THE BIOT-SAVART OPERATOR AND ELECTRODYNAMICS ON BOUNDED SUBDOMAINS OF THE THREE-SPHERE

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ABSTRACT

The Biot-Savart operator and electrodynamics on bounded subdomains of the three-sphere

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We study the generalization of the Biot-Savart law from electrodynamics in the presence of curvature. We define the integral operator BS acting on all vector fields on subdomains of the three-dimensional sphere, the set of points in R^4 that are one unit away from the origin. By doing so, we establish a geometric setting for electrodynamics in positive curvature. When applied to a vector field, the Biot-Savart operator behaves like a magnetic field; we display suitable electric fields so that Maxwell's equations hold. Specifically, the Biot-Savart operator applied to a "current" V is a right inverse to curl; thus BS is important in the study of curl eigenvalue energy-minimization problems in geometry and physics. We show that the Biot-Savart operator is self-adjoint and bounded. The helicity of a vector field, a measure of the coiling of its flow, is expressed as an inner product of BS(V) with V. We find upper bounds for helicity on the three-sphere; our bounds are not sharp but we produce explicit examples within an order of magnitude. In all instances, the formulas we give are geometrically meaningful: they are preserved by orientation-preserving isometries of the three-sphere. Applications of the Biot-Savart operator include plasma physics, geometric knot theory, solar physics, and DNA replication.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

n.b., This is an updated version as of November 2005; it is not the version submitted to Penn's faculty in April 2004. The update consists of correcting a few minor typographical errors.

The Biot-Savart law in electrodynamics calculates the magnetic field B arising from a current flow V in a smoothly bounded region Ω of $\mathbb{R}^3$. Taking the curl of B recovers the flow V, provided there is no time-dependence for this system. The Biot-Savart law can be extended to an operator which acts on all smooth vector fields V defined in Ω. Cantarella, DeTurck, and Gluck investigated its properties in [5] and have found numerous connections to ideas in geometric knot theory, to energy minimization problems for vector fields, to plasma physics, and to DNA structure [4, 6, 8, 9, 10]. This dissertation investigates how this story changes in the presence of curvature by looking at subdomains Ω of the three-dimensional sphere, S^3.

In this work, we develop an approach to electrodynamics on such bounded subdomains via the Biot-Savart operator, which we define on $\Omega \subset S^3$. We provide integral formulas for Maxwell's equations and derive a useful correlation between the Biot-Savart and curl operators. We investigate applications to the helicity of vector fields and provide upper bounds on helicity values. We conclude by mentioning possible applications to energy-minimization problems for vector fields and also to a problem in solar physics.

Our formulas are geometrically meaningful, in that their integrands are preserved by orientation-preserving isometries of S^3. Though verifying that Maxwell's equations hold on orientable 3-manifolds is an elementary exercise in differential forms, neither a literature search nor a search via Google uncovered any geometric formulas for electrodynamics in the presence of curvature.

The Biot-Savart operator in Euclidean space is defined, for $x, y \in \mathbb{R}^3$ as

$$BS(V)(y) = \int_\Omega V(x) \times \nabla\phi(x,y)\, dvol_x \quad ,$$

for a current flow V on a compact subdomain Ω. The function $\phi(x,y) = -\frac{1}{4\pi}\frac{1}{|y-x|}$ is the fundamental solution to the Laplacian.

The integral formula for the Biot-Savart operator in Euclidean space requires the addition of vectors lying in different tangent spaces. To obtain an analogous formula on the 3-sphere, we must decide how to move tangent vectors among tangent spaces. Two natural choices exist: parallel transport along a minimal geodesic or left translation (or right translation) using the group structure of S^3 viewed as the group of unit quaternions or $SU(2)$. Each method has its advantages and disadvantages; wherever convenient, we use the more illustrative method. We define the Biot-Savart operator on the 3-sphere as an integral using each transport method.

$$\begin{aligned} BS(V)(y) &= \int_\Omega P_{yx}V(x) \times \nabla\phi(\alpha)\, dx \\ BS(V)(y) &= \int_\Omega L_*V(x) \times \nabla\phi_0(\alpha)\, dx - \frac{1}{4\pi^2}\int_\Omega L_*V(x)\, dx + 2\nabla \int_\Omega L_*V(x) \times \nabla\phi_1(\alpha)\, dx \end{aligned}$$

Here, P_{yx} denotes parallel transport from x to y and L_* denotes left-translation from x to y. Let $\alpha(x,y)$ be the distance on the three-sphere between x and y; then the potential functions above are

$$\begin{aligned} \phi(\alpha(x,y)) &= -\frac{1}{4\pi^2}(\pi - \alpha)\csc(\alpha) \\ \phi_0(\alpha(x,y)) &= -\frac{1}{4\pi^2}(\pi - \alpha)\cot(\alpha) \\ \phi_1(\alpha(x,y)) &= -\frac{1}{16\pi^2}\alpha(2\pi - \alpha) \quad . \end{aligned}$$

Electrodynamics on the entire 3-sphere was developed in [14], with applications to geometric knot theory and to the helicity of vector fields. Electrodynamics on compact subdomains (with boundary) of S^3 is the correct analogue of the Euclidean setting, and it raises a rich and interesting set of issues:

- The Hodge Decomposition Theorem for vector fields is more complicated than for the three-sphere, because curl is no longer a self-adjoint operator and divergence is no longer the (negative) adjoint of gradient.
- Current flows on bounded domains can deposit electric charge on boundaries and thereby affect Maxwell's equations.
- Nonsingular current flows can be restricted to tubular neighborhoods of knots, enabling connections between the writhing number of the core knot and both the helicity and flux of these flows.

Using these explicit formulas, we obtain integral versions of Maxwell's equations; in particular,

Theorem 1.1. *The divergence of $BS(V)$ is zero.*

Theorem 1.2.

$$\begin{aligned}\nabla_y \times BS(V)(y) \quad &= \quad \begin{pmatrix} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{pmatrix} \\ &- \nabla_y \int_\Omega (\nabla_x \cdot V(x))\, \phi_0 \; dvol_x \\ &+ \nabla_y \int_{\partial\Omega} (V(x) \cdot \hat{n})\, \phi_0 \; darea_x\end{aligned}$$

A useful consequence of this theorem is that for V divergence-free and tangent to the boundary, the curl operator acts as a left inverse to the Biot-Savart operator. Any such V in this space that is also an eigenfield of BS must furthermore be an eigenfield of curl. The eigenvalue for curl is precisely the reciprocal of the eigenvalue for Biot-Savart, i.e., if $BS(V) = \lambda V$, then $\nabla \times V = \frac{1}{\lambda} V$.

We also show that BS is a bounded, self-adjoint operator. We describe its image and find its kernel.

Theorem 1.3. *The kernel of the Biot-Savart operator on Ω is precisely the subspace of gradients that are always orthogonal to the boundary $\partial\Omega$.*

The *helicity* of a vector field measures the extent to which the field lines wrap and coil around each other. It was introduced by Woltjer [28] in 1958 and named by Moffatt [19] in 1969. Helicity is conveniently expressed in Euclidean space as the L^2 inner product $H(V) = \langle V, BS(V)\rangle$. We present the corresponding integral formula for the helicity of vector fields on S^3; the formula is again invariant under isometries.

The helicity of a vector field is bounded by its L^2 energy:

Theorem 1.4. *Let R be the radius of a ball in S^3 with the same volume as Ω. Then for any vector field $V \in VF(\Omega)$, we have bounds on $BS(V)$ and the helicity of V as follows:*

$$|H(V)| \quad \leq \quad N(R)\langle V, V\rangle \quad ,$$

where $N(R) = \frac{1}{\pi}\left[2(1-\cos R) + (\pi - R)\sin R\right]$.

Chapter 2

Background and history

This chapter begins with a brief history of the 19th century experiments that led to the Biot-Savart law in electrodynamics. Next is a description of the extension of this law to an operator on all vector fields on $\mathbb{R}^3$; we mention several results about the Biot-Savart operator in Euclidean space. In the next section, we discuss the helicity of vector fields on $\mathbb{R}^3$ and its connections to the Biot-Savart operator. Again several results about helicity are described for Euclidean vector fields. Finally, the chapter ends with a section on electrodynamics results on the three-sphere and hyperbolic three-space.

2.1 Electrodynamics history: Biot and Savart's work

Little was known until 1820 about the interplay between electric current and magnetism. That year, Oersted discovered that moving an electric charge generated an effect on compass needles; indeed compass needles had been previously observed to wobble during thunderstorms. His discovery was communicated to the French Academie des Sciences on September 11, 1820. Within a week, Ampere showed that two parallel wires carrying currents would attract each other if the currents flowed in the same direction, and would repel each other if the currents flowed in opposing directions.

On October 30, 1820, Jean-Baptiste Biot (1774-1862) and his junior colleague Felix Savart (1791-1841) performed a landmark experiment, described in [3]. Starting with a long vertical wire and a magnetic needle some horizontal distance apart, they showed that running a current through the wire caused the needle to move. After a suitable transient time, the needle settled into a stable position resulting from the magnetic force induced by the current. They showed that this force was perpendicular to the plane spanned by the wire and the line connecting needle to wire, and furthermore that the intensity of the force was inversely proportional to the distance between wire and needle.

After these observations, Biot shared them with Laplace, and they deduced the force exerted by each small section of the wire. Biot and Savart conducted a second experiment to test these ideas. This time they used a bent wire in their setup. They knew that the force intensity at the needle location y due to the current through a point x was $\dfrac{f(\theta)}{R^2}$, where R is the distance $|x-y|$, and θ is the angle between the wire and the vector $x-y$.

In modern times, this result carries their names as the Biot-Savart law: For a steady current J inside a region $\Omega \subset \mathbb{R}^3$, where Ω could represent a curve, surface, or volume, the magnetic field associated to J is

$$B(y) = \frac{\mu_0}{4\pi} \int_\Omega J(x) \times \frac{y-x}{|y-x|^3}\, dx \quad . \tag{2.1}$$

See Figure 2.1 for a depiction of the integrand. The constant μ_0 represents the *permeability of free space*, $\mu_0 = 4\pi \times 10^{-7}\, N/A^2$ (Newtons per amp squared). Magnetic force is measured in terms of Teslas, $T = N/(A \cdot m)$. The earth's magnetic field is approximately $5 \times 10^{-5}\, T$; one Tesla represents a strong magnetic field one might encounter in a laboratory. We choose to work in units such that $\mu_0 = 1$. In addition, we will always think of Ω as being three-dimensional, whether it is considered as a subset of $\mathbb{R}^3$ as in this section or as a subset of S^3 as in later chapters.

Ampere conjectured, correctly, that all magnetic effects are due to a current flow. Another of his contributions is Ampere's Law, which we state in two different versions. Viewed as a differential,

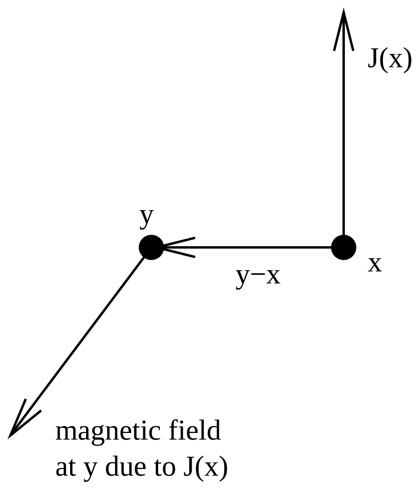

Figure 2.1: The integrand of the Biot-Savart law.

Ampere's Law says

$$\nabla \times B = \mu_0 J \quad .$$

Integrating both sides over the domain Ω and invoking Stokes' Theorem produces the integral version of Ampere's Law:

$$\oint_{C=\partial\Sigma} B \cdot d\vec{s} \;=\; \mu_0 \int_{\Sigma} J \cdot \hat{n}\; dx \quad .$$

In other words, the circulation of the magnetic field around a closed curve C is equal to the flux of the current J through any surface Σ bounded by C.

Specifically for a closed curve C' on the boundary of Ω, we may choose Σ as a surface lying outside Ω; then the right-hand side of Ampere's Law vanishes. Thus, a magnetic field has zero circulation about a curve on the boundary,

$$\oint_{C'} B \cdot d\vec{s} = 0 \quad .$$

Electrodynamics continued to develop in the early 19th century. In 1831, Faraday discovered that moving a magnetic field generates an electric field; 11 years earlier Oersted had discovered that the motion of electric charge generates a magnetic field. Maxwell and Lorentz added further results and polish to the subject. Maxwell's equations describe the curl and divergence of electric and magnetic fields. The differential version of Ampere's Law appears as one of them. We list Maxwell's equations in section 4.5 and show that through our definitions they hold on the three-sphere and its subdomains.

For a more thorough history of electrodynamics, see Tricker's book [25]; this book along with an unpublished historical report by Cantarella, DeTurck, and Gluck are the primary sources for the material in this section.

2.2 The Biot-Savart operator on $\mathbb{R}^3$

Let J be a smooth current contained in a subdomain Ω of $\mathbb{R}^3$. Currents obey the continuity equation,

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} \quad ,$$

where $\rho(x,t)$ is the volume charge density. The continuity equation implies that all steady currents are divergence-free. To be contained in Ω, the current is also tangent to the boundary. We call **fluid knots** those vector fields that are both divergence-free and tangent to the boundary. The current J then is a fluid knot. Fluid knots cannot have any gradient component due to the Hodge Decomposition Theorem for vector fields; see [7] for a recent exposition. We make use of this theorem on S^3 later in section 3.4.

Cantarella, DeTurck, and Gluck [5] extended the Biot-Savart formula on currents to be an integral operator acting on all smooth vector fields defined on Ω, a space we denote $VF(\Omega)$. This Biot-Savart operator, $BS : VF(\Omega) \to VF(\Omega)$, is expressed as

$$BS(V)(y) = \frac{1}{4\pi} \int_{\Omega} V(x) \times \frac{y-x}{|y-x|^3} \, dx \quad . \tag{2.2}$$

They prove four main theorems and provide explicit versions of Maxwell's equations.

Proposition 2.1 (CDG, [5]). *Let $V \in VF(\Omega)$.*

$$\begin{aligned}\nabla_y \times BS(V)(y) &= \begin{pmatrix} V(y) & \text{inside } \Omega \\ & \\ 0 & \text{outside } \Omega \end{pmatrix} \\ &\quad + \nabla_y \int_{\Omega} \frac{\nabla_x \cdot V(x)}{|y-x|}\, dvol_x \\ &\quad - \nabla_y \int_{\partial\Omega} \frac{V(x)\cdot \hat{n}}{|y-x|}\, darea_x\end{aligned}$$

This proposition proves one direction of the following theorem.

Theorem 2.2 (CDG, [5]). *The equation $\nabla \times BS(V) = V$ holds in Ω if and only if V is divergence-free and tangent to the boundary.*

Known for almost two centuries, Ampere's Law guarantees that curl is a left inverse to BS for a fluid knot V. Theorem 2.2 states that this is the only case when curl acts as a left inverse. Therefore the eigenvalue problems for the Biot-Savart operator, which arise in studying helicity and in plasma physics, cannot be converted in general to eigenvalue problems for the curl operator. However, when we restrict to vector fields that are fluid knots, we can convert eigenvalue problems from Biot-Savart to curl, as Arnold does in [1].

Theorem 2.3 (CDG, [5]). *The kernel of the Biot-Savart operator is precisely the space of gradient vector fields that are orthogonal to the boundary of Ω.*

Theorem 2.4 (CDG, [5]). *The image of the Biot-Savart operator is a proper subspace of the image of curl, and its orthogonal projection into the subspace of "fluxless knots" is injective.*

The subspace of *fluxless knots* are defined as fluid knots which have zero flux through every cross-sectional surface $(\Sigma, \partial\Sigma) \subset (\Omega, \partial\Omega)$.

