suppose on a busy night, every room is occupied. A traveler arrives late in the evening, and the manager, who's a mathematician, finds a way to give her one of the rooms without forcing any guests to leave the hotel. Describe how the manager could do this. (Some guests may have to change rooms, but no one is forced to share a room.)

Problem 3: Hilbert's Hotel again

That same night, all of the presenters at the Infinitely-Long Awards Show descend on the Hilbert Hotel and need rooms. There are as many presenters as natural numbers. Describe how our intrepid hotel manager could accommodate all of the presenters in rooms, without forcing anyone to leave.

Are the rational numbers countable?

We'll work with just the positive rational numbers for now

$$Q_+ = \left\{ \frac{a}{b} : a, b \in \mathbb{N}; gcd(a, b) = 1 \right\}$$

Consider the ordered pairs (a, b) – these all sit in the first quadrant. Let's count them along diagonal lines.

Okay, is everything countable???

Georg Cantor, 1845-1918

Cantor, a German mathematician, was the first to show that $\mathbb{Q}$ is countable (1873).

That year he grappled with the question of whether $\mathbb{R}$ was countable.

Using the *Cantor diagonalization argument*, he was able to show $\mathbb{R}$ was not countable.

Cantor's work was not universally accepted; he suffered from depression.

Definition

Set A is **uncountable** *if it is not countable*

Cantor's Diagonalization Argument

A *proof by contradiction*: we assume the opposite (the logical negation) of what we're trying to prove, and show that it leads to a contradiction.

Theorem. (Cantor, 1873) The interval $[0, 1]$ of $\mathbb{R}$ is uncountable.

Suppose the opposite is true. Then we should be able to list all of the real numbers in one countable sequence. Call the first one on the list x_1, the next one x_2, etc.

[on board]

We can construct a number y not equal to any x_n (for any n). To construct y, we choose its nth decimal digit to be

$$\begin{array}{ll} 1 & \text{if } n\text{th digit of } x_n \neq 1 \\ 3 & \text{if } n\text{th digit of } x_n \text{ equals } 1 \end{array}$$

So y and x_n differ in the nth digit, so they cannot be equal.

Infinite cardinalities

Mathematicians use the Hebrew letter $\aleph_0$ to denote the countably infinite cardinality,

$$\text{card}(\mathbb{N}) = \aleph_0$$

Fact: $\text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(\mathbb{R})$,
this is known as the *cardinality of the continuum* $\mathbf{c}$

Here's an infinite list of sets of increasing uncountable cardinalities:

$$\mathbb{R},\ \mathcal{P}(\mathbb{R}),\ \mathcal{P}\left(\mathcal{P}(\mathbb{R})\right),\ \mathcal{P}\left(\mathcal{P}\left(\mathcal{P}(\mathbb{R})\right)\right), \ldots$$

Are there other cardinalities?

Continuum Hypothesis

Cantor wondered, are there any levels of infinity between $\mathbb{N}$ and $\mathbb{R}$.

We could imagine listing cardinalities

$$0, 1, 2, 3, 4, \ldots, \ \aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots, \aleph_\infty, \ \ldots$$

So do the real numbers have cardinality $\aleph_1$?

Continuum Hypothesis

The real numbers have cardinality $\aleph_1$; there's no levels of infinity between $\mathbb{N}$ and $\mathbb{R}$.

Godel's Incompleteness Theorem

The mathematician Kurt Godel proved an important result with deep philosophical meaning.

A set of axioms is called **inconsistent** if you can prove that some statement both is true and is false using those axioms.
It is **incomplete** if there are statements that are incapable of being proven true or false.
Example: A weak set of geometry axioms (in 2-d) is incomplete: you cannot distinguish between normal Euclidean geometry and hyperbolic geometry (or spherical geometry).

Godel's Incompleteness Theorem

Any set of axioms (which is 'strong enough') is either *incomplete* or *inconsistent*

Which one do you think mathematicians prefer? Politicians?

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Continuum Hypothesis is neither true nor false

Continuum Hypothesis

The real numbers have cardinality $\aleph_1$; there's no levels of infinity between $\mathbb{N}$ and $\mathbb{R}$.

- In the 1940's, Godel showed that the Continuum Hypothesis could never be proven to be false.
  (This doesn't mean it's true.)
- In the 1950's, Paul Cohen showed that the Continuum Hypothesis could never be proven to be true.

So the Continuum Hypothesis is an example of how modern set theory of integers and real numbers is *incomplete*. It does remain consistent.