Name: ____________________

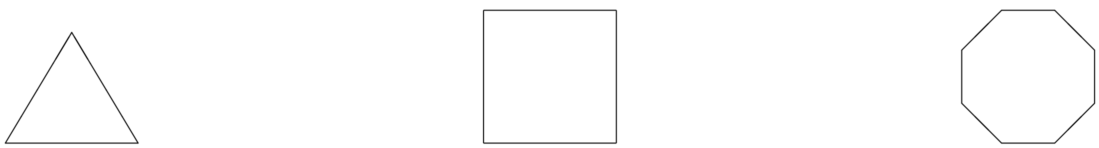
Platonic Solids

Math 165, class exercise, Sept. 16, 2010

1. Introduction

In two dimensions, one can form a *equilateral triangle* by using three sides of the same length (and three equal angles). One can also form a *square*: four equal sides, four equal angles. And a *regular pentagon*: five equal sides, five equal angles. These are all called **regular polygons**. A stop sign is a red *regular octagon*.

For any number of sides $N > 2$, one can make a *regular N-gon*, with N equal sides and N equal angles.



A **regular polyhedron** is the three-dimensional analog of a regular polygon; it is an object whose sides are congruent regular polygons which come together at equal angles. How many regular polyhedra would you guess that there are? One for each N?

The regular polyhedra are called the **Platonic Solids** since they were discovered by Plato and the ancient Greeks. Today you will build a few of the Platonic Solids. You should build these in groups of 2, but each person should complete a worksheet.

Some vocabulary:

- a **face** of a polyhedron is a regular polygon; for example, a cube has six faces (top, bottom, left, right, front, back)
- an **edge** of a polyhedron is a line segment along which two faces meet
- a **vertex** is a corner of a polyhedron; it is where three or more edges meet

2. Cube

- Using the toothpicks and gumdrops, build a cube.
- In the first results table (on the next page), record the shape of the faces under which polygon?; record the number of faces F, edges E, and vertices V. Compute $V - E + F$.
- In the second results table, record the number of faces that meet at a vertex of your figure. Record the number of edges that meet at each vertex.
- Also in the second results table, record in your table the interior angle of the polygon used to construct this figure. Add up the interior angles for each face that meets at a vertex, and enter this into the table. Is it more or less than 360°? Why?

3. Results Tables

Fill in the following information for each Platonic solid.

Platonic Solid	$F = \#$ faces	$E = \#$ edges	$V = \#$ vertices	$V - E + F$
tetrahedron				
cube				
octahedron				
dodecahedron				
icosahedron				

Platonic Solid	which polygon?	interior angle of polygon	# faces at vertex	how many edges meet at vertex?	sum of int. angles at vertex
tetrahedron					
cube					
octahedron					
dodecahedron					
icosahedron					

4. Tetrahedron

- The **tetrahedron** is a figure with 4 triangles as its faces. There is only one way that the four equilateral triangles can fit together.
- First make one equilateral triangle. Lay it flat on the table.
- Now make another equilateral triangle, where one of the sides is a side of the triangle you just made. (This new triangle will be sticking up in the air.)
- Can you place one more stick into the shape so that it becomes a shape with 4 equilateral triangles as its sides?
- Complete all boxes in the results tables for the tetrahedron.

5. Octahedron

- The octahedron is a figure with eight triangles as its faces. See if you can make this figure. (Hint 1: it is not as simple as just attaching something to the tetrahedron.)
- Hint 2: Begin with a square and then attach triangles above it and below it.
- Complete all boxes in the results tables for the octahedron.

6. Dodecahedron

- This one is harder than the first three to make. We'll have a couple models in class.
- Start by making two separate pentagons. One will form the top of the dodecahedron; the other forms the bottom.
- One obvious way to connect the two pentagons is with vertical sticks. Why doesn't that give us a regular polyhedron? [look for squares]
- Try building a pentagon using two vertices of the bottom pentagon. Can you attach another pentagon that shares an edge with this one and the bottom one?
- Keep going with this process to build 5 pentagons sticking up from the bottom pentagon. (hint: They will have to lean out.)
- Can you find a way to attach the top pentagon?
- Complete all boxes in the results tables for the dodecahedron.

7. Are there other Platonic Solids?

Some of the items below ask questions; you should answer those and complete the results table for this worksheet.

(1) When we make a polyhedron, we attach several polygons at each vertex. If this attachment occurs in a plane, you can see that the interior angles of the polygons at this vertex must add up to 360°.