Theorem 2.5 (CDG, [5]). *The Biot-Savart operator is a bounded operator; hence it extends to a bounded operator on the L^2 completion of its domain, where it is both compact and self-adjoint.*

As an application, we show that the Biot-Savart operator is manifest in Gauss's formula for linking number. In a half-page paper [17] in 1833, Gauss gave the linking number of two knots (simple closed curves) K_1, K_2 in $\mathbb{R}^3$ as

$$L(K_1, K_2) = \frac{1}{4\pi} \int_{K_1 \times K_2} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x-y}{|x-y|^3} \, ds \, dt$$

By manipulating the integral, we see the Biot-Savart integrand taken over the curve K_1.

$$\begin{aligned} L(K_1, K_2) &= \int_{K_2} \left[\frac{1}{4\pi} \int_{K_1} \frac{dx}{ds} \times \frac{y-x}{|x-y|^3} \, ds \right] \cdot \frac{dy}{dt} \, dt \\ L(K_1, K_2) &= \int_{K_2} BS\left(\frac{dx}{ds}\right) \cdot \frac{dy}{dt} \, dt \end{aligned}$$

In the last equation, we loosen the definition of BS so that the domain of integration of $BS\left(\frac{dx}{ds}\right)$ is the curve K_1, rather than a three-dimensional subdomain of Euclidean space.

2.3 Helicity

The **helicity** of a vector field on a domain Ω in $\mathbb{R}^3$ is a measure of the extent to which the field lines wrap and coil around one another. Denote the L^2 inner product of vector field on Ω as $\langle V, W \rangle = \int_\Omega V \cdot W \, dx$. Helicity can be defined in terms of the Biot-Savart operator:

$$\begin{aligned} H(V) &= \langle V, BS(V) \rangle \\ H(V) &= \frac{1}{4\pi} \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} \, dx \, dy \end{aligned}$$

Helicity was introduced by Woltjer [28] in 1958 and named by Moffatt [19] in 1969. For divergence-free vector fields, helicity is the same as Arnold's asymptotic Hopf invariant, described in [1]. It has many applications in plasma physics, geometric knot theory, magnetohydrodynamics, and energy minimization problems for vector fields.

There is an analogous concept to helicity of vector fields for curves. The **writhing number** of a smooth, simple curve $K \subset \mathbb{R}^3$, which is parameterized by arclength, is

$$Wr(K) = \frac{1}{4\pi} \int_{K \times K} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x-y}{|x-y|^3} \, ds \, dt \quad .$$

It measures the extent to which K wraps and coils around itself. The writhing number was introduced by Călugăreanu ([11, 12, 13]) in 1959-1961 and was named by Fuller [16] in 1971. It has applications in studying how knotting of DNA affects its replication [23].

Both helicity and writhing number are analogues of Gauss' linking integral formula mentioned above. However, unlike the linking number, neither helicity nor writhing number is necessarily integer-valued.

Călugăreanu proved a relation between linking and writhing numbers, namely

$$Link \;=\; Twist \;+\; Writhe \quad ,$$

which was generalized in [27]. The specific setup is as follows: take a smooth knot $K \subset \mathbb{R}^3$, which is parameterized by arclength s. Let $\nu(s)$ be a normal vector field along K. Construct a smooth ribbon from K by extending it some small distance in the direction of $\nu(s)$. The new edge of the ribbon defines a new knot K'. Călugăreanu showed that the difference between the linking number $L(K, K')$ of the two curves on the ribbon and the writhing number of K was equal to the **twist** of K, which is defined as

$$Tw(K, \nu) = \frac{1}{2\pi} \int_K \frac{dx}{ds} \times \nu(s) \cdot \frac{d\nu}{ds}\, ds \quad . \tag{2.3}$$

In 1984, Berger and Field proved a formula connecting helicity and writhing number. Let Ω_K be a tubular neighborhood of a smooth knot K; let V_K be a smooth vector field on Ω_K that is parallel to K and depends only on the distance from K. Such a vector field is necessarily divergence-free. Then,

Theorem 2.6 (Berger-Field, [2]).

K

$$H(V) \;=\; Flux(V)^2\; Wr(K) \quad .$$

Here

If a plasma is injected into a containment vessel Ω, it will turbulently flow for a short time and quickly shed some of its energy. This flow is described by a version of the Navier-Stokes equations that takes magnetic effects into account. Eventually, the plasma reaches a minimal energy state and stabilizes to a steady flow. This process known as **plasma relaxation**. During this process, the helicity of the plasma decays on a much slower time scale than the energy does; it is fair to approximate helicity as a constant during plasma relaxation.

We therefore may solve for the steady plasma flow as the vector field on Ω with minimum energy $\langle V, V \rangle$ for a fixed helicity value. This is known as the **Woltjer problem**, named after the astrophysicist Ludewijk Woltjer. We ask that V be divergence-free and tangent to $\partial\Omega$ so that it properly models a steady plasma flow. The Woltjer Problem can be solved via Lagrange multipliers and is far more tractable than using the magnetohydrodynamics version of the Navier-Stokes equations to determine the plasma flow. The Woltjer Problem has been solved analytically for spherically symmetric domains and for solid tori in Euclidean space [10, 6].

For Ω not simply-connected, an additional constraint beyond helicity is required to properly model the stable plasma flow. Fix the **flux** of V through a basis of cross-sectional surfaces for $H_2(\Omega, \partial\Omega)$. Then the stable plasma flow resulting on Ω is well approximated by the solution to the **Taylor problem:** *among all divergence-free vector fields, tangent to $\partial\Omega$, with fixed flux and helicity, find the one with minimum energy.* Taylor [24] gave solutions to this problem on a solid flat torus and showed that they exhibited a **reversed field pinch**, a plasma flow where the paritcles near the boundary move in the reverse direction to the main axis of the flow. The reversed field pinch has been observed experimentally but does not appear in Woltjer solutions. Taylor's results were extended in [21].

One more energy-minimization problem allows for choice of domain. **Optimal domains problem:** *Among all compact subdomains $\Omega \subset \mathbb{R}^3$ of a given volume, and among all divergence-free vector fields tangent to $\partial\Omega$ with prescribed helicity, find the vector field with the minimum energy and find the domain containing it.*

Cantarella, DeTurck, and Gluck [9] proved several results regarding the optimal domains problem; together with them, we have numerical results and a conjectured solution.

2.4 Electrodynamics results on S^3 and H^3

Electrodynamics on the three-sphere appears in a forthcoming paper by DeTurck and Gluck [14]. The author worked closely with them in the early stages of this project as they defined the Biot-Savart operator on all of the three-sphere. The resulting formulas, which appear in section 4.2, are rightly credited to them, but we furnish our own independent proofs in that section.

Once these were established, the three of us split our labor. In this work, the author has focused on developing electrodynamics on bounded subdomains of the three-sphere, which introduces significant obstacles not found on the whole of S^3. They have focused on applications of the Biot-Savart operator to linking, writhing, and twisting in S^3. Furthermore they prove analogous results for hyperbolic three-space H^3. We summarize their results in this section.

We present their formulas for linking, writhing, and twisting in the next three results.

Theorem 2.8 (Linking integrals on S^3 and H^3). (DG, [14]) *Let K_1 and K_2 be smooth knots on the appropriate space; let $\alpha(x, y)$ be the distance between two points x and y in the appropriate space. Then their linking numbers are calculated by the following integrals.*

1. On the three-sphere using left-translation to move vector fields:

$$\begin{aligned} L(K_1, K_2) \quad = \quad & \frac{1}{4\pi^2} \int_{K_1 \times K_2} (L_{yx^{-1}})_* \frac{dx}{ds} \times \frac{dy}{dt} \cdot \nabla_y \phi(x, y) \; ds\, dt \\ & - \frac{1}{4\pi^2} \int_{K_1 \times K_2} (L_{yx^{-1}})_* \frac{dx}{ds} \cdot \frac{dy}{dt} \; ds\, dt \end{aligned}$$

Here, $\phi(\alpha(x, y)) = (\pi - \alpha) \cot \alpha$.

2. On the three-sphere using parallel transport to move vector fields:

$$L(K_1, K_2) \quad = \quad \frac{1}{4\pi^2} \int_{K_1 \times K_2} P_{yx} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \nabla_y \phi(x, y) \; ds\, dt$$

Here, $\phi(\alpha(x,y)) = (\pi - \alpha)\csc\alpha$.

3. On hyperbolic three-space using parallel transport to move vector fields:

$$L(K_1, K_2) \quad = \quad \frac{1}{4\pi}\int_{K_1\times K_2} P_{yx}\frac{dx}{ds}\times\frac{dy}{dt}\cdot\nabla_y\phi(x,y)\; ds\, dt$$

Here, $\phi(\alpha(x,y)) = \operatorname{csch}\alpha$.

The linking number is the same, of course, whether we use parallel transport or left-translation to move vector fields. Now the writhing of a curve can be defined by similar formulas.

Definition 2.9 (Writhing integrals on S^3 and H^3). (DG, [14]) *Let K be a smooth knot on*

The two definitions of twist on S^3 produce different values.

$$Tw_L(K,\nu) = Tw_P(K,\nu) - \frac{\ell}{2\pi}$$

But notice that the two methods do agree on the sum of twist and writhe.

$$Wr_L(K) + Tw_L(K,\nu) = Wr_P(K) + Tw_P(K,\nu)$$

DeTurck and Gluck conclude by extending Călugăreanu's result to S^3 and H^3. Let K be a smooth knot in the appropriate space. Define a ribbon about K via the normal field $\nu(s)$ and call the new edge K' as in section 2.3.

Theorem 2.11 (DG, [14]). *Link* equals *twist* plus *writhe.*

$$L(K,K') = Tw_*(K,\nu) + Wr_*(K)$$

The subscript $*$ in the above formula indicates that we are allowed to choose consistently either the parallel transport or the left-translation formulas on S^3.

Chapter 3

Vector calculus on S^3

Much of our work on the three-sphere is to be performed locally. The three-sphere S^3 is blessed with structure; it can be viewed as a subset of $\mathbb{R}^4$ or as a Lie group, either as $SU(2)$ or as the group of unit quaternions, $Sp(1)$. Taking advantage of these structures, we work with three orthonormal frames on S^3: (1) left-invariant vector fields, (2) spherical coordinates, and (3) toroidal coordinates.

In this chapter, we describe each of these frames and how they relate to one another. Next, we give the formulas for the operators gradient, divergence, curl, and Laplacian in terms of the three frames. Then, we describe the Hodge Decomposition Theorem for vector fields on S^3. After that, we detail the notion of a triple product of three vectors in $\mathbb{R}^4$. Following that, we describe two means of transporting vector fields between different tangent spaces on the three-sphere; either use the group structure to left translate vector fields or use the Riemannian connection to parallel transport them. In the next two sections, we develop a version of the Laplacian which operates on vector fields and describe its behavior. For closed manifolds, the inverse to this vector Laplacian operator exists and is known as the Green's operator. We also note that Appendix A contains a list of vector identities involving the vector operators which are then proven for all orientable Riemannian 3-manifolds.

3.1 Preliminaries

Let M^3 be an orientable, Riemannian 3-manifold possibly with boundary. Call $VF(M)$ the space of smooth vector fields on M. Denote the L^2 inner product of two functions f, g on a manifold M as $\langle f, g\rangle = \int_M fg\, dvol$, and denote the induced L^2 norm as $\|f\|$. Define the L^2 inner product of two vector fields $V, W \in VF(M)$ as $\langle V, W\rangle = \int_M V \cdot W\, dvol$, and denote the induced L^2 norm as $\|V\|$. We reserve the notation $|V(x)|$ to represent the length of V at the point x.

Let $[f]$ denote the average value of the function f on the three-sphere, i.e.,

$$[f] := \frac{1}{\mathrm{vol}(S^3)} \int_{S^3} f(x)\, dx = \frac{1}{2\pi^2} \int_{S^3} f(x)\, dx \quad .$$

Begin by viewing $S^3 \subset \mathbb{R}^4$. Let $\{x, y, u, v\}$ be standard Euclidean coordinates on $\mathbb{R}^4$. By writing $z = x + iy$ and $w = u + iv$, we can view $S^3 \subset \mathbb{C}^2$ as the set of points where $|z|^2 + |w|^2 = 1$. Also consider S^3 as the group $SU(2)$; the point $(z, w) \in S^3$ corresponds to the matrix

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \quad .$$

In particular, the point $(1, 0)$, the *north pole*, corresponds to the identity matrix in $SU(2)$.

We will often work with a *subdomain* Ω of S^3, by which we mean that $\Omega \subset S^3$ is a compact 3-manifold with piecewise smooth boundary.

Often we encounter functions $f(x, y)$ and vector fields $V(x, y)$ that depend upon two points in S^3. When performing the vector operations gradient, divergence, and curl, it is necessary to indicate at which point the differentiation should occur. We accomplish this by adding a subscript to the nabla operator. For example, $\nabla_y f(x, y)$ indicates that we take the gradient with respect to y coordinates; the resulting vector field lies in $T_y S^3$. We adopt this notation consistently throughout this work and apologize for any confusion with covariant derivatives.

3.2 Orthonormal frames on S^3

3.2.1 Left-invariant frame

The first orthonormal frame we consider comes from the Lie algebra $\mathfrak{su}(2)$. The basis of $\mathfrak{su}(2)$ given by

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & i \\ +i & 0 \end{bmatrix},$$

corresponds to three orthogonal tangent vectors at the north pole $(1,0) \in \mathbb{C}^2$ of S^3. Choose three left-invariant vector fields $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ so that they agree at the north pole with the above basis. In Euclidean coordinates on $\mathbb{R}^4$, this left-invariant frame is given by

$$\begin{aligned} \hat{u}_1 &= -y\,\hat{x} + x\,\hat{y} + v\,\hat{u} - u\,\hat{v} \\ \hat{u}_2 &= -u\,\hat{x} - v\,\hat{y} + x\,\hat{u} + y\,\hat{v} \\ \hat{u}_3 &= -v\,\hat{x} + u\,\hat{y} - y\,\hat{u} + x\,\hat{v} \end{aligned}$$

This framing induces the natural orientation on S^3 embedded in $\mathbb{R}^4$. These vector fields are known as **Hopf fields**. Let $\{\omega_1, \omega_2, \omega_3\}$ denote the corresponding orthonormal coframe field, e.g., $\omega_1 = -y\,dx + x\,dy + v\,du - u\,dv$. The volume form on S^3 is $dvol = \omega_1 \wedge \omega_2 \wedge \omega_3$.

Figure 3.1 depicts orbits of the Hopf field $\hat{u}_1$, as viewed in $\mathbb{R}^3$. It has one orbit along the circle $x^2 + y^2 = 1$, and another along the circle $u^2 + v^2 = 1$, which is projected onto the z-axis in the sketch.

Remark 3.1. The Lie bracket $[\hat{u}_i, \hat{u}_j] = 2\sigma_{ijk}\hat{u}_k$, where σ_{ijk} is the sign of the permutation (ijk) and is zero if (ijk) is not a permutation. Its nontrivial Lie brackets imply that the left-invariant frame does not form a coordinate system on S^3.

Remark 3.2. One could just as easily choose a frame consisting of right-invariant vector fields that agree at the north pole with the basis of $\mathfrak{su}(2)$ given above.

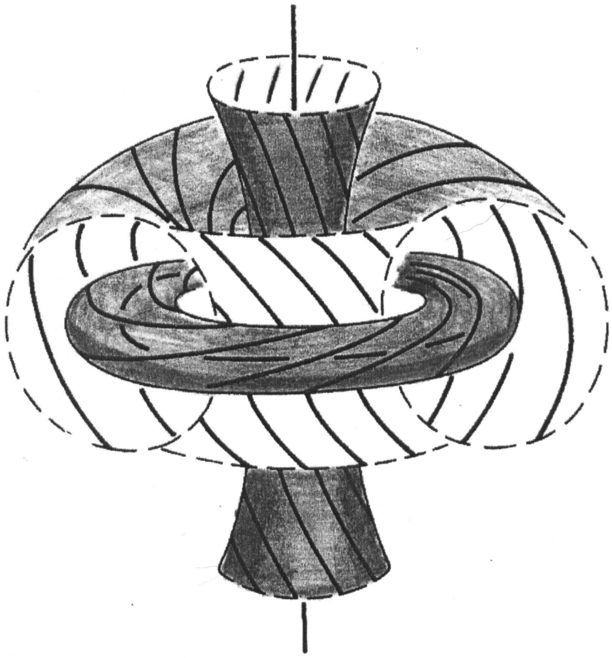

Figure 3.1: Orbits of the Hopf field $\hat{u}_1$.

