The Mathematics of Voting

Math 165

Jason Parsley

WAKE FOREST
UNIVERSITY

Department of Mathematics

Winston Salem, NC

28 October 2010

WAKE FOREST
UNIVERSITY
Department of Mathematics

Voting for 2 candidates

Today, we talk about voting, which may not seem mathematical.

President of the Math TA's

Let's say there's an election which has just 2 candidates running for office.
Say it's *President of the Math TA's*, a highly-coveted (and highly-fictional) office, and our two candidates are the 113 TA's,

Austin Quinn

How do we decide who wins?

Everyone gets to vote, and the most votes wins. This is

Majority rule – a candidate wins if and only if he has a majority of the votes cast.

Wake Forest University
Department of Mathematics

Other 2 candidate voting systems

There are other options besides majority rule:

1. we could require that a candidate needs 60% of the vote to win; otherwise the position is undecided until the next election. (a *quota system*)
2. Maybe Dr. Rouse's grant is funding the President TA's salary. It might be reasonable for him to just choose the winner. In this case, we would call him a *dictator*.
3. Maybe the candidates have different thresholds. For example, to override a presidential veto, the House must have 2/3 majority; meanwhile only 33.4% is needed to confirm the veto.

- An **anonymous** system treats all voters the same.
- A **neutral** system treats all candidates the same.
- In a **monotone** system, if a winning candidate were to suddenly receive more votes, she must still win. (Example: Survivor & voting off the island.)

May's Theorem

But for 2-candidate elections, everything works well if we impose these 3 conditions.

May's Theorem

For a 2-candidate election, any voting system that is anonymous, neutral, and monotone, must be some quota system.

However, for 3 (or more) candidates, voting becomes really complicated.

We will give 5 analogous conditions to anonymous, neutral, and monotone. It is *impossible* for any voting system to satisfy all 5, a result known as Arrow's Impossibility Theorem.

Wake Forest University
Department of Mathematics

Examples of voting systems for ≥ 3 candidates

We could look at the voters' top choice among the candidates. This leads to 3 systems:

1. **Majority Rule** A candidate wins if she gets a majority of the first-place votes. If no candidate gets a majority, the position is undecided until the next election (a *runoff*).
2. **Plurality Rule** A candidate wins if she gets the most first-place votes (a *plurality* of the votes).
3. **Plurality Rule with quota** A candidate wins if she gets the most first-place votes (a *plurality* of the votes), and gets at least 40% of the total. (NC uses this for primary elections.)

Voting for sports teams

If we asked voters (usually sportswriters) to vote for the best college basketball team, they provide more than their first choice. They list their top 25, in order.

Example.

First-place votes are worth 25 points, second-place votes are worth 24 pts, . . . , 25th place votes are worth 1 point. The team with the most points is ranked first.

Top-25 polls like this are modified examples of **Borda count**.

Borda count

For n candidates, each voter ranks all n candidates in order.

First-place votes are worth n, second-place votes $n-1$, etc. The candidate with the most points wins.

Kiribati uses Borda count to elect their president.

Our example: voting for Halloween costumes

As a class, we're going to vote today. Rather than voting for politicians or sports teams, let's consider Halloween costumes. National Geographic published the top 10 most popular costumes for 2010. We will decide between

`cat, pirate, vampire, witch`

The others on their list are (in order)

nurse, vixen, zombie, fairy, athlete, Batman, Dracula

Our vote

Suppose a (female) friend of yours is looking for a costume idea, and you want to give her your favorite suggestions, out of these four options. We will pass around ballots, and you should rank the four costumes in order.

We will examine these using plurality rule, Borda count, and a few other voting systems.

Instant-Runoff Voting

In Instant-Runoff Voting*, we look only at first-place votes.

- The candidate(s) with the fewest number of first place votes are eliminated in the first round
- Then, everyone's preference list is renumbered from 1 to $n-1$, with the eliminated candidate removed. The voters who chose the eliminated candidate now have their vote count for their second-choice.
- Repeat until only 1 candidate is left; she wins.

abbreviated IRV, also called *Hare method*

Wake Forest University
Department of Mathematics

So why use IRV?

In the 2008 Democratic primary for Secretary of Labor, 4 candidates ran. Each received 24-27% of the vote.

The top 2 competed in a runoff election in mid-June.
Donnan, the original winner, still won (by a larger margin).

Negatives:
only 63,662 people voted
running an election is expensive – this runoff cost state taxpayers $3.5-4 million.

This runoff would have been avoided had we used instant-runoff voting

- NC is using IRV for judicial races this fall
- Olympics select host city using (slightly modified) IRV method.

Wake Forest University
Department of Mathematics

Pairwise comparison

We look at each candidate's record head-to-head versus each other. The one with the best record wins.

What is the optimal voting system?

Many other voting systems exist to decide a winner. Which one is best? (It would be great to find some criteria like we did for 2-candidate elections, and show these lead us to an optimal solution.)

Here are some criteria, which we hope any system satisfies.

- universality
- monotonicity
- Independence of Irrelevant Alternatives (IIA)
- un-imposed
- no dictator

IIA considers how we rank a pair candidates A and B. If a voter adjusts her preference of a 3rd candidate C on her ballot, this should not affect how A does against B.

Wake Forest University
Department of Mathematics

Arrow's Impossibility Theorem

Kenneth Arrow, a mathematically-inclined economist at Stanford, proved in the 1960's that it was impossible to satisfy all of these.

Arrow's Impossibility Theorem

For an election with more than 2 candidates, it is impossible to satisfy

- universality (all ranking orders are possible)
- monotonicity
- Independence of Irrelevant Alternatives (IIA)
- un-imposed (no outcome is known in advance)
- no dictator

He won the 1972 Nobel Prize in Economics, based in part on this result.

Wake Forest University
Department of Mathematics

Best-Worst Defeat

Some of the other voting systems have a mathematical or economic flair. Let's look at one of my favorites:

Best-Worst Defeat

We will make a *comparison matrix* comparing all the candidates versus each other:

	C	P	V	W
C	0			
P		0		
V			0	
W				0

Comparison Matrix for our election

Best-Worst Defeat

	C	P	V	W
C	0			
P		0		
V			0	
W				0

In the row with C and column with P record the difference between the number of voters who preferred C over P minus the number who preferred P over C.

(This produces a *skew-symmetric matrix*.)

Take the lowest number in each row ... this is the candidate's *worst defeat*. The winner is the one with the highest of these, that is, the best *worst defeat*.