Algebra and Geometry

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Rational numbers

Definition

A rational number is a number of the form a/b, where a and b are integers (and $b \neq 0$).

- Examples: 1, -1, 0, $2/5$, $3/7$, $-31/27$.

- Fact: A real number is rational if and only if its decimal expansion is either finite, or eventually repeating.

Properties of rational numbers

- If x and y are rational numbers (and $y \neq 0$), then so are

$$x + y, x - y, x \cdot y, \text{ and } \frac{x}{y}.$$

- Examples of irrational numbers:

$$\sqrt{2}, \pi, e, \sqrt{2}^{\sqrt{2}}.$$

Fields

Definition

A field F is a collection of numbers so that if x and y are in F, then

- $x + y$ *is in* F,
- $x - y$ *is in* F,
- $x \cdot y$ *is in* F, *and*
- x/y *is in* F *(if* $y \neq 0$*).*

Examples of fields

- The set of rational numbers $\mathbb{Q}$ is a field.

- The collection $\{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.

- This is because

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2},$$

and

$$\frac{1}{a + b\sqrt{2}} = \frac{(a - b\sqrt{2})}{(a + b\sqrt{2})(a - b\sqrt{2})} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}.$$

Algebraic numbers

Definition

An algebraic number is a number that is a root of a polynomial $p(x)$ with integer coefficients.

- The degree of an algebraic number α is the smallest number n so that there is a polynomial $p(x)$ of degree n with $p(\alpha) = 0$.

- The set of all algebraic numbers is a field.

Examples

- If $\alpha = \sqrt{2}$, the degree of α is 2 and $p(x) = x^2 - 2$.
- If $\alpha = \sqrt{2} + \sqrt{3}$, the degree of α is 4 and $p(x) = x^4 - 10x^2 + 1$.
- If $\alpha = 2\cos(2\pi/9)$, the degree of α is 3 and $p(x) = x^3 - 3x + 1$.

Geometric constructions

- The Greeks were interested in constructing various shapes and figures using only a straightedge and compass.

- The only constructions they allowed were the following:
  - Start with the points $(0, 0)$ and $(1, 0)$.
  - Given two points, draw the line between them.
  - Given three points A, B and C, draw a circle centered at A with radius $|BC|$.
  - Mark the intersection of two lines, a line and a circle, or two circles.

- A number x is constructible if there is a finite sequence of the steps above that will mark the point $(x, 0)$.

Properties

- The set of constructible numbers forms a field!

- Fact: If x is constructible, so is $\sqrt{x}$.

- The set of constructible numbers is countably infinite, so there are many numbers that aren't constructible.

Regular polygons

- Go to `http://www.geogebra.org/webstart/geogebra.html`.
- Construct an equilateral triangle.
- You are only allowed to mark the points $(0, 0)$ and $(1, 0)$, and intersections of lines and circles you draw.
- You may use the perpendicular line, parallel line, perpendicular bisector, and angle bisector tools.

Homework

- Construct a regular pentagon.

- Hint:

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}.$$

- You should turn in (or e-mail to Dr. Rouse) a picture, as well as the sequences of moves you made.

- Please figure this out yourself (or with a friend), don't look it up online.

Questions

- Can you trisect a sixty degree angle?
- Can you double a cube? (Is $\sqrt[3]{2}$ constructible?)
- Can you square a circle? (Is $\sqrt{\pi}$ constructible?)
- Which regular polygons can you construct?

Answers (1/2)

• If α is a constructible number, then α is algebraic, and the degree of α is a power of two.

• Since $\cos(2\pi/9)$ is algebraic and has degree 3, it isn't constructible. Thus, it's not possible to trisect a sixty degree angle.

• Since $\sqrt[3]{2}$ is a root of $x^3 - 2$, it has degree 3 and so it isn't possible to double the cube.

Answers (2/2)

- In 1885, Lindemann showed that π is not an algebraic number. This implies that it isn't possible to square the circle.

- Gauss constructed a regular 17-gon. He also proved this.

Theorem

A regular n-gon is constructible with a ruler and compass if and only if

$$n = 2^a \cdot p_1 \cdot p_2 \cdots p_k$$

where the p_1, p_2, p_3, ..., p_k are distinct primes that are each one more than a power of 2.