We proceed to prove two results of interest for later work. For both of them, let Ω be a subdomain of the three-sphere.

Proposition 3.3. *Let U be a left-invariant smooth vector field on Ω and let W be any smooth vector field on Ω. Then,*

$$\nabla_W U = W \times U \quad .$$

Proof. Write U in terms of the left-invariant basis: $U = \sum_i a_i \, \hat{u}_i$, where the a_i are real constants. Also write W in terms of this basis: $W = W(x) = \sum_i w_i(x) \, \hat{u}_i$, where the $w_i(x)$ are real-valued functions.

We claim that $\nabla_{\hat{u}_i} \hat{u}_j = \sigma_{ijk} \hat{u}_k$, where σ_{ijk} is the sign of the permutation.

By the symmetries of the left-invariant fields $\hat{u}_i$, the covariant derivative anti-commutes; for instance, $\nabla_{\hat{u}_1} \hat{u}_2 = -\nabla_{\hat{u}_2} \hat{u}_1$. Thus, the Lie bracket

$$[\hat{u}_1, \hat{u}_2] = \nabla_{\hat{u}_1} \hat{u}_2 - \nabla_{\hat{u}_2} \hat{u}_1 = 2\nabla_{\hat{u}_1} \hat{u}_2 = 2\hat{u}_3 \quad .$$

As mentioned earlier, $[\hat{u}_1, \hat{u}_2] = 2\hat{u}_3$; hence, we have shown that $\nabla_{\hat{u}_1} \hat{u}_2 = \hat{u}_3$, and the other permutations follow likewise. Since $\nabla_{\hat{u}_i} \hat{u}_i = 0$, the claim is complete.

Now we can prove the proposition. We utilize the convention of summing over all repeated indices.

$$\begin{aligned} \nabla_W U &= \nabla_{w_i \hat{u}_i} a_j \hat{u}_j \\ &= w_i \nabla_{\hat{u}_i} a_j \hat{u}_j \\ &= w_i a_j \nabla_{\hat{u}_i} \hat{u}_j + w_i \hat{u}_i(a_j) \, \hat{u}_j \\ &= w_i a_j \sigma_{ijk} \hat{u}_k + 0 \end{aligned}$$

This last term is easily recognized as the cross-product $W \times U$, and the proof is complete. □

Corollary 3.4. *Let U be a left-invariant field defined on Ω, and let G be a gradient defined on Ω. Then,*

$$\nabla(U \cdot G) = [U, G] \quad .$$

Proof. Use vector identity (4) from Appendix A to begin:

$$\nabla(U \cdot G) \;=\; U \times (\nabla \times G) \;+\; G \times (\nabla \times U) \;+\; \nabla_U G \;+\; \nabla_G U$$

By the preceding proposition, the last term is $G \times U$. The first term on the right-hand side vanishes since $\nabla \times G = 0$. In the second term on the right, $\nabla \times U = -2U$, since U is a left-invariant field (see Table 3.1). Then,

$$\begin{aligned}
\nabla(U \cdot G) &= 0 \;+\; (G \times -2U) \;+\; \nabla_U G \;+\; G \times U \\
&= \nabla_U G \;-\; G \times U \\
&= \nabla_U G \;-\; \nabla_G U \\
&= [U, G]
\end{aligned}$$

□

3.2.2 Spherical coordinates

An n-dimensional sphere can be parameterized in terms of n different angular coordinates. We define the spherical coordinates $\{\alpha, \beta, \gamma\}$ as follows. Let α represent the distance on S^3 from a point to the north pole $(1, 0, 0, 0) \in \mathbb{R}^4$. The set of points that are the same distance from the north pole describes a two-sphere in S^3; let β and γ be the standard spherical coordinates on these two-spheres. In terms of Euclidean coordinates we have

$$\begin{aligned}
x &= \cos\alpha \\
y &= \sin\alpha\,\cos\beta \\
u &= \sin\alpha\,\sin\beta\,\cos\gamma \\
v &= \sin\alpha\,\sin\beta\,\sin\gamma \quad ,
\end{aligned}$$

where $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq \pi$, and $0 \leq \gamma \leq 2\pi$.

The standard vectors given by these coordinates do not have unit length. To establish an orthonormal frame, define

$$\hat{\alpha} = \frac{\partial}{\partial\alpha}, \quad \hat{\beta} = \frac{1}{\sin\alpha}\frac{\partial}{\partial\beta}, \quad \hat{\gamma} = \frac{1}{\sin\alpha\,\sin\beta}\frac{\partial}{\partial\gamma} \quad .$$

The corresponding orthonormal coframe is

$$\{d\alpha,\ \sin\alpha\, d\beta,\ \sin\alpha\,\sin\beta\, d\gamma\} \quad .$$

The volume form given by these coordinates is $dvol = \sin^2\alpha \sin\beta\, d\alpha \wedge d\beta \wedge d\gamma$. The volume of S^3 is

$$\int_{\gamma=0}^{2\pi}\int_{\beta=0}^{\pi}\int_{\alpha=0}^{\pi} \sin^2\alpha \sin\beta\ d\alpha\, d\beta\, d\gamma = 2\pi^2 \quad .$$

The transformation from spherical coordinates to the left-invariant frame is given by the orthogonal matrix M_1:

$$M_1 = \begin{bmatrix} \cos\beta & -\cos\alpha\sin\beta & -\sin\alpha\sin\beta \\ \sin\beta\cos\gamma & \cos\alpha\cos\beta\cos\gamma + \sin\alpha\sin\gamma & \sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma \\ \sin\beta\sin\gamma & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\cos\gamma & \sin\alpha\cos\beta\cos\gamma + \cos\alpha\cos\gamma \end{bmatrix} \quad ,$$

where

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} = M_1 \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = M_1 \begin{bmatrix} d\alpha \\ \sin\alpha\, d\beta \\ \sin\alpha\,\sin\beta\, d\gamma \end{bmatrix} \quad .$$

To transform from the left-invariant frame to spherical coordinates, simply use the transpose, M_1^T. The determinant of M_1 is $+1$, and so spherical coordinates preserve the natural orientation on S^3.

3.2.3 Toroidal coordinates

For our third orthonormal frame, view a point $(z, w) \in S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$. Write $z = x + iy$ and $w = u + iv$, or in polar form, $z = re^{i\theta}$ and $w = \rho e^{i\phi}$. Then the three-sphere is the set of points in $\mathbb{C}^2$ with $r^2 + \rho^2 = 1$. By letting $r = \cos\sigma$ and $\rho = \sin\sigma$, we establish toroidal coordinates $\{\sigma, \theta, \phi\}$

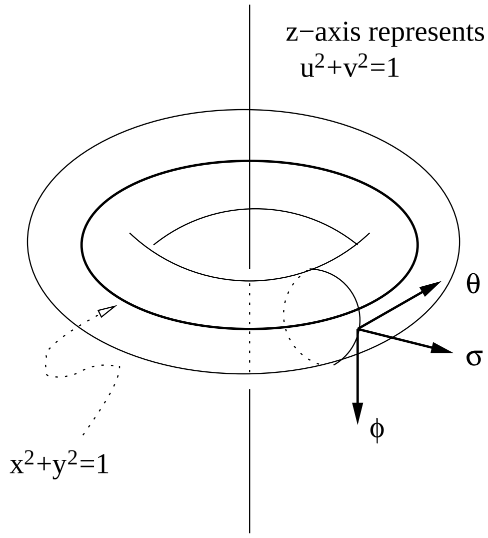

Figure 3.2: Toroidal coordinates.

on S^3. These are represented terms of Euclidean coordinates $(x, y, u, v) \in \mathbb{R}^4$ as

$$\begin{aligned} x &= \cos\sigma\cos\theta \\ y &= \cos\sigma\sin\theta \\ u &= \sin\sigma\cos\phi \\ v &= \sin\sigma\sin\phi \quad , \end{aligned}$$

where $0 \le \sigma \le \pi/2,\ 0 \le \theta \le 2\pi$, and $0 \le \phi \le 2\pi$. Figure 3.2 shows toroidal coordinates represented in $\mathbb{R}^3$; note that $\{\hat\sigma, \hat\theta, \hat\phi\}$ defines a *left-handed* orientation.

The coordinate σ foliates the three-sphere into tori, from which these coordinates obtain their name. For example, the *Clifford torus* in S^3 is the set of points where $r^2 = \rho^2 = 1/2$; in toroidal coordinates, the Clifford torus is given as $\{\sigma = \pi/4\}$. In the cases when $\sigma \equiv 0$ or $\sigma \equiv \pi/2$, the torus degenerates into a circle, either $x^2 + y^2 = 1$ or $u^2 + v^2 = 1$, respectively. These tori are integrable surfaces for the left-invariant vector field $\hat u_1$, defined in Section 3.2.1; Figure 3.1 depicts the flow of $\hat u_1$. In toroidal coordinates, $\hat u_1 = \cos\sigma\,\hat\theta - \sin\sigma\,\hat\phi$.

The standard vectors given by these coordinates do not have unit length. To establish an orthonormal frame, define

$$\hat{\sigma} = \frac{\partial}{\partial \sigma}, \quad \hat{\theta} = \frac{1}{\cos\sigma}\frac{\partial}{\partial \theta}, \quad \hat{\phi} = \frac{1}{\sin\sigma}\frac{\partial}{\partial \phi} \quad .$$

The corresponding orthonormal coframe is

$$\{d\sigma,\, \cos\sigma\, d\theta,\, \sin\sigma\, d\phi\} \quad .$$

The volume form given by these coordinates is $dvol = \cos\sigma \sin\sigma\, d\sigma \wedge d\theta \wedge d\phi$. As a check, the volume of S^3 again computes to $2\pi^2$.

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi}\int_{\sigma=0}^{\pi/2} \cos\sigma \sin\sigma \; d\sigma\, d\theta\, d\phi = 2\pi^2$$

The transformation from toroidal coordinates to the left-invariant frame is given by the orthogonal matrix M_2:

$$M_2 = \begin{bmatrix} 0 & \cos\sigma & -\sin\sigma \\ \cos(\theta-\phi) & \sin\sigma\sin(\theta-\phi) & \cos\sigma\sin(\theta-\phi) \\ -\sin(\theta-\phi) & \sin\sigma\cos(\theta-\phi) & \cos\sigma\cos(\theta-\phi) \end{bmatrix} \quad ,$$

where

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} = M_2 \begin{bmatrix} \hat{\sigma} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = M_2 \begin{bmatrix} d\sigma \\ \cos\sigma\, d\theta \\ \sin\sigma\, d\phi \end{bmatrix} \quad .$$

To transform from the left-invariant frame to toroidal coordinates, simply use the transpose, M_2^T. The determinant of M_2 is -1, which means that (σ, θ, ϕ) defines a left-handed frame on S^3. One can avoid this by using the frame (σ, ϕ, θ).

To go from spherical coordinates to toroidal ones, we can multiply by the orthogonal matrix $M_2^T M_1$.

3.3 Vector calculus formulas on S^3

On any Riemannian 3-manifold (M^3, g) there exists the following 1-1 correspondence between the space of vector fields $VF(M)$ and smooth differential forms $\Lambda^*(M)$:

$$\begin{array}{ccccccc}
\Lambda^0(M) & \xrightarrow{d} & \Lambda^1(M) & \xrightarrow{d} & \Lambda^2(M) & \xrightarrow{d} & \Lambda^3(M) \\
\| \uparrow & & \uparrow \Psi_1 & & \uparrow \Psi_2 & & \uparrow * \\
C^\infty(M) & \xrightarrow[grad]{} & VF(M) & \xrightarrow[curl]{} & VF(M) & \xrightarrow[div]{} & C^\infty(M)
\end{array}$$

The map Ψ_1 sends a vector field V to the 1-form $\Psi_1(V)(\cdot) = \omega_V(\cdot) = g(V, \cdot)$, hence the need for the Riemannian metric on M. The map Ψ_2 sends a vector field V to the 2-form $\Psi_2(V) = i_V dvol = dvol(V, \cdot, \cdot)$; it requires only the volume form $dvol$ on M. The map from functions to 3-forms is given by the Hodge star operator $* : f \mapsto f\, dvol$. Note that $\Psi_2 = *\Psi_1$, where here the Hodge star maps between 1-forms and 2-forms.

The formulas for the vector operators gradient, divergence, curl, and Laplacian are written as

$$\nabla f = \Psi_1^{-1}(df) \tag{3.1}$$

$$\nabla \cdot V = *d\,[\Psi_2(V)] = *d * \Psi_1(V) \tag{3.2}$$

$$\nabla \times V = \Psi_2^{-1}\,(d\Psi_1(V)) = \Psi_1^{-1}\,(*d\Psi_1(V)) \tag{3.3}$$

$$\Delta f = *d\left[\Psi_2(\Psi_1^{-1}(df))\right] = *d * df \tag{3.4}$$

Formulas for these operators appear in Table 3.1 for each of our three frame fields. All of the calculations are straightforward via formulas (3.1)-(3.4). We compute the divergence and curl of $\hat{u}_1$ as a sample calculation.

To calculate the divergence and curl of $\hat{u}_i$, we utilize the orthonormal left-invariant coframe $\{\omega_1, \omega_2, \omega_3\}$. These forms are not derived from a coordinate system, so they are not necessarily exact. In fact, $d\omega_1 = -2\omega_2 \wedge \omega_3$, $d\omega_2 = -2\omega_3 \wedge \omega_1$, and $d\omega_3 = -2\omega_1 \wedge \omega_2$. Recall, the volume form is $dvol = \omega_1 \wedge \omega_2 \wedge \omega_3$.

First compute the divergence, $\nabla \cdot \hat{u}_1 = *d\Psi_2(\hat{u}_1)$. To begin, $\Psi_1(\hat{u}_1) = \omega_1$ and

$$
\begin{aligned}
\Psi_2(\hat{u}_1) &= *\Psi_1(\hat{u}_1) \;=\; *\omega_1 \;=\; \omega_2 \wedge \omega_3 \\
d\Psi_2(\hat{u}_1) &= d\omega_2 \wedge \omega_3 + \omega_2 \wedge d\omega_3 \\
d\Psi_2(\hat{u}_1) &= (-2\omega_3 \wedge \omega_1) \wedge \omega_3 + \omega_2 \wedge (-2\omega_1 \wedge \omega_2) \\
d\Psi_2(\hat{u}_1) &= 0 \quad .
\end{aligned}
$$

Therefore $\hat{u}_1$ is divergence-free; so are $\hat{u}_2$ and $\hat{u}_3$. Thus, any left-invariant field is divergence-free.

We now compute $\nabla \times \hat{u}_1$:

$$
\begin{aligned}
\nabla \times \hat{u}_1 &= \Psi_2^{-1}\left(d\Psi_1(\hat{u}_1)\right) \\
&= \Psi_2^{-1}\left(d\omega_1\right) \\
&= \Psi_2^{-1}(-2\omega_2 \wedge \omega_3) \\
&= -2\hat{u}_1
\end{aligned}
$$

Similarly, $\nabla \times \hat{u}_2 = -2\hat{u}_2$ and $\nabla \times \hat{u}_3 = -2\hat{u}_3$. Any left-invariant vector field U then is an eigenfield of curl with eigenvalue -2.

Table 3.1: Vector calculus formulas on S^3

Left-invariant frame $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$

Write $\vec{V} = v_1\,\hat{u}_1 + v_2\,\hat{u}_2 + v_3\,\hat{u}_3$.