(2) For a polyhedron, is the sum of the interior angles at a vertex greater than, less than, or equal to 360 degrees? (Refer back to your results tables.) Why do you think this is? Does the sum of the angles have something to do with how ‘pointy’ the corner is?

(3) Notice that the only shapes we used to make these four Platonic Solids were the square, triangle, and pentagon.

(4) Let’s look at the triangles: In the tetrahedron, three triangles meet at each vertex. What was the sum of the interior angles? Was it less than 360°?

(5) For the octahedron, how many triangles meet at a vertex? What was the sum of the interior angles? Was it less than 360°?

(6) Each interior angle of a triangle measures 60°. To have a vertex, we need at least three faces meeting. How many triangles can meet at a vertex while keeping the sum of the interior angles less than 360°? (There are multiple answers, find them all.)

(7) Have we built all of these? If not, which one(s) are we missing?

(8) Why can’t you have 6 triangles meeting at a vertex of a Platonic solid?

(9) Use the sums of the interior angles to figure out why you can only have one figure, the cube, with squares as its sides.

(10) Now use the sum of the interior angles to figure out why you can only have one solid, the dodecahedron, with pentagons as its sides.

(11) Why aren't there any regular solids with hexagons for sides? Or with octagons for sides? Explain.

(12) Now, compare the results tables for the cube and the octahedron. Do you notice any sort of swapping between them?

(13) Once you have built the icosahedron, come back to this question. Compare the results tables for the dodecahedron and the icosahedron. Do you notice any sort of swapping between them? Is it similar to the pattern between the cube and octahedron?

8. Summary

We have so far constructed 4 Platonic Solids. You should find that there is one more missing from our list, one where five triangles meet at each vertex. This is called an **icosahedron**. It has 20 faces and is rather tough to build, so we save it for last.

These Platonic Solids can only be built from triangles (tetrahedron, octahedron, icosahedron), squares (cube), and pentagons (dodecahedron). This is far different from the world of regular polygons, where there were infinitely many different n-gons (one for each $n \geq 3$).

9. Duality for Platonic solids

For any polyhedron, we can construct its **dual polyhedron** in the following way

- Put a vertex in the center of each face of the original polyhedron.
- If there is an original edge between two original faces, draw a new edge between their corresponding new vertices.
- If you're unsure of what the new faces are, they exist where original faces would meet at an original vertex.

Exercise. Try this for a cube. How many vertices does the dual have? Edges? Faces? Can you recognize it? (The dual should be a familiar solid.)

Exercise. Try this for an octahedron. How many vertices does the dual have? Edges? Faces? Can you recognize it? (The dual should be a familiar solid.)

Exercise. Try this for a tetrahedron. How many vertices does the dual have? Edges? Faces? Can you recognize it? (The dual should be a familiar solid.)

Exercise. Look back at the results tables. How do dual solids pair up in terms of V, E, F? What do you think the dual solid of a dodecahedron is? The dual solid of an icosahedron?

10. A table for later

Consider regular polygons in a regular pattern. Let n be the number of sides on a polygon. Let m be the number of polygons meeting at each vertex. Consider this table

	$n=2$	3	4	5	6	7	$\cdots$
$m=2$	S	S	S	S	S	S	$\cdots$
3	S	Tet	Cube	Do	E		
4	S	Oct	E				
5	S	Icos					
6	S	E					
$\vdots$	$\vdots$						$\ddots$

- The 5 Platonic solids are shown
- E refers to a tiling of the Euclidean plane, by either 4 squares, 6 triangles, or 3 hexagons
- S refers to polygons which tile a sphere
- All of the remaining spaces will be explained in a later week – they represent a new type of geometry, *hyperbolic geometry.*

11. 165 problems

The following count as problems for the math 165 course:

Problem 1: Construct, ideally using better materials than toothpicks and candy, a dodecahedron and an icosahedron.

Problem 2: What is the dual of the dual of a Platonic solid? How does it compare in size with the original solid? What if you take the dual of the dual of it (i.e., four iterations of duals)? Draw pictures of this for a tetrahedron, and a cube.

Problem 3: (a) In the table above, describe how the cases $m = 2$ and $n = 2$ represent polygons which tile the sphere. (*hint:* one of them represent bi-gons, and the other is dual to bi-gons) Can you draw pictures?
(b) Explain how duality is expressed in the table.

Other problems may be announced for this topic.