Conway's Soldiers

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Math 165: Freshman-only Math Seminar

Materials and goals

- Each of you should get a copy of the instructions, a game board, and at least 20 paper clips.

- Please let me know if you get a piece 4 squares above the line. If you do, write down how you did it (HW problem).

- There is a 100 dollar prize for the first student to get a piece 5 square above the line. (The prize may only be claimed before I explain it).

Questions

- How far above the line were you able to get a piece?
- Do you think you can get a piece as high as you like?
- What does this game have to do with mathematics?

Potential

- Let's determine what is necessary to get a piece 5 square above the line.

- To do this, we're going to define the *potential* of an arrangement of pieces, and show that the potential doesn't increase in the process of making a move.

Definitions

Definition

If $a_1, a_2, \ldots$ is a sequence of real numbers, the infinite series $\sum_{n=}^{\infty} a_n$ is defined by

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) \, .$$

If the limit exists, the series converges, otherwise the series diverges.

- Let $\phi = \frac{\sqrt{5}-1}{2} \approx 0.618$. Then,

$$\phi^2 + \phi = \frac{5+1-2\sqrt{5}}{4} + \frac{\sqrt{5}-1}{2} = \frac{6-2\sqrt{5}}{4} + \frac{-2+2\sqrt{5}}{4} = 1.$$

Definition of potential (1/2)

- Pick a square S five squares above the line. The distance from a square to S is the number of horizontal or vertical moves necessary to go from there to S.

			0			
		2	1	2		
		3	2	3		
		4	3	4		
		5	4	5		
	7	6	5	6	7	
	8	7	6	7	8	

Definition of potential (2/2)

- The potential of an arrangement is the quantity

$$\sum_{i=0}^{\infty} n(i)\phi^{i}$$

where $n(i)$ is the number of pieces on squares with a distance i to S.

- Example (on the board).

Potential doesn't decrease

- Fact: After a move is made, the potential is either the same, or it goes down.

- Reason: If we the jump is made in the direction of S, we'll start with two pieces with distances m and $m-1$. After, we'll have one piece with distance $m-2$.

- Since $\phi^2 + \phi = 1$, if we multiply by ϕ^{m-2} we get

$$\phi^m + \phi^{m-1} = \phi^{m-2}.$$

Biggest potential

- How big can the potential of an initial configuration be?

- The biggest possible potential happens when *every* square below the line has a piece on it.

- The middle column has potential

$$\phi^5 + \phi^6 + \phi^7 + \phi^8 + \cdots .$$

Summation formula (1/2)

- Fact: If $-1 < x < 1$, then

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

- Proof: Let $S_n = 1 + x + x^2 + \cdots + x^n$. The left hand side is $\lim_{n\to\infty} S_n$.

- We have

$$\begin{aligned} xS_n &= \qquad\qquad x + x^2 \quad + x^3 + \cdots + x^n + x^{n+1} \\ -S_n &= -1 \qquad -x - x^2 \quad -x^3 - \cdots - x^n \\ (x-1)S_n &= x^{n+1} - 1 \\ S_n &= \frac{x^{n+1}-1}{x-1}. \end{aligned}$$

Summation formula (2/2)

- As $x \to \infty$, $x^{n+1} \to 0$, and so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{x^{n+1} - 1}{x - 1} = -\frac{1}{x - 1}.$$

- We have $-\frac{1}{x-1} = \frac{1}{1-x}$. QED

Potential computation (1/2)

- Thus, the middle column has potential

$$\begin{aligned}\phi^5 + \phi^6 + \cdots &= \phi^5 \left(1 + \phi + \phi^2 + \cdots\right) \\ &= \phi^5 \cdot \frac{1}{1-\phi} \\ &= \phi^5 \cdot \frac{1}{\phi^2} = \phi^3.\end{aligned}$$

- The columns next to that have potential

$$\phi^6 + \phi^7 + \cdots = \phi(\phi^5 + \phi^6 + \cdots +) = \phi(\phi^3) = \phi^4.$$

Potential computation (2/2)

- The total potential is

$$\begin{aligned}\phi^3 + 2\phi^4 + 2\phi^5 + 2\phi^6 + \cdots &= \phi^3 + 2\phi^4(1 + \phi + \phi^2 + \cdots) \\ &= \phi^3 + \frac{2\phi^4}{1-\phi} \\ &= \phi^3 + \frac{2\phi^4}{\phi^2} \\ &= \phi^3 + 2\phi^2 \\ &= (\phi^3 + \phi^2) + \phi^2 \\ &= \phi + \phi^2 \\ &= 1.\end{aligned}$$

- This is the same as $\phi^0 = 1$, the potential of the arrangement of a single piece S.

Summary

- The largest the potential of any initial arrangement can be is 1.

- The potential doesn't go down in the process of making a move.

- The only way to get a piece to the 5th square is to start with a piece on ALL of the squares below the line.

- Whether or not you can do it depends on how you define the rules for the infinite game.

Homework problems

- Get a piece 4 squares above the line. Indicate the initial starting configuration, and at each step which piece moves where.

- Read the webpage
`http://www.chiark.greenend.org.uk/~sgtatham/solarmy/`
and summarize their argument that the 5th square can be reached with an infinite number of moves. Comment on whether or not you think their formulation of the problem is reasonable (in particular, whether the move sequence they call a "whoosh" is valid).