Here $\hat{u}_i(f)$ denotes the action of the vector field $\hat{u}_i$ on the function f.

Gradient:	∇f	$=$	$\hat{u}_1(f)\,\hat{u}_1 \;+\; \hat{u}_2(f)\,\hat{u}_2 \;+\; \hat{u}_3(f)\,\hat{u}_3$
Divergence:	$\nabla\cdot\vec{V}$	$=$	$\hat{u}_1(v_1) \;+\; \hat{u}_2(v_2) \;+\; \hat{u}_3(v_3)$
Curl:	$\nabla\times\vec{V}$	$=$	$[\hat{u}_2(v_3) - \hat{u}_3(v_2)]\;\hat{u}_1 \;+\; [\hat{u}_3(v_1) - \hat{u}_1(v_3)]\;\hat{u}_2$
			$+\;[\hat{u}_1(v_2) - \hat{u}_2(v_1)]\;\hat{u}_3 \;-\; 2\vec{V}$
Laplacian:	Δf	$=$	$\hat{u}_1\left(\hat{u}_1(f)\right) \;+\; \hat{u}_2\left(\hat{u}_2(f)\right) \;+\; \hat{u}_3\left(\hat{u}_3(f)\right)$

Spherical coordinates $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\}$

Write $V = f\,\hat{\alpha} + g\,\hat{\beta} + h\,\hat{\gamma}$.

Gradient: $$\nabla f = f_\alpha\,\hat{\alpha} \;+\; \frac{f_\beta}{\sin\alpha}\,\hat{\beta} \;+\; \frac{f_\gamma}{\sin\alpha\sin\beta}\,\hat{\gamma}$$

Divergence: $$\nabla\cdot V = f_\alpha \;+\; \frac{2\cos\alpha}{\sin\alpha}f \;+\; \frac{1}{\sin\alpha}g_\beta \;+\; \frac{\cos\beta}{\sin\alpha\sin\beta}g \;+\; \frac{1}{\sin\alpha\sin\beta}h_\gamma$$

Curl: $$\nabla\times V = \frac{(h\sin\beta)_\beta - g_\gamma}{\sin\alpha\sin\beta}\,\hat{\alpha} \;+\; \frac{f_\gamma - (h\sin\alpha)_\alpha\sin\beta}{\sin\alpha\sin\beta}\,\hat{\beta} \;+\; \frac{(g\sin\alpha)_\alpha - f_\beta}{\sin\alpha}\,\hat{\gamma}$$

Laplacian: $$\Delta f = f_{\alpha\alpha} \;+\; \frac{2\cos\alpha}{\sin\alpha}f_\alpha \;+\; \frac{1}{\sin^2\alpha}f_{\beta\beta} \;+\; \frac{\cos\beta}{\sin^2\alpha\sin\beta}f_\beta \;+\; \frac{1}{\sin^2\alpha\,\sin^2\beta}f_{\gamma\gamma}$$

Table 3.1, continued.

Toroidal coordinates $\left\{\hat{\sigma}, \hat{\theta}, \hat{\phi}\right\}$

Write $V = f\hat{\sigma} + g\hat{\theta} + h\hat{\phi}$. Note: $\{\hat{\sigma}, \hat{\theta}, \hat{\phi}\}$ is a *left-handed frame*. When taking cross-products, use the right-handed frame $\left\{\hat{\sigma}, -\hat{\theta}, \hat{\phi}\right\}$.

Gradient:	∇f	$=$	$f_\sigma\,\hat{\sigma} + \dfrac{f_\theta}{\cos\sigma}\,\hat{\theta} + \dfrac{f_\phi}{\sin\sigma}\,\hat{\phi}$
Divergence:	$\nabla \cdot V$	$=$	$f_\sigma + \dfrac{2\cos 2\sigma}{\sin 2\sigma} f \;+\; \dfrac{1}{\cos\sigma} g_\theta \;+\; \dfrac{1}{\sin\sigma} h_\phi$
Curl:	$\nabla \times V$	$=$	$\left[\dfrac{g_\phi}{\sin\sigma} - \dfrac{h_\theta}{\cos\sigma}\right]\hat{\sigma} \;+\; \left[\dfrac{(h\sin\sigma)_\sigma - f_\phi}{\sin\sigma}\right]\hat{\theta} + \left[\dfrac{f_\theta - (g\cos\sigma)_\sigma}{\cos\sigma}\right]\hat{\phi}$
Laplacian:	Δf	$=$	$f_{\sigma\sigma} + \dfrac{2\cos 2\sigma}{\sin 2\sigma} f_\sigma \;+\; \dfrac{1}{\cos^2\sigma} f_{\theta\theta} \;+\; \dfrac{1}{\sin^2\sigma} f_{\phi\phi}$

3.4 The Hodge Decomposition Theorem

We make frequent use of the Hodge Decomposition Theorem for vector fields applied both to subdomains of the three-sphere and to the three-sphere itself. Cantarella, DeTurck, and Gluck provide a detailed treatment of the Hodge Decomposition Theorem for subdomains of $\mathbb{R}^3$ in [7]. The result and proof for subdomains of S^3 is analogous to the Euclidean case; we state the theorem but refer you to their work for a proof. After presenting an example, we depict how the theorem changes for a closed manifold M^3.

Theorem 3.5 (Hodge Decomposition Theorem for $\mathbb{R}^3$). *Let Ω be a compact three-dimensional submanifold of S^3 with $\partial\Omega$ piecewise smooth. Then, there exists a decomposition of $VF(\Omega)$ into five mutually orthogonal subspaces,*

$$VF(\Omega) = FK \oplus HK \oplus CG \oplus HG \oplus GG \quad ,$$

where,

$$\begin{aligned}
FK &= \text{fluxless knots} &&= \{\nabla \cdot V = 0,\ V \cdot n = 0,\ \text{all interior fluxes} = 0\} \\
HK &= \text{harmonic knots} &&= \{\nabla \cdot V = 0,\ V \cdot n = 0,\ \nabla \times V = 0\} \\
CG &= \text{curly gradients} &&= \{V = \nabla\phi,\ \nabla \cdot V = 0,\ \text{all boundary fluxes} = 0\} \\
HG &= \text{harmonic gradients} &&= \{V = \nabla\phi,\ \nabla \cdot V = 0,\ \phi \text{ locally constant on } \partial\Omega\} \\
GG &= \text{grounded gradients} &&= \{V = \nabla\phi,\ \phi|_{\partial\Omega} = 0\}
\end{aligned}$$

The subspaces HK and HG are finite dimensional and

$$\begin{aligned}
HK &\cong H_1(\Omega, \mathbb{R}) \cong \mathbb{R}^{\text{genus } \partial\Omega} \\
HG &\cong H_2(\Omega, \mathbb{R}) \cong \mathbb{R}^{|\text{components of } \partial\Omega| - |\text{components of } \Omega|}
\end{aligned}$$

Furthermore,

$$
\begin{array}{llccccccc}
\text{ker div} & = & FK & \oplus & HK & \oplus & CG & \oplus & HG \\
\text{image curl} & = & FK & \oplus & HK & \oplus & CG & & \\
\text{ker curl} & = & & & HK & \oplus & CG & \oplus & HG & \oplus & GG \\
\text{image grad} & = & & & & & CG & \oplus & HG & \oplus & GG
\end{array}
$$

Definition 3.6. *Define* **fluid knots** *(which is often truncated to "knots") to be the subspace $K(\Omega) = FK \oplus HK$. Similarly, define* **gradients** *to be the subspace $G(\Omega) = CG \oplus HG \oplus GG$.*

Definition 3.7. *A vector field V is said to be* **Amperian** *if it has zero circulation around every closed curve C on $\partial\Omega$ that bounds a surface outside Ω, i.e., $\oint_C V \cdot ds = 0$.*

Example 3.8. Let Ω be a tubular neighborhood of the circle $x^2 + y^2 = 1$ in the three-sphere $S^3 = \{(x, y, u, v)|x^2 + y^2 + u^2 + v^2 = 1\}$. Define the tube using toroidal coordinates as $\Omega = \{(\sigma, \theta, \phi) : 0 \leq \sigma\sigma_a\}$ for some angle σ_a. Let $a = \sin\sigma_a$. The boundary of Ω is a torus defined by the circles $u^2 + v^2 = a^2$ and $x^2 + y^2 = 1 - a^2$, or simply by the toroidal coordinate $\sigma = \sigma_a = \arcsin a$.

Then the Hodge Decomposition Theorem implies that the harmonic knots on Ω are one-dimensional since $\partial\Omega$ has genus one. The vector field given by $W = \dfrac{1}{\cos\sigma}\hat{\theta}$ is divergence-free, curl-free, and tangent to the boundary; thus W is a generator of $HK(\Omega)$.

Now, consider the left-invariant field $\hat{u}_1$ on Ω. Recall, in toroidal coordinates,

$$\hat{u}_1 = \cos\sigma\,\hat{\theta} - \sin\sigma\,\hat{\phi} \quad .$$

It is divergence-free and tangent to the boundary, thus $\hat{u}_1$ is a fluid knot. We decompose $\hat{u}_1$ into its fluxless knot and harmonic knot components: $\hat{u}_1 = u_F + u_H$. The harmonic component must be a multiple of W, so

$$u_H = cW = \frac{c}{\cos\sigma}\hat{\theta}$$

Also, u_H must contain all the flux of $\hat{u}_1$ through a cross-sectional disk of the solid torus Ω, i.e.,

$F(u_H) = F(\hat{u}_1)$. This flux condition determines the constant c. First calculate, the flux of $\hat{u}_1$:

$$\begin{aligned}
F(\hat{u}_1) &= \int_{\phi=0}^{2\pi}\int_{\sigma=0}^{\arcsin a} \hat{u}_1 \cdot \hat{\theta}\, \sin\sigma \, d\sigma \, d\phi \\
F(\hat{u}_1) &= 2\pi \int_{\sigma=0}^{\arcsin a} \sin\sigma \cos\sigma \, d\sigma \\
F(\hat{u}_1) &= \pi a^2
\end{aligned}$$

Next calculate the flux of u_H:

$$\begin{aligned}
F(u_H) &= 2\pi \int_{\sigma=0}^{\arcsin a} c \,\frac{\sin\sigma}{\cos\sigma}\, d\sigma \\
F(u_H) &= -\pi c \, \ln(1-a^2)
\end{aligned}$$

Therefore, $c = \dfrac{-a^2}{\ln(1-a^2)}$. Note that $\lim_{a\to 0} c = 1$ and $c = \sqrt{1-a^2} + O(a^4)$. When $a = 1/\sqrt{2}$, then $\sigma_a = \pi/4$ and the region Ω describes the solid Clifford torus. In that case, $c = 1/(2\ln 2) \approx 0.721$.

To conclude the example, the decomposition of $\hat{u}_1$ into fluxless and harmonic components is

$$\begin{aligned}
u_H &= \frac{-a^2}{\ln(1-a^2)}\frac{1}{\cos\sigma}\,\hat{\theta} \\
u_F &= \left(\cos\sigma - \frac{-a^2}{\ln(1-a^2)}\frac{1}{\cos\sigma}\right)\hat{\theta} \;-\; \sin\sigma\,\hat{\phi} \quad .
\end{aligned}$$

Now consider the case of a closed manifold M. The Hodge Decomposition Theorem is simpler and involves only three components. We express it in terms of vector fields; for a thorough treatment of the theorem in terms of differential forms, see chapter 6 of Warner's book [26].

Theorem 3.9 (Hodge Decomposition Theorem for M closed). *Let M be a closed orientable Riemannian manifold. Then, there exists a decomposition of $VF(M)$ into three mutually orthogonal subspaces,*

$$VF(\Omega) = FK \oplus HK \oplus G \quad ,$$

where,

$$\begin{array}{lcll} FK & = & \text{fluxless knots} & = \{\nabla \cdot V = 0, \text{ all fluxes} = 0\} \\ HK & = & \text{harmonic knots} & = \{\nabla \cdot V = 0, \ \nabla \times V = 0\} \\ G & = & \text{gradients} & = \{V = \nabla \phi\} \end{array}$$

The subspace HK is finite dimensional and $HK(M) \cong H_1(M, \mathbb{R})$. Furthermore,

$$\begin{array}{lcccccc} \text{ker div} & = & FK & \oplus & HK & & \\ \text{image curl} & = & FK & & & & \\ \text{ker curl} & = & & & HK & \oplus & G \\ \text{image grad} & = & & & & & G \end{array}$$

Fluxless knots, more specifically have zero flux through every closed surface Σ contained in M. The isomorphism $HK \cong H_1(M, \mathbb{R})$ is given by viewing $V \in HK$ as a functional on 1-forms; i.e., $V : \Lambda^1(M) \to \mathbb{R}$, where $V : \alpha \mapsto \int_M \alpha(V)\, dvol$.

For M closed, the gradients behave analogously to the grounded gradients defined previously; the subspaces CG and HG have no analogue for closed manifolds.

Proposition 3.10. *For M closed, curl is a self-adjoint operator; also, divergence and gradient are negative adjoints of each other.*

Proof. Via identity 6 in the Appendix, and the Divergence Theorem,

$$\begin{aligned} \int_M (\nabla \times V) \cdot W\, dvol &= \int_M \nabla \cdot (V \times W)\, dvol + \int_M (\nabla \times W) \cdot V\, dvol \\ \int_M (\nabla \times V) \cdot W\, dvol &= 0 + \int_M (\nabla \times W) \cdot V\, dvol \end{aligned}$$

Thus, $\langle \nabla \times V, W \rangle = \langle \nabla \times W, V \rangle$, so curl is self-adjoint.

To see divergence and gradient are negative adjoints, we utilize identity 5 from the Appendix and the Divergence Theorem:

$$\begin{aligned}
\int_M f(\nabla \cdot V)\, dvol &= \int_M \nabla \cdot (fV)\, dvol - \int_M \nabla f \cdot V\, dvol \\
\int_M f(\nabla \cdot V)\, dvol &= 0 - \int_M \nabla f \cdot V\, dvol
\end{aligned}$$

Thus $\langle f, \nabla \cdot V \rangle = -\langle \nabla f, V \rangle$, where the first L^2 inner product is in $C^\infty(M)$ and the second is in $VF(M)$. Hence, $(\text{div})^* = -\text{grad}$. $\square$

Since it arises so often in this work, here is the Hodge Decomposition Theorem for the three-sphere.

Corollary 3.11. *The Hodge Decomposition Theorem on S^3 decomposes $VF(S^3)$ into only two nontrivial subspaces. The subspace $HK(S^3)$ is trivial. Thus, all knots are fluxless knots, i.e., $K(S^3) = FK(S^3)$. Thus,*

$$VF(S^3) = K(S^3) \oplus G(S^3) \quad .$$

Furthermore,

$$\begin{aligned}
K(S^3) &= \text{ker div} &&= \text{image curl} \\
G(S^3) &= \text{image grad} &&= \text{ker curl} \quad .
\end{aligned}$$

3.5 Triple products

Let A, B, C be vectors (or vector fields) on $\mathbb{R}^4$. Let $\alpha \in [0, \pi]$ be the angle between vectors A and B.

Definition 3.12. *The* **triple product** of A, B, and C is the vector

$$[A, B, C] = det \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ \hat{x_1} & \hat{x_2} & \hat{x_3} & \hat{x_4} \end{bmatrix} \quad .$$

The triple product of three vectors in $\mathbb{R}^4$ is the analogue of the cross product of two vectors in $\mathbb{R}^3$. Indeed, the product of $n-1$ vectors in $\mathbb{R}^n$ is similarly defined as the determinant of an $n \times n$ matrix.

Three useful properties of triple products are that

1. $[A, B, C] = [B, C, A] = -[A, C, B]$.

