Permutations, Parity, and Puzzles

Jeremy Rouse

WAKE FOREST UNIVERSITY

21 Oct 2010
Math 165: Freshman-only Math Seminar

Magic trick

- I'm going to start with a magic trick.

Definition

Definition

If S is a finite set, a permutation of S is a function that rearranges the elements of S. This means that $\pi : S \to S$ is a function and

- *If $x \neq y$, $\pi(x) \neq \pi(y)$.*

Example: Let $S = \{1, 2, 3, 4, 5\}$ and define π by

$$\begin{aligned} \pi(1) &= 3 \\ \pi(2) &= 1 \\ \pi(3) &= 4 \\ \pi(4) &= 2 \\ \pi(5) &= 5. \end{aligned}$$

Composing permutations

- If π and τ are two permutations of the same set we can *compose* them and get another permutation, $\pi \circ \tau$ (this means "do π" and then "do τ").

- We define

$$(\pi \circ \tau)(n) = \tau(\pi(n)).$$

(Sometimes we write $\pi\tau$ for short).

- Define τ by

$$\tau(1) = 5 \quad \tau(2) = 4 \quad \tau(3) = 3 \quad \tau(4) = 2, \text{ and } \quad \tau(5) = 1.$$

Example

The new permutation is given by

$$
\begin{aligned}
(\pi \circ \tau)(1) &= \tau(\pi(1)) = \tau(3) = 3 \\
(\pi \circ \tau)(2) &= \tau(\pi(2)) = \tau(1) = 5 \\
(\pi \circ \tau)(3) &= \tau(\pi(3)) = \tau(4) = 2 \\
(\pi \circ \tau)(4) &= \tau(\pi(4)) = \tau(2) = 4 \\
(\pi \circ \tau)(5) &= \tau(\pi(5)) = \tau(5) = 1.
\end{aligned}
$$

Cycle notation

- Mathematicians write permutations as products of *cycles*.

- A typical cycle looks $(1, 3, 4, 2)$. This means the permutation sends $1 \to 3$, $3 \to 4$, $4 \to 2$, and $2 \to 1$. (This is the permutation π from the previous slide).

- Each permutation can be written uniquely as a product of cycles (where each number appears in only one cycle).

- For example,

$$\tau = (1, 5)(2, 4).$$

(If a number if fixed by a permutation, we don't write it).

Example of composing permutations in cycle notation

- Let $a = (1, 2, 5, 4)$ and $b = (1, 4, 2)(3, 5)$.

- We have

$$ab = (1)(2, 3, 5)(4) = (2, 3, 5).$$

Group

Definition

A group G is a collection of permutations with the property that if a and b are both in G, so is $a \circ b$.

• Example: The collection of all permutations of $\{1, 2, 3, 4, 5\}$ is a group, called S_5. It contains 120 permutations.

• Example: If $a = 1$ (the identity permutation), $b = (1, 2)(3, 4)$, $c = (1, 3)(2, 4)$ and $d = (1, 4)(2, 3)$, then $\{a, b, c, d\}$ is a group.

$$a^2 = b^2 = c^2 = d^2 = a$$
$$bc = cb = d, bd = db = c, cd = dc = b.$$

Transpositions

Definition

A transposition is a permutation that swaps two numbers, and leaves the rest fixed.

- Examples: $(1,3)$, $(2,5)$, $(4,5)$, $(1,2)$.

- Fact: Every permutation can be written as a composition of permutations.

- Example:

$$(1,2,5,4) = (1,2)(1,5)(1,4).$$

Parity

Definition

We call a permutation odd if it can be written as a product of an odd number of transpositions. We call a permutation even if it can be written as a product of an even number of transpositions.

- Fact: No permutation is both even and odd.

- If we look at the cycle decomposition, the permutation is even if there are an even number of cycles of even length (and odd otherwise).

Explanation

- For example, the permutation

$$\sigma = (1, 3, 5)(2, 4, 6, 8)(7, 9)$$

is even, because there are two cycles with even length.

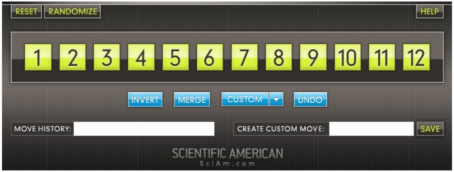
- I did the magic trick by using the fact that I knew if the permutation was even or odd.

- The collection of all even permutations of $\{1, 2, 3, 4, 5\}$ is a group. It's called A_5 and contains 60 permutations.

Puzzles to play (1/2)

- The 15-puzzle

- The M_{12}-puzzle



Puzzles to play (2/2)

- Top spin puzzle

- The Rubik's cube.

Format

- The puzzle is scrambled.
- The goal is to unscramble it by using a sequence of basic moves.
- The collection of all possible moves is a group.

Homework

1. Solve the 15 puzzle in 200 or fewer moves.

2. Read the solution to the M_{12} puzzle and then solve it.

In each case, get a picture of the solved puzzle (and the move count for the 15 puzzle) and hand it in or e-mail it to me.