2. $[A, B, C]$ is orthogonal to A, B, and C. If A, B, and C are linearly independent, then $\{A, B, C, [A, B, C]\}$ forms a basis that agrees with the standard orientation on $\mathbb{R}^4$. If A, B, and C are linearly dependent, then $[A, B, C] = 0$.

3. If A is a point in S^3 (i.e., $|A| = 1$), and B and C are tangent to the three-sphere at A, i.e., $B, C \in T_A S^3$, then $[A, B, C]$ can be viewed as a vector in $T_A S^3$, where it is equal to the cross product $B \times C$.

   More generally, for $A \in S^3$, the triple product $[A, B, C] = B^\perp \times C^\perp$, where $B^\perp$ (and likewise for $C^\perp$) is the component of B perpendicular to A.

$$B^\perp = B - (A \cdot B)\, A$$

Lemma 3.13 (DG, [14]). *Let $y \in S^3$. For vector fields A, B in $\mathbb{R}^4$ that do not depend upon y,*

$$\nabla_y \times [A, B, y] = 2(A \cdot y)\, B - 2(B \cdot y)\, A \ .$$

Often, calculations require a formula for an iterated double product. Our following result generalizes such a result from [14].

Lemma 3.14. *Let A, B, C be vectors in $\mathbb{R}^4$. Let $C^\perp$ represent the component of C which is orthogonal to the plane spanned by A and B.*

$$[A, B, [A, B, C]] = -|A|^2\, |B|^2 \sin^2 \alpha\; C^\perp$$

Equivalently,

$$
\begin{aligned}
[A, B, [A, B, C]] \;\; = \;\; & \big(|B|^2 \,(A \cdot C) - |A|\,|B| \cos\alpha \,(B \cdot C) \big) \; A \\
& + \big(|A|^2 \,(B \cdot C) - |A|\,|B| \cos\alpha \,(A \cdot C) \big) \; B \\
& - |A|^2 \,|B|^2 \sin^2\alpha \; C
\end{aligned}
$$

Proof. Use Gram-Schmidt to find $B^{\perp}$ orthogonal to A, and to find $C^{\perp}$ orthogonal to both A and $B^{\perp}$:

$$
\begin{aligned}
B^{\perp} &= B \sin\alpha \\
C^{\perp} &= C - \frac{(B \cdot B)\,(A \cdot C) - (A \cdot B)\,(B \cdot C)}{(A \cdot A)\,(B \cdot B) - (A \cdot B)^2} \, A \; - \; \frac{(A \cdot A)\,(B \cdot C) - (A \cdot B)\,(A \cdot C)}{(A \cdot A)\,(B \cdot B) - (A \cdot B)^2} \, B \\
C^{\perp} &= C - \frac{|B|^2 \,(A \cdot C) - |A|\,|B| \cos\alpha \,(B \cdot C)}{|A|^2 \,|B|^2 \sin^2\alpha} \, A - \frac{|A|^2 \,(B \cdot C) - |A|\,|B| \cos\alpha \,(A \cdot C)}{|A|^2 \,|B|^2 \sin^2\alpha} \, B \qquad (3.5)
\end{aligned}
$$

Let $D = [A, B, C]$. Assume $\{A, B, C\}$ are linearly independent, else $D = 0$. Then D is orthogonal to the span of A, B, C and the basis $\{A, B, C, D\}$ has positive orientation in $\mathbb{R}^4$. The length of D is

$$
|D| = |A| \, \left|B^{\perp}\right| \, \left|C^{\perp}\right| = |A|\,|B| \, \left|C^{\perp}\right| \, \sin\alpha \;\; .
$$

Let $E = [A, B, D] = [A, B, [A, B, C]]$. Then E is orthogonal to D, so it is a linear combination of A, B, and C. Since E is also orthogonal to A and B, it must be a multiple of $C^{\perp}$. The basis $\{A, B, D, E\}$ must have positive orientation in $\mathbb{R}^4$, which forces the vector E to point in the direction of $-C^{\perp}$, i.e.,

$$
E \; = \; -\frac{|E|}{|C^{\perp}|} \, C^{\perp}
$$

The length of E is

$$
\begin{aligned}
|E| \;\; &= \;\; |A|\,|B|\,|D| \, \sin\alpha \\
|E| \;\; &= \;\; |A|^2 \,|B|^2 \, \left|C^{\perp}\right| \, \sin^2\alpha
\end{aligned}
$$

Thus we conclude that

$$
E \; = \; -|A|^2 \,|B|^2 \sin^2\alpha \; C^{\perp} \;\; .
$$

$\square$

3.6 Transport methods for vector fields on S^3

In the next chapter, we will define on S^3 the analogue to the Biot-Savart integral operator from Euclidean space,

$$BS(V)(y) = \frac{1}{4\pi} \int_{\Omega} V(x) \times \frac{y-x}{|y-x|^3} \, dvol_x \quad .$$

This integral formula requires the addition of vectors $V(x)$ lying in different tangent spaces. Truly, we must move all the vectors to one single tangent space before summing (or integrating) them. In Euclidean space, this is hardly an issue; we simply drag the vectors to one common base point. The vectors themselves do not change by this dragging; they are *free* of their base points. On arbitrary manifolds, this free vector property is lost; a tangent vector at one point of S^3 will not necessarily be tangent if considered at a different point of S^3.

To obtain a Biot-Savart formula on the three-sphere, we must decide how to move tangent vectors to a common tangent space, that is so that they remain tangent. Two natural choices exist: parallel transport along a minimal geodesic and left (or right) translation using the group structure of S^3 viewed as $SU(2)$ or as the group of unit quaternions. Each has its advantages and disadvantages; wherever convenient, we use the more illustrative method; sometimes we use each method and provide two different proofs, e.g., Theorem 4.6.

In this section, we describe each transport method and detail its properties. Later, in chapter 4, we will define the Biot-Savart operator as an integral using each transport method.

3.6.1 Left translation

Consider the three-sphere as the group $SU(2)$ (or as the unit quaternions). For any two points $x, y \in S^3$, we can map x to y via the left group action: $L_{yx^{-1}} : x \mapsto (yx^{-1})x = y$. If $V(x)$ is a tangent vector at the point x, then the push forward of the left-translation map moves $V(x)$ to the tangent space at y, e.g., $(L_{yx^{-1}})_* V(x) \in T_y S^3$.

This setup is quite valuable, especially when utilizing the left-invariant orthonormal frame, defined in Section 3.2.1. A left-invariant vector field $U(x)$ is one with the property that $(L_{yx^{-1}})_* U(x) = U(y)$. Any left-invariant $U(x)$ is divergence-free and is an eigenfield of curl: $\nabla \times U(x) = -2U(x)$.

Remark 3.15. Alternatively, one could use the right group action $R_{x^{-1}y} : x \mapsto x(x^{-1}y) = y$ and define the right-translation of $V(x)$ as $(R_{x^{-1}y})_* V(x)$. Then one would prefer the right-invariant orthonormal frame, see Remark 3.2. All right-invariant vector fields $W(x)$ are still divergence-free and curl eigenfields, but the eigenvalue of curl switches sign from -2 to $+2$: $\nabla \times W(x) = +2W(x)$.

Let $V(x)$ be a smooth vector field and $f(x)$ a smooth function on S^3. The left-translation of $f(x)V(x)$ is $(L_{yx^{-1}})_* (f(x)V(x)) = f(x)(L_{yx^{-1}})_* V(x)$. Express V in the left-invariant frame as $V(x) = v_1(x)\hat{u}_1 + v_2(x)\hat{u}_2 + v_3(x)\hat{u}_3$. Now V is left-invariant if and only if each function $v_i(x)$ is constant. Define the notation $[V]$ as the left-invariant field

$$[V] = [v_1]\hat{u}_1 + [v_2]\hat{u}_2 + [v_3]\hat{u}_3 \quad .$$

Proposition 3.16. *Let $V \in VF(S^3)$. Then, $[V]$ depicts the L^2 projection of V onto the three-dimensional space of left-invariant vector fields, and is expressed by the formula*

$$[V] = \frac{1}{2\pi^2} \int_{S^3} (L_{yx^{-1}})_* V(x) \, dx \quad .$$

Proof. Express V in terms of the left-invariant frame as above. We show the formula first:

$$\begin{aligned}
\int_{S^3} (L_{yx^{-1}})_* V(x) \, dx &= \int_{S^3} (L_{yx^{-1}})_* \left(v_1(x) \, \hat{u}_1(x) + v_2(x) \, \hat{u}_2(x) + v_3(x) \, \hat{u}_3(x) \right) \, dx \\
&= \int_{S^3} v_1(x) \, \hat{u}_1(y) + v_2(x) \, \hat{u}_2(y) + v_3(x) \, \hat{u}_3(y) \, dx \\
&= \left(\int_{S^3} v_1(x) \, dx \right) \hat{u}_1(y) + \left(\int_{S^3} v_2(x) \, dx \right) \hat{u}_2(y) + \left(\int_{S^3} v_3(x) \, dx \right) \hat{u}_3(y) \\
&= 2\pi^2 \, [v_1] \, \hat{u}_1(y) + 2\pi^2 \, [v_2] \, \hat{u}_2(y) + 2\pi^2 \, [v_3] \, \hat{u}_3(y) \\
&= 2\pi^2 \, [V]
\end{aligned}$$

The L^2 projection of V onto the space of left-invariant fields is

$$\text{proj } V = \frac{\langle V, \hat{u}_1 \rangle}{\langle \hat{u}_1, \hat{u}_1 \rangle} \, \hat{u}_1 + \frac{\langle V, \hat{u}_2 \rangle}{\langle \hat{u}_2, \hat{u}_2 \rangle} \, \hat{u}_2 + \frac{\langle V, \hat{u}_3 \rangle}{\langle \hat{u}_3, \hat{u}_3 \rangle} \, \hat{u}_3 \quad .$$

The inner product $\langle \hat{u}_i, \hat{u}_j \rangle = 2\pi^2 \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol. Thus,

$$\langle \hat{u}_2, \hat{u}_2 \rangle = \int_{S^3} v_i(x) \hat{u}_i \cdot \hat{u}_i \, dx = \int_{S^3} v_i(x) \, dx = 2\pi^2 [v_i] \quad .$$

We conclude that

$$\text{proj } V = [v_1]\hat{u}_1 + [v_2]\hat{u}_2 + [v_3]\hat{u}_3 = [V] \quad . \qquad \square$$

3.6.2 Parallel transport

On a Riemannian manifold, the parallel transport of a vector V at one point x moves V along a minimal geodesic $\gamma(t)$ to another point y. Let $\gamma(0) = x$. Parallel transport is determined by breaking V into two components, one parallel to the geodesic at x, i.e., in the direction $\gamma'(0)$, and the other perpendicular to the geodesic. The component of V parallel to the geodesic at x follows the geodesic and remains parallel to the geodesic at y. The component of V perpendicular to the geodesic at x remains perpendicular at y. Let $P_{yx}V$ denote the parallel transport of V from x to y.

Three important disadvantages of parallel transport in comparison with left translation on the three-sphere are

1. Parallel transport from x to its antipode $-x$ is not well-defined.

2. Parallel transport is not multiplicative in the sense that $P_{zx}V$ does not necessarily equal $P_{zy}(P_{yx}V)$.

3. The vector field created by taking the parallel transport of a vector at a point is neither divergence-free or a curl eigenfield. By left translating a vector at a point, we get a left-invariant field; all left-invariant fields are both divergence-free and curl eigenfields.

Some advantages of parallel transport over left-translation are

1. Parallel transport is available on all Riemannian 3-manifolds whereas few of these manifolds have Lie group structures. For our calculations to generalize most easily, we attempt to use parallel transport whenever possible.

2. As evident in chapter 4, the Biot-Savart operator is expressed more conveniently in parallel transport form.

3. Parallel transport allows us to utilize the ambient Euclidean space $\mathbb{R}^4$ to facilitate certain calculations, especially ones where the cross-product of vector fields on S^3 can be exchanged for a triple product in $\mathbb{R}^4$.

3.6.3 The calculus of parallel transport

A substantial portion of this calculus was first developed by Dennis DeTurck for work on [14].

Let x, y be non-antipodal points on S^3; we will view them as unit vectors in $\mathbb{R}^4$. If x and y are orthogonal, then $G(t) = x \cos t + y \sin t$ determines the unique minimal geodesic on S^3 between them. Let α be the distance on S^3 between them. Then $(x \cdot y) = \cos \alpha$. The component of y perpendicular to x is $w = (y - \cos \alpha\, x)$; its length is $|w| = \sin \alpha$. The unique minimal geodesic between x and y is

$$G(t) = \cos t\ \hat{x} + \sin t\ \frac{y - x \cos \alpha}{\sin \alpha} \quad .$$

Notice $G(\alpha) = y$. The derivative of $G(t)$ is a tangent vector in $T_{G(t)}S^3$ given by

$$G'(t) = -\sin t\ \hat{x} + \cos t\ \frac{y - x \cos \alpha}{\sin \alpha} \quad .$$

The gradient of $\alpha = \alpha(x, y)$ is often needed for calculations. Recall from section 3.1 that $\nabla_y \alpha(x, y)$ denotes the gradient of α with respect to y variables.

The gradient of α with respect to x must point away from y along the geodesic and vice-versa. Thus

$$\nabla_x \alpha(x, y) = -G'(0) = \frac{x \cos \alpha - y}{\sin \alpha} \tag{3.6}$$

$$\nabla_y \alpha(x, y) = G'(\alpha) = \frac{y \cos \alpha - x}{\sin \alpha} \tag{3.7}$$

Two properties are worth noting. First, $\nabla_x \alpha$ is orthogonal to x and so $\nabla_x \alpha \in T_x S^3$; also, $\nabla_y \alpha$ is orthogonal to y and so $\nabla_x \alpha \in T_y S^3$. Second, $\nabla_x \alpha$ and $\nabla_y \alpha$ are both unit vectors.

For any two unit vectors $x, y \in \mathbb{R}^n$ such that $x \neq \pm y$, the unique map $M \in SO(n)$ that maps x to y and that fixes all vectors orthogonal to both x and y is

$$M(v) = v - \frac{(v \cdot (x+y))}{1+(x \cdot y)}\, x + \frac{(v \cdot x)\ (1 + 2(x \cdot y)) - (v \cdot y)}{1 + (x \cdot y)}\, y$$

The derivation of M is straightforward and omitted.

For v a tangent vector at $x \in S^3$, this map M precisely describes its parallel transport to the tangent space at y. The expression above simplifies to

$$P_{yx}(v) = v - \frac{(v \cdot y)}{1 + (x \cdot y)}\ (x+y) \tag{3.8}$$

As an exercise, we show $P_{yx}(\nabla_x \alpha) = -\nabla_y \alpha$.

$$\begin{aligned}
P_{yx}(\nabla_x \alpha) &= \nabla_x \alpha - \frac{(\nabla_x \alpha \cdot y)}{1 + (x \cdot y)}\ (x+y) \\
&= \frac{\cos\alpha}{\sin\alpha}\, x - \frac{1}{\sin\alpha}\, y - \frac{\cos\alpha\,(x \cdot y) - (y \cdot y)}{(1+\cos\alpha)\sin\alpha}\ (x+y) \\
&= \frac{\cos\alpha\,(1+\cos\alpha)}{(1+\cos\alpha)\sin\alpha}\, x - \frac{1}{\sin\alpha}\, y - \frac{\cos^2\alpha - 1}{(1+\cos\alpha)\sin\alpha}\ (x+y) \\
&= \frac{\cos\alpha + \cos^2\alpha + \sin^2\alpha}{(1+\cos\alpha)\sin\alpha}\, x + \frac{-(1+\cos\alpha) + \sin^2\alpha}{(1+\cos\alpha)\sin\alpha}\, y \\
&= \frac{1}{\sin\alpha}\, x - \frac{\cos\alpha}{\sin\alpha}\, y \\
&= -\nabla_y \alpha
\end{aligned}$$

Remark 3.17. For a vector v at $x \in S^3$ that points parallel to the geodesic γ running through x and $y \in S^3$, left-translation from x to y is exactly the same as parallel transport. The two methods differ only in how they treat components that are perpendicular to the geodesic γ. Hence,

$$\nabla_y \alpha = -(L_{yx^{-1}})_* \nabla_x \alpha \quad . \tag{3.9}$$

As an exercise, the reader is invited to show this directly using the group structure of S^3 and equations (3.6) and (3.7).

Remark 3.18. For any function $f(\alpha)$ that depends only upon the distance $\alpha(x, y)$ from x to y, its gradient with respect to x variables is $\nabla_x f(\alpha) = f'(\alpha)\nabla_x \alpha$. Thus the methods of transporting its gradient vector are equivalent.

$$\nabla_y f(\alpha) \;=\; -P_{yx}\nabla_x f(\alpha) \;=\; -(L_{yx^{-1}})_* \nabla_x f(\alpha)$$

3.7 Vector Laplacian operator

In performing calculus on vector fields, one needs the analogue of the Laplacian applied to a vector field. In this section we define such an operator, the vector Laplacian $L(V)$, and discuss some of its properties. The next section is devoted to finding its kernel and image.

Let M^3 be a compact, orientable, Riemannian manifold possibly with smooth boundary ∂M. In the next section we consider separately cases where M is closed and also where M is compact with boundary. Let $VF(M)$ be the space of smooth vector fields defined on M.

Definition 3.19. *The* **vector Laplacian** $L : VF(M) \to VF(M)$ *is given by*

$$L(V) = -\nabla \times (\nabla \times V) + \nabla\,(\nabla \cdot V) \quad .$$

For Cartesian coordinates in Euclidean space, the vector Laplacian acts by applying the (scalar) Laplacian to each component function. If $V = \sum_i f_i \frac{\partial}{\partial x_i}$, then $L(V) = \sum_i (\Delta f_i) \frac{\partial}{\partial x_i}$. The scalar Laplacian is always lurking in a coordinate-specific formula for the vector Laplacian.

Example 3.20. *Cylindrical coordinates* (r, ϕ, z) *on* $\mathbf{R^3}$.

Let $V = u(r,\phi,z)\,\hat{r} + v(r,\phi,z)\,\hat{\phi} + w(r,\phi,z)\,\hat{z}$. Then,

$$L(V) \quad = \quad \left(\Delta u - \frac{1}{r^2}u - 2v_\phi\right)\,\hat{r} \,+\, \left(\Delta v - \frac{1}{r^2}v\right)\,\hat{\phi} \,+\, \Delta w\,\hat{z}$$

Alas, the computed formulas for the vector Laplacian in most coordinate systems, even spherical coordinates on $\mathbb{R}^3$ and S^3, become much more complicated.

We assume that the Hodge Decomposition Theorem splits $VF(M)$ into five mutually orthogonal subspaces $FK \oplus HK \oplus CG \oplus HG \oplus GG$, as it does for subdomains of $\mathbb{R}^3$ and S^3.

The vector Laplacian, as defined above, corresponds naturally with the definition of the Laplace operator on forms. Denote the k-forms on M^n as $\Lambda^k(M)$, define the operator

$$\delta = (-1)^{n(k+1)} * d * : \Lambda^k(M) \to \Lambda^{k-1}(M) \ \ .$$

Then the Laplace-Beltrami operator on k-forms is

$$\Delta = \delta d + d\delta = (-1)^{nk} * d * d + (-1)^{n(k+1)} d * d * \ \ .$$

For functions, we once again obtain equation (3.4), $\Delta = *d * d$.

From the commutative diagram given in Section 3.3, to each $V \in VF(M^3)$ there is associated a canonical 1-form $\Psi_1(V) = \omega_V \in \Lambda^1(M)$, given by $\omega_V(W) = \langle V, W\rangle$. Then $\nabla \cdot V$ is associated to $\delta\omega_V = *d * \omega_V$ and $\nabla \times V$ is associated to $*d\omega_V$. Thus, $L(V)$ is associated to the 1-form $(- * d * d + d * d*)\,\omega_V$, which is precisely the formula for $\Delta\omega_V$.

Proposition 3.21. *Let $f \in C^\infty(M)$ and $V \in VF(M)$. The Laplacian, taken on vectors or scalars as need be, commutes with the gradient, divergence, and curl operators.*

$$\begin{array}{lrcl} (a) & L(\nabla f) & = & \nabla(\Delta f) \\ (b) & \Delta(\nabla \cdot V) & = & \nabla \cdot L(V) \\ (c) & L(\nabla \times V) & = & \nabla \times L(V) \end{array}$$

This result implies that the vector Laplacian sends divergence-free vector fields to divergence-free vector fields, gradients to gradients, and curl-free vector fields to curl-free vector fields.

Proof. (a) $L(\nabla f) = -\nabla \times \nabla \times \nabla f + \nabla(\nabla \cdot \nabla f) = 0 + \nabla(\Delta f)$.

(b) $\nabla \cdot L(V) = \nabla \cdot (-\nabla \times \nabla \times V + \nabla(\nabla \cdot V)) = 0 + \Delta(\nabla \cdot V)$.

(c) The left-hand side is

$$L(\nabla \times V) = -\nabla \times \nabla \times (\nabla \times V) + \nabla(\nabla \cdot \nabla \times V) = \nabla \times (-\nabla \times \nabla \times V) + 0 \ \ .$$

The right-hand side is

$$\nabla \times L(V) = \nabla \times (-\nabla \times \nabla \times V) + \nabla \times \nabla(\nabla \cdot V) = \nabla \times (-\nabla \times \nabla \times V) + 0 \ \ .$$

The two sides are equal, so $L(\nabla \times V) = \nabla \times L(V)$. $\square$

Now we work towards showing the vector Laplacian is self-adjoint. That follows as a corollary to the following proposition.

Proposition 3.22. *Let M be as described above. Let $V, W \in VF(M)$. Let $\hat{n}$ denote the unit outward normal vector to ∂M. Then,*

$$\begin{aligned} -\langle L(V), W\rangle &= \langle \nabla \cdot V, \nabla \cdot W\rangle + \langle \nabla \times V, \nabla \times W\rangle \\ &\quad + \int_{\partial M} ((\nabla \times V) \times W) \cdot \hat{n}\, d(area) \\ &\quad - \int_{\partial M} (\nabla \cdot V)\, W \cdot \hat{n}\, d(area) \end{aligned}$$

Proof. Begin by writing out $L(V)$:

$$-\langle L(V), W\rangle = \int_M \nabla \times \nabla \times V \cdot W - \nabla(\nabla \cdot V) \cdot W\, dvol$$

We need the following two vector identities (see Appendix A):

$$\begin{aligned} (\nabla \times \nabla \times V) \cdot W &= \nabla \times V \cdot \nabla \times W + \nabla \cdot ((\nabla \times V) \times W) \\ \nabla(\nabla \cdot V) \cdot W &= (\nabla \cdot V)(\nabla \cdot W) - \nabla \cdot (\nabla \cdot V) W \end{aligned}$$

Plugging these both into the equation above,

$$\begin{aligned} -\langle L(V), W\rangle &= \int_M \nabla \times V \cdot \nabla \times W + \nabla \cdot ((\nabla \times V) \times W)\, dvol \\ &\quad + \int_M (\nabla \cdot V)(\nabla \cdot W) - \nabla \cdot (\nabla \cdot V) W\, dvol \\ &= \langle \nabla \times V, \nabla \times W\rangle + \int_{\partial M} ((\nabla \times V) \times W) \cdot \hat{n}\, d(area) \\ &\quad \langle \nabla \cdot V, \nabla \cdot W\rangle - \int_{\partial M} (\nabla \cdot V)\, W \cdot \hat{n}\, d(area) \end{aligned}$$

This concludes the proof. $\square$

Corollary 3.23. *Let M be as above and also closed. Then, the vector Laplacian is self-adjoint on $VF(M)$ and*

$$-\langle L(V), V\rangle = \|\nabla \cdot V\|^2 + \langle \nabla \times V, \nabla \times V\rangle \ .$$

Proof. For M closed the last two terms drop out of Proposition 3.22, which becomes

$$-\langle L(V), W\rangle = \langle \nabla \cdot V, \nabla \cdot W\rangle + \langle \nabla \times V, \nabla \times W\rangle \quad .$$

Then clearly $\langle L(V), W\rangle = \langle V, L(W)\rangle$, so the vector Laplacian is self-adjoint for M^3 closed. $\square$

Thus on a closed manifold, the vector Laplacian vanishes if and only if the vector field is both curl-free and divergence-free, i.e., a harmonic knot.

3.8 Kernel and image of vector Laplacian

We now turn to understanding the kernel and image of the vector Laplacian. We examine two cases: closed manifolds and compact manifolds with boundary.

3.8.1 M closed

Consider M^3, a closed 3-manifold. Decompose $VF(M) = FK(M) \oplus HK(M) \oplus G(M)$ by the Hodge Theorem. How does the vector Laplacian act on each subspace?

Corollary3.23 implies that the kernel of the vector Laplacian is precisely the harmonic knots HK. For a simply-connected manifold, such as the three-sphere, the vector Laplacian has a trivial kernel.

Consider a fluxless knot V. Then V is divergence free and so is $L(V) = -\nabla \times \nabla \times V$. The curl operator maps the space of fluxless knots bijectively onto itself, hence so does the vector Laplacian.

In Proposition 3.21 we showed that the vector Laplacian sends gradients to gradients, but is it onto? Consider a gradient, ∇f; we seek another gradient ∇u such that $L(\nabla u) = \nabla f$. Note $L(\nabla u) = \nabla(\Delta u)$. On a closed manifold, the scalar Laplacian is invertible for functions with average value 0; i.e., there exists a function u such that $\Delta u = f - [f]$. Then $\nabla(\Delta u) = \nabla f$, and we conclude that $L(G) = G$.

In the paragraphs above, we have proven the following theorem:

Theorem 3.24. *Let M be closed. The vector Laplacian respects the Hodge decomposition of $VF(M)$. Its kernel is $HK(M)$, and it maps the other subspaces bijectively to themselves,*

$$\begin{aligned} L(FK) &= FK \\ L(HK) &= 0 \\ L(G) &= G \end{aligned}$$

For a closed manifold with $H_1(M, \mathbb{R}) = 0$, such as the three-sphere, the vector Laplacian has trivial kernel and maps $VF(M)$ bijectively to itself. We use that fact to define an inverse to L in section 3.9.

3.8.2 M compact with boundary

Let M^3 be compact with piecewise smooth ∂M. We assume that the Hodge Decomposition Theorem for vector fields on M is analogous to Theorem 3.5, i.e., the space $VF(M)$ decomposes into five orthogonal subspaces: $VF(M) = FK \oplus HK \oplus CG \oplus HG \oplus GG$.

Theorem 3.25. *The vector Laplacian on $VF(M)$ has the following kernel:*

$$\ker L \quad = \quad HK \oplus CG \oplus HG \oplus C_L \quad ,$$

where $C_L \subset FK \oplus GG$ is defined to be the space

$$C_L = \{V_f - \nabla g_f \mid V_f \in FK, \nabla g_f \in GG, -\nabla \times \nabla \times V = \nabla(\Delta g_f) \in CG\} \quad .$$

Proof. Any vector field V that lies in the kernels of both the curl and divergence operators will necessarily have $L(V) = 0$. So the kernel of L includes the subspace $HK \oplus CG \oplus HG$.

Now consider the space of fluxless knots. For $V \in FK$, then we have $L(V) = -\nabla \times \nabla \times V$. A subspace of FK is sent by the curl operator into the space $HK \oplus CG$; call this subspace $FK_L \subset FK$. For $V \in FK_L$, its curl $\nabla \times V$ lies in the kernel of curl, so $L(V) = 0$. Thus, FK_L must lie in the kernel of L.

Furthermore, these are the only fluxless knots in the kernel. All other fluxless knots V have some component of their curl that lies in FK; thus $\nabla \times \nabla \times V = L(V)$ does not vanish.

Let's now examine $\nabla g \in GG$; we see that $L(\nabla g) = \nabla \Delta g$. For this to vanish, Δg must be a constant. So the subset of GG in the kernel is $\{\nabla g \in GG(M) \mid \Delta g = constant\}$; denote this as GG_L.

Now, we turn to understanding C_L. The next theorem shows that the vector Laplacian maps a subspace of the fluxless knots onto the curly gradients. Also, the vector Laplacian maps a subspace of the grounded gradients onto the curly gradients. Thus any curly gradient ∇f has two corresponding preimages via the vector Laplacian. We define $C_L \subset FK \oplus GG$ as the subspace of differences of these preimages: any vector field in C_L is the difference of a fluxless knot V_f and a grounded gradient ∇g_f, which are both mapped to the same curly gradient via the vector Laplacian, i.e.,

$$L(V_f) = L(\nabla g_f) = \nabla f \in CG \quad .$$

As a subset of $FK \oplus GG$, the space C_L is orthogonal to the remainder of the kernel, $HK \oplus CG \oplus GG$. Moreover, the subspaces FK_L and GG_L are trivially contained in C_L.

Gathering the pieces, we have that $\ker L = HK \oplus CG \oplus HG \oplus C_L$; the theorem is proven. $\square$

Theorem 3.26. *For M as above, the vector Laplacian is surjective, i.e.,* $\text{Image}\,(L) = VF(M)$.

Proof. We first show L is onto the space of fluid knots, $K(M)$. For any $W \in \text{Image}\,(curl)$, we can find a $X \in FK$ such that $\nabla \times X = W$; we can also find $V \in FK$ such that $-\nabla \times V = X$. Thus $L(V) = -\nabla \times \nabla \times V = W$. In conclusion, $L(FK) = FK \oplus HK \oplus CG$.

Let $W = \nabla g$ be an arbitrary gradient on M, where g can be chosen so that its average value on M is zero. We seek a gradient $\nabla u \in GG$ such that $L(\nabla u) = \nabla g$. Since $\nabla u \in GG$, we have that $u|_{\partial M} = 0$. Since $L(\nabla u) = \nabla \Delta u$, the desired condition is tantamount to the Dirichlet problem:

$$\begin{aligned} \Delta u &= g \ \ (\text{on } M) \\ u|_{\partial M} &= 0 \end{aligned}$$

The Dirichlet problem has a unique solution, and we conclude $L(GG) = CG \oplus HG \oplus GG$. Having calculated the image of the subspaces FK and GG, we recall that the other three subspaces $HK \oplus CG \oplus HG$ lie in the kernel of L, as proved in Theorem 3.25. Therefore, the result is shown:

$$\begin{aligned} L\,(VF(M)) &= L(FK) \cup L(GG) \\ &= FK \oplus HK \oplus CG \ \cup \ CG \oplus HG \oplus GG \\ &= VF(M) \end{aligned}$$

$\square$

3.9 Green's operator

Assume M^3 is closed. The vector Laplacian bijectively maps fluxless knots and gradients on M to themselves, and we can define the inverse of the vector Laplacian. Call this the **Green's operator**

$$Gr : FK(M) \oplus G(M) \to FK(M) \oplus G(M) \quad ;$$

use similar notation $Gr(f)$ to denote the inverse Laplacian for functions. As the inverse to the vector Laplacian, the Green's operator is a bounded, linear, self-adjoint operator and $Gr(L(V)) = L(Gr(V)) = V$. It maps fluxless knots to fluxless knots and gradients to gradients. Also, Gr commutes with the gradient, divergence, and curl operators:

Proposition 3.27. *Let M be closed and simply-connected. Let $f \in C^\infty(M)$ have average value 0. Then,*

$$\begin{array}{lrcl} (1) & Gr(\nabla f) & = & \nabla(Gr(f)) \\ (2) & Gr(\nabla \cdot V) & = & \nabla \cdot Gr(V) \\ (3) & Gr(\nabla \times V) & = & \nabla \times Gr(V) \end{array}$$

Proof. For (1) apply L to each side: the left-hand side becomes $L(Gr(\nabla f)) = \nabla f$. The right-hand side becomes $L(\nabla Gr(f)) = \nabla \Delta(Gr(f)) = \nabla f$. Since L is one-to-one, equation (1) must hold.

For equation (2), apply the (scalar) Laplacian to both sides. The left-hand side gives

$$\Delta\left(Gr(\nabla \cdot V)\right) = \nabla \cdot V - [\nabla \cdot V] \quad ,$$

since $\nabla \cdot V$ has average value 0 on a closed manifold. The right-hand side becomes, by Proposition 3.21,

$$\Delta\left(\nabla \cdot Gr(V)\right) = \nabla \cdot L\left(Gr(V)\right) \quad .$$

Thus, after applying the Laplacian, the two sides are equal, so $Gr(\nabla \cdot V)$ and $\nabla \cdot Gr(V)$ must differ only by a harmonic function. On a closed manifold, the only harmonic functions are constants. Since both terms have average value zero on M, we conclude that they are indeed equal.

For equation (3), apply L to each side: the left-hand side becomes $L(Gr(\nabla \times V)) = \nabla \times V$. The right-hand side becomes $L(\nabla \times Gr(V)) = \nabla \times L(Gr(V)) = \nabla \times V$. Again since L is one-to-one, equation (3) must hold. $\square$

Now we determine the Green's operator for S^3. All results in the remainder of this section are due to DeTurck and Gluck [14].

In $\mathbb{R}^3$, the Green's operator is the vector potential defined on a subdomain Ω as

$$Gr(V) := A(V, \phi)(y) = \int_\Omega \phi(x, y)V(x)\, dx \quad ,$$

where $\phi(x, y) = -\frac{1}{4\pi}\frac{1}{|x - y|}$ is the fundamental solution to the Laplacian. We leave it as an exercise to check that $L(A(V, \phi)) = V$.

On S^3, we expect the Green's operator to also be some sort of convolution operator for a suitably chosen function ϕ. Alas, the operator $A(V, \phi)$ no longer inverts the vector Laplacian on the three-sphere. So we are forced to consider other vector convolution operators:

$$\begin{aligned}
A(V, \phi)(y) &= \int_{S^3} \phi(x, y)(L_{yx^{-1}})_* V(x)\, dx \\
B(V, \phi)(y) &= \int_{S^3} (L_{yx^{-1}})_* V(x) \times \nabla_y \phi(x, y)\, dx \\
G(V, \phi)(y) &= \nabla_y \int_{S^3} (L_{yx^{-1}})_* V(x) \cdot \nabla_y \phi(x, y)\, dx
\end{aligned}$$

They are linear in both V and ϕ. The vector Laplacian applied to each is

$$\begin{aligned}
L(A(V, \phi)) &= A(V, \Delta\phi) \;-\; 4A(V, \phi) \;-\; 2B(V, \phi) \\
L(B(V, \phi)) &= 2A(V, \Delta\phi) \;+\; B(V, \Delta\phi) \;-\; 2G(V, \phi) \\
L(G(V, \phi)) &= G(V, \Delta\phi)
\end{aligned}$$

Let $\alpha = \alpha(x, y)$ be the distance between two points x and y on the three-sphere. We choose potential functions ϕ_0, ϕ_1, ϕ_2 as

$$\begin{aligned}
\phi_0(\alpha) &= -\frac{1}{4\pi^2}\,(\pi-\alpha)\cot\alpha \\
\phi_1(\alpha) &= -\frac{1}{16\pi^2}\,\alpha(2\pi-\alpha) \\
\phi_0(\alpha) &= -\frac{1}{192\pi^2}\,[\,3\alpha(2\pi-\alpha)+2\alpha(\pi-\alpha)(2\pi-\alpha)\cot\alpha]
\end{aligned}$$

They have been selected so that

$$\begin{aligned}
\Delta\phi_0 &= \delta(\alpha)-[\delta] &&= \delta(\alpha)-\frac{1}{2\pi^2} \\
\Delta\phi_1 &= \phi_0-[\phi_0] &&= \phi_0+\frac{1}{8\pi^2} \\
\Delta\phi_2 &= \phi_1-[\phi_0] &&= \phi_1+\left(\frac{1}{24}+\frac{1}{32\pi^2}\right)
\end{aligned}$$

Here $\delta(\alpha)=\delta(x,y)$ is the Dirac delta distribution: $\int_{S^3} f(x)\delta(x,y)\,dx = f(y)$.

The idea now is to take a linear combination of A, B, and G operators with suitable potential functions from above in order to produce the Green's operator. For a constant function c, we calculate $B(V,c)=G(V,c)=0$ and $A(V,c)=2\pi^2 c\,[V]$.

Now take $L\left(A(V,\phi_0)\right)$:

$$\begin{aligned}
L(A(V,\phi_0)) &= A\left(V(x),\delta(x,y)\right)-A\left(V,\tfrac{1}{2\pi^2}\right)-4A(V,\phi_0)-2B(V,\phi_0) \\
L(A(V,\phi_0)) &= V(y)-[V]-4A(V,\phi_0)-2B(V,\phi_0)
\end{aligned}$$

This has almost produced the Green's operator; we need only remove the extra three terms, so introduce $2B(V,\phi_1)$ and take its Laplacian:

$$\begin{aligned}
2L(B(V,\phi_1)) &= 4A(V,\phi_0)+4A\left(V,\tfrac{1}{8\pi^2}\right)+2B(V,\phi_0)+2B\left(V,\tfrac{1}{8\pi^2}\right)-4G(V,\phi_1) \\
2L(B(V,\phi_1)) &= 4A(V,\phi_0)+[V]+2B(V,\phi_0)+0-4G(V,\phi_1) \\
L(A(V,\phi_0))+2L(B(V,\phi_1)) &= V-4G(V,\phi_1)
\end{aligned}$$

All three extra terms from $L(A)$ have cancelled, but one more is introduced. Fortunately, adding a $4G(V,\phi_2)$ term cancels it:

$$\begin{aligned} L\left(4G(V,\phi_2)\right) &= 4G(V,\phi_1) - 4G(V,[\phi_1]) \\ L\left(4G(V,\phi_2)\right) &= 4G(V,\phi_1) - 0 \\ L(A(V,\phi_0)) + 2L(B(V,\phi_1)) + L\left(4G(V,\phi_2)\right) &= V \end{aligned}$$

We have successfully calculated the Green's operator.

Proposition 3.28 (DG, [14]). *The Green's operator on S^3 is*

$$\begin{aligned} Gr(V) &= A(V,\phi_0) \, + \, 2B(V,\phi_1) \, + \, 4G(V,\phi_2) \\ Gr(V) &= \int_{S^3} \phi_0 (L_{yx^{-1}})_* V(x) \, dx \\ &\quad + 2 \int_{S^3} (L_{yx^{-1}})_* V(x) \times \nabla_y \phi_1 \, dx \\ &\quad + 4 \, \nabla_y \int_{S^3} (L_{yx^{-1}})_* V(x) \cdot \nabla_y \phi_2 \, dx \end{aligned}$$

Chapter 4

The Biot-Savart operator

In this chapter, we define the Biot-Savart operator on the three-sphere and on its subdomains. Two formulas are given, one for moving vector fields via parallel transport and one for moving via left-translation. DeTurck and Gluck have developed the Biot-Savart operator and electrodynamics on S^3 in [14]. We develop the analogous story on subdomains in this dissertation; the subdomain case is more challenging and allows for a comparison of results with those for bounded Euclidean domains.

This chapter begins with some preliminaries. Then we present an exposition of the Biot-Savart operator on S^3 with original proofs. In the next section follows the BS operator on subdomains. Then we present a quite useful lemma, which we call the Key Lemma. Next, we describe how Maxwell's Equations still hold in this setting. Finally we describe the kernel and image of the Biot-Savart operator and give other useful results about it.

4.1 Preliminaries

To define the Biot-Savart operator, we need the notions of parallel transport and left-translation of vector fields, as described in section 3.6. Recall, P_{yx} denotes parallel transport from x to y and $(L_{yx^{-1}})_*$ denotes left-translation from x to y. Let $\alpha(x, y)$ be the distance on the three-sphere between x and y.

We will express the Biot-Savart operator as a convolution of V, thought of as an electrical current, with an appropriate potential function ϕ. The left-translation version of BS requires two different convolutions; the two potential functions are

$$\begin{aligned}\phi_0(\alpha(x,y)) &= -\frac{1}{4\pi^2}(\pi-\alpha)\cot(\alpha)\\ \phi_1(\alpha(x,y)) &= -\frac{1}{16\pi^2}\alpha(2\pi-\alpha) \quad .\end{aligned}$$

We encountered both of these functions when describing the Green's operator in section 3.9. The function ϕ_0 is the fundamental solution of the Laplacian on S^3.

$$\Delta\phi_0 = \delta(\alpha) - \frac{1}{2\pi^2}$$

Here $\delta(\alpha)$ represents the Dirac delta function. The constant $1/2\pi^2$ appears so that the right-hand side has average value zero; the Laplacian of a function must have average value zero on a closed manifold.

The function ϕ_1 is defined so that

$$\Delta\phi_1 = \phi_0 - [\phi_0] \quad .$$

The parallel translation version of BS requires just one potential function,

$$\phi(\alpha(x,y)) = -\frac{1}{4\pi^2}(\pi-\alpha)\csc(\alpha) \quad .$$

This function ϕ is the fundamental solution of the shifted Laplacian.

$$\Delta\phi - \phi = \delta(\alpha)$$

Note that $\phi_0(\alpha) = \phi(\alpha)\cos\alpha$.

4.2 Defining the Biot-Savart operator on S^3

To define the Biot-Savart operator on the three-sphere, consider a current J existing in a compact region $\Omega \subset \mathbb{R}^3$. The Biot-Savart law (2.1) from magnetostatics defines the magnetic field B associated to J. In three-space, the notion of a magnetic field extends to the Biot-Savart operator, which acts on all vector fields on Ω. We seek the corresponding Biot-Savart operator defined on the three-sphere. In the next section, we consider the Biot-Savart operator on subdomains of S^3. In order to accomplish this, we must comprehend what essential properties a magnetic field in $\mathbb{R}^3$ possesses.

For J smooth, its magnetic field is smooth except across $\partial\Omega$, where the field remains continuous; magnetic fields are linear in J. Therefore we will work in the category of smooth linear operators. The first essential property is that magnetic fields have finite energy in an L^2 sense, provided that the current J does. Second, magnetic fields are divergence-free. Third, Ampere's Law dictates that the curl of a magnetic field must return the current J to which it is associated.

For magnetostatics on Ω, currents are standardly considered to be steady, hence divergence-free; since contained in Ω, the currents are tangent to the boundary $\partial\Omega$. As they have no gradient component, currents are fluid knots by the Hodge Decomposition Theorem, section 3.4. How should the Biot-Savart operator act when extended to act on a gradient? In subdomains of three-space, the kernel of BS consists of harmonic gradients and grounded gradients. On S^3, all gradients behave like grounded gradients, so a fourth requirement is that BS must vanish when applied to a gradient field on the three-sphere.

In the following proposition, we define the Biot-Savart operator via these four properties; they are sufficient to guarantee uniqueness. To show existence, two different formulas for $BS(V)$ are derived thereafter. The results in this section first appeared in [14]; herein we furnish independent proofs.

Proposition 4.1 (DG, [14]). *The* **Biot-Savart operator** $BS : VF(S^3) \to VF(S^3)$ *is defined to be the smooth linear operator satisfying these four properties:*

1.	BS has finite energy	$\langle BS(V), BS(V)\rangle < \infty$
2.	BS is divergence free	$\nabla \cdot BS(V) = 0$
3.	For V a fluid knot, curl inverts $BS(V)$	$\nabla \times BS(V) = V$
4.	BS vanishes on gradients	$BS(\nabla f) = 0 \quad \forall \nabla f \in VF(S^3)$

It is uniquely determined among smooth linear operators from $VF(\Omega)$ to $VF(\Omega)$.

Proof. Suppose B_1 and B_2 are two operators from $VF(S^3)$ to $VF(S^3)$ satisfying the four properties above. We show the operator $B_1 - B_2$ must be zero on the space of fluid knots. Each operator is itself zero on the space of gradients.

Let V be a fluid knot. Both $B_1(V)$ and $B_2(V)$ are divergence-free on the three-sphere, hence both are fluid knots. The curl of both $B_1(V)$ and $B_2(V)$ is V; thus $(B_1 - B_2)(V)$ lies in the kernel of curl on S^3, which is the gradients. Since the field $(B_1 - B_2)(V)$ is both a fluid knot and a gradient, it must be trivial. Therefore $B_1 = B_2$. □

To show existence, we define two integral formulas for of the Biot-Savart operator on the three-sphere. One moves vector fields via parallel transport and is a consequence of the Key Lemma, and the other moves them via left-translation.

Theorem 4.2 (DG, [14]). *The Biot-Savart operator can be written as an integral where vector fields are moved via parallel transport as*

$$BS(V)(y) \quad = \quad \int_{S^3} P_{yx} V(x) \times \nabla_y \phi(x, y)\, dx \tag{4.1}$$

Proof. We verify the four properties given in Proposition 4.1; three of them are proved later in this work. In the next chapter, we show that $BS(V)$ is bounded (see Equation 5.1) in terms of $\|V\|$, so the first property holds. In Proposition 4.4, we show that the parallel transport formula for BS is always divergence-free, whether V is defined on the entire three-sphere or only on a subdomain Ω, so the second property holds. For the third property, we prove Ampere's Law in Theorem 4.6; thus for V a fluid knot on S^3, we see $\nabla \times BS(V) = V$.

Lastly, we must show that BS as defined above vanishes for gradients. Let ∇f be a smooth vector field on the three-sphere. Then Theorem 4.6 on Maxwell's Equation implies

$$\nabla \times BS(\nabla f) \quad = \quad \nabla f(y) \; - \; \nabla_y \int_{S^3} \Delta f(x)\, \phi_0(x,y)\; dx$$

Recall ϕ_0 was defined as the fundamental solution to the Laplacian, so $\int_{S^3} \Delta f(x)\, \phi_0(x,y)\; dx = f(y) - [f]$, where $[f]$ is the average value of f on the three-sphere. Thus,

$$\begin{aligned} \nabla \times BS(\nabla f) \quad &= \quad \nabla f(y) \; - \; \nabla_y \left(f(y) - [f] \right) \\ \nabla \times BS(\nabla f) \quad &= \quad 0 \end{aligned}$$

So $BS(\nabla f)$ is curl-free; it is also divergence-free by the second property. On the three-sphere, the only trivial vector fields are both curl-free and divergence-free, so $BS(\nabla f) = 0$. $\square$

Now we shift towards finding a formula in terms of left-translation of vector fields. In Euclidean space, the Biot-Savart operator is the negative curl of the Green's operator,

$$BS(V) = -\nabla \times Gr(V) \quad .$$

We now calculate an explicit formula for $-\nabla \times Gr(V)$ on the three-sphere and show that it does indeed represent the Biot-Savart operator.

Proposition 3.28 gives the Green's operator on S^3 as $Gr(V) = A(V,\phi_0) + 2B(V,\phi_1) + 4G(V,\phi_2)$. Calculate the curl of each convolution operator individually.

The operator G is a gradient; its curl vanishes. The curl of A is straightforward:

$$\nabla \times A(V,\phi_0) = -2A(V,\phi_0) - B(V,\phi_0) \quad .$$

Calculating the curl of $B(V,\phi_1)$ is more complicated. To obtain Maxwell's Equation for $\nabla \times BS$ in Theorem 4.6, we calculated the curl of $B(V,\phi_0)$. The proof did not rely upon the choice of ϕ_0, so we are free to replace it by ϕ_1:

$$\begin{aligned}\nabla \times B(V, \phi_1) &= A(V, \Delta\phi_1) - G(V, \phi_1) \\ \nabla \times B(V, \phi_1) &= A(V, \phi_0) + \tfrac{1}{4}[V] - G(V, \phi_1)\end{aligned}$$

We conclude that minus the curl of $Gr(V)$ is

$$-\nabla \times Gr(V) = B(V, \phi_0) \; - \; \tfrac{1}{2}[V] \; + \; 2G(V, \phi_1)$$

We now show that $-\nabla \times Gr(V)$ does, in fact, represent the Biot-Savart operator on the three-sphere.

Theorem 4.3 (DG, [14]). *The Biot-Savart operator can be written as an integral where vector fields are moved via left-translation as $BS(V) = -\nabla \times Gr(V)$, which expanded is*

$$\begin{aligned} BS(V)(y) \quad = \quad & \int_{S^3} (L_{yx^{-1}})_* V(x) \times \nabla_y \phi_0 \; dx \\ & -\frac{1}{4\pi^2} \int_{S^3} (L_{yx^{-1}})_* V(x) \; dx \\ & +2\nabla_y \int_{S^3} (L_{yx^{-1}})_* V(x) \cdot \nabla_y \phi_1 \; dx \end{aligned} \tag{4.2}$$

Proof. Verify the four conditions of Proposition 4.1. The operator $-\nabla \times Gr(V)$ has finite energy due to the regularity of the Green's operator and the compactness of S^3. It is divergence-free because it is the curl of a vector field. Taking the curl of this operator yields

$$\begin{aligned}\nabla \times (-\nabla \times Gr(V)) &= L(Gr(V)) - \nabla\left(Gr(\nabla \cdot V)\right) \\ \nabla \times (-\nabla \times Gr(V)) &= V - \nabla\left(Gr(\nabla \cdot V)\right)\end{aligned} \tag{4.3}$$

So if V is divergence-free, then the curl of this operator returns V.

The last condition states that the operator should vanish on gradients. All gradients are mapped to zero because $-\nabla \times Gr(\nabla f) = -Gr(\nabla \times \nabla f) = -Gr(0) = 0$. □

4.3 Defining the Biot-Savart operator on subdomains of S^3

On a subdomain Ω of the three-sphere, we define BS to be the same operator as defined on S^3, only with the region of integration restricted to Ω. Any vector field $V \in VF(\Omega)$ is assumed to extend continuously to a vector field defined on all of S^3 which vanishes outside of Ω; this is in keeping with electrodynamical setups where a current flow is defined inside a particular region. This produces an operator $BS : VF(\Omega) \to VF(\Omega)$, given by the formulas

$$\begin{aligned} BS(V)(y) &= \int_\Omega P_{yx}V(x) \times \nabla_y \phi(x,y)\, dx \\ BS(V)(y) &= \int_\Omega (L_{yx^{-1}})_* V(x) \times \nabla_y \phi_0 \, dx \\ &\quad -\frac{1}{4\pi^2} \int_\Omega (L_{yx^{-1}})_* V(x)\, dx \\ &\quad +2\nabla_y \int_\Omega (L_{yx^{-1}})_* V(x) \cdot \nabla_y \phi_1 \, dx \end{aligned}$$

For $y \notin \Omega$, these formulas also define the behavior of $BS(V)$ outside of the domain.

Proposition 4.4. *BS is divergence-free, whether defined on Ω or on S^3.*

In the following proof, we do not use any facts about BS save its parallel transport formula. It is therefore fine to cite this proof when proving that BS was represented by this formula in Proposition 4.2.

Proof. In this argument, we calculate the divergence of BS directly from the parallel transport formula. We use $\Omega \subseteq S^3$ to denote the domain.

$$\begin{aligned} \nabla_y \cdot BS(V)(y) &= \nabla_y \cdot \int_\Omega P_{yx}V(x) \times \nabla_y \phi \, dx \\ &= \int_\Omega \nabla_y \cdot (P_{yx}V(x) \times \nabla_y \phi)\, dx \end{aligned}$$

Now apply vector identity 6 from Appendix A:

$$\begin{aligned} \nabla_y \cdot BS(V)(y) &= \int_\Omega \nabla_y \phi \cdot (\nabla_y \times P_{yx}V(x)) - P_{yx} \cdot (\nabla_y \times \nabla_y \phi)\, dx \\ \nabla_y \cdot BS(V)(y) &= \int_\Omega \nabla_y \phi \cdot (\nabla_y \times P_{yx}V(x)) \quad , \end{aligned}$$

since the curl of $\nabla_y \phi$ is zero.

Now we think of the vectors as being in $\mathbb{R}^4$. Recall,

$$\begin{aligned} P_{yx}V(x) &= V(x) - \frac{V \cdot y}{1+\cos\alpha}\,(x+y) \\ \nabla_y \phi &= \phi'(\alpha)\,\nabla_y \alpha \;=\; \phi'(\alpha)\,\frac{y\cos\alpha - x}{\sin\alpha} \end{aligned}$$

Now,

$$\begin{aligned} \nabla_y \times P_{yx}V(x) &= \nabla_y \times V(x) - \nabla_y \times \frac{V \cdot y}{1+\cos\alpha}\,(x+y) \\ &= 0 - \nabla_y \left(\frac{V \cdot y}{1+\cos\alpha}\right) \times (x+y) \;-\; \frac{V \cdot y}{1+\cos\alpha}\,\nabla_y \times (x+y) \end{aligned}$$

The cross product above is taken in the tangent space at y, so a vector crossed with y contributes nothing. Similarly $\nabla_y \times y = 0$. Since x is fixed with respect to y, $\nabla_y \times x = 0$. Thus,

$$\nabla_y \times P_{yx}V(x) \;=\; -\nabla_y \left(\frac{V \cdot y}{1+\cos\alpha}\right) \times x$$

Now calculate the gradient above:

$$\nabla_y \left(\frac{V \cdot y}{1+\cos\alpha}\right) \;=\; \frac{1}{1+\cos\alpha}\nabla_y\,(V \cdot y) \,+\, \frac{V \cdot y}{(1+\cos\alpha)^2}\nabla_y \cos\alpha$$

To proceed, we need to calculate $\nabla_y\,(V \cdot y)$. It should point along the component of V that lies perpendicular to y. Let θ be the angle between $V(x)$ and y in $\mathbb{R}^4$. Then $(V \cdot y) = |V|\cos\theta$, and we calculate

$$\begin{aligned} \nabla_y\,(V \cdot y) &= \nabla_y\,(|V|\cos\theta) \\ \nabla_y\,(V \cdot y) &= \tfrac{d}{d\theta}\,(|V|\cos\theta)\,\nabla_y \theta \\ \nabla_y\,(V \cdot y) &= -|V|\sin\theta\,\frac{y\cos\theta - V/|V|}{\sin\theta} \\ \nabla_y\,(V \cdot y) &= V - (V \cdot y)\,y \end{aligned} \tag{4.4}$$

Indeed this formula holds for the gradient of the inner product of any two vectors in $\mathbb{R}^n$.

Also, $\nabla_y \cos\alpha \;=\; \nabla_y\,(x \cdot y) \;=\; x - (x \cdot y)$. Then,

$$\nabla_y \left(\frac{V \cdot y}{1+\cos\alpha}\right) \;=\; \frac{1}{1+\cos\alpha}\,(V - (V \cdot y)\,y) \;-\; \frac{V \cdot y}{(1+\cos\alpha)^2}\,(x - (x \cdot y)\,y)$$

Now that we have calculated this gradient, express the curl of $P_{yx}V$ as

$$\nabla_y \times P_{yx}V(x) \quad = \quad \left(-\frac{1}{1+\cos\alpha}\,(V-(V\cdot y)\,y)\;+\;\frac{V\cdot y}{(1+\cos\alpha)^2}\,(x-(x\cdot y)\,y)\right)\times x$$

Express this to a triple product, described in section 3.5.

$$\nabla_y \times P_{yx}V(x) \quad = \quad \left[\,y,\;-\left(-\frac{1}{1+\cos\alpha}\,(V-(V\cdot y)\,y)\;+\;\frac{V\cdot y}{(1+\cos\alpha)^2}\,(x-(x\cdot y)\,y)\right),\;x\,\right]$$

A triple product is zero if the three vectors are not linearly independent, so all x and y vectors in the middle term vanish.

$$\nabla_y \times P_{yx}V(x) \quad = \quad \frac{1}{1+\cos\alpha}\,[y,V,x]$$

Let $x^\perp$ denote the component of x that is perpendicular to y, i.e., $x^\perp = x-(x\cdot y)\,y$ is in T_yS^3. Similarly define $V^\perp \in T_yS^3$. The triple product is unchanged if we switch to these vectors: $[y,V,x]=[y,V^\perp,x^\perp]$. Again, let the triple product again represents a cross product.

$$\nabla_y \times P_{yx}V(x) = \left[y,V^\perp,x^\perp\right] = V^\perp \times x^\perp \in T_yS^3$$

Returning to the divergence of $BS(V)$,

$$\begin{aligned}
\nabla_y \cdot BS(V)(y) \quad &= \quad \int_\Omega \nabla_y\phi\cdot(\nabla_y \times P_{yx}V(x)) \\
&= \quad \int_\Omega \phi'(\alpha)\,\nabla_y\alpha\cdot\left(\frac{1}{1+\cos\alpha}\,V^\perp\times x^\perp\right)\,dx \\
&= \quad \int_\Omega \frac{\phi'(\alpha)}{1+\cos\alpha}\,\frac{((x\cdot y)\,y-x)}{\sin\alpha}\cdot\left(\frac{1}{1+\cos\alpha}\,V^\perp\times x^\perp\right)\,dx \\
&= \quad \int_\Omega \frac{\phi'(\alpha)}{(1+\cos\alpha)\sin\alpha}\left(-x^\perp\right)\cdot\left(V^\perp\times x^\perp\right)\,dx
\end{aligned}$$

The product $x^\perp\cdot\left(V^\perp\times x^\perp\right)\equiv 0$, so the integrand vanishes identically at each point. We conclude that $\nabla_y\cdot BS(V)(y)=0$. Hence the parallel transport formula for BS is always free. $\square$

4.4 Key Lemma

In this section, we prove an important lemma relating vector fields and functions on S^3. For a specific choice of function, this lemma specializes to a pointwise version of Maxwell's Equation

$$\nabla \times B = J + \frac{\partial E}{\partial t} \quad .$$

Lemma 4.5. *[Key Lemma] Let x, y be two non-antipodal points in S^3. Let $\phi = \phi(\alpha)$ be a function, depending only on α, which may have a singularity at $\alpha = 0$ but is otherwise smooth. Let $V(x)$ be a tangent vector at $x \in S^3$. Then,*

$$\nabla_y \times \{P_{yx} V(x) \times \nabla_y \phi\} - \nabla_y \{V(x) \cdot \nabla_x (\phi \cos \alpha)\} = (\Delta\phi - \phi)(V(x) - (V(x) \cdot y)\, y)$$

We consider both inner products to be defined on $\mathbb{R}^4$. The one on the left-hand side is defined on $T_x S^3$, but $V(x) \in T_x S^3$ can equivalently be viewed on $\mathbb{R}^4$.

Proof. Denote the two terms on the left-hand side of the Key Lemma as terms S and T. We will show that $S + T = (\Delta\phi - \phi)(V(x) - (V(x) \cdot y)\, y)$.

$$\begin{aligned} S &= \nabla_y \times \{P_{yx} V(x) \times \nabla_y \phi\} \\ T &= -\nabla_y \{V(x) \cdot \nabla_x (\phi \cos \alpha)\} \end{aligned}$$

The gradient of $\phi(\alpha)$ is calculated using equation (3.7)

$$\begin{aligned} \nabla_y \phi(\alpha) &= \phi'(\alpha)\, \nabla_y \alpha \\ &= \phi'(\alpha)\, \frac{y \cos \alpha - x}{\sin \alpha} \quad . \end{aligned}$$

Then S becomes

$$\begin{aligned} S &= \nabla_y \times \left(P_{yx} V(x) \times \phi'(\alpha) \frac{y \cos \alpha - x}{\sin \alpha} \right) \\ S &= \nabla_y \times \left(\frac{\phi'(\alpha)}{\sin \alpha} \{P_{yx} V(x) \times (y \cos \alpha - x)\} \right) \quad . \end{aligned} \tag{4.5}$$

Now convert the cross product, taken at $y \in S^3$, to a triple product and use equation (3.8) to express the parallel transport of $V(x)$.

$$\begin{aligned} P_{yx}V(x) \times (y\cos\alpha - x) &= [y,\, P_{yx}V(x),\, y\cos\alpha - x] \\ &= \left[y, \left(V(x) - \frac{V\cdot y}{1+\langle x,y\rangle}(x+y)\right), -x\right] \\ &= [y,\, V(x),\, -x] \\ &= [y,\, x,\, V(x)] \end{aligned}$$

Insert the triple product into Equation 4.5 and apply vector identity 7 from Appendix A: $\nabla \times fA = \nabla f \times A + f(\nabla \times A)$. Then, we obtain

$$S = \nabla_y \times \frac{\phi'(\alpha)}{\sin\alpha}[y,x,V(x)] \tag{4.6}$$

$$S = \left\{\nabla_y\left(\frac{\phi'(\alpha)}{\sin\alpha}\right) \times [y,x,V(x)]\right\} + \frac{\phi'(\alpha)}{\sin\alpha}\nabla_y \times [y,x,V(x)] \tag{4.7}$$

Call these two terms S_1 and S_2 respectively, so $S = S_1 + S_2$.

$$S_1 = \nabla_y\left(\frac{\phi'(\alpha)}{\sin\alpha}\right) \times [y,x,V(x)] \tag{4.8}$$

$$S_2 = \frac{\phi'(\alpha)}{\sin\alpha}\nabla_y \times [y,x,V(x)] \tag{4.9}$$

We analyze these two terms separately. The first one, S_1, is converted to a double triple product.

$$\begin{aligned} S_1 &= \left[y, \nabla_y\frac{\phi'(\alpha)}{\sin\alpha}, [y,x,V(x)]\right] \\ S_1 &= \left[y, \frac{d}{d\alpha}\left(\frac{\phi'(\alpha)}{\sin\alpha}\right)\nabla_y\alpha, [y,\, x,\, V(x)]\right] \\ S_1 &= \frac{d}{d\alpha}\left(\frac{\phi'}{\sin\alpha}\right)\left[y, \frac{y\cos\alpha - x}{\sin\alpha}, [y,\, x,\, V(x)]\right] \\ S_1 &= -\frac{\phi''\sin\alpha - \phi'\cos\alpha}{\sin^3\alpha}[y,\, x, [y,\, x,\, V(x)]] \end{aligned}$$

Now we utilize Lemma 3.14 to evaluate the double triple product. But first we must calculate $V^\perp$, the component of $V(x)$ orthogonal to x and y from equation (3.5).

$$V^\perp = V - \frac{V\cdot y}{\sin^2\alpha}\,y + \frac{\cos\alpha}{\sin^2\alpha}(V\cdot y)\,x$$

Then S_1 becomes

$$\begin{aligned}
S_1 &= -\frac{\phi'' \sin\alpha - \phi' \cos\alpha}{\sin^3\alpha} \left(-\sin^2\alpha\, V^{\perp}\right) \\
S_1 &= \frac{\phi'' \sin\alpha - \phi' \cos\alpha}{\sin^3\alpha} \left(\sin^2\alpha\, V + (V\cdot y)\, y - \cos\alpha\,(V\cdot y)\, x\right) \\
S_1 &= \left(\phi'' - \phi' \,\frac{\cos\alpha}{\sin\alpha}\right) V \\
&\quad + \left(-\phi''\,\frac{1}{\sin^2\alpha} + \phi' \,\frac{\cos\alpha}{\sin^3\alpha}\right)(V\cdot y)\; y \\
&\quad + \left(\phi''\,\frac{\cos\alpha}{\sin^2\alpha} - \phi' \,\frac{\cos^2\alpha}{\sin^3\alpha}\right)(V\cdot y)\; x
\end{aligned} \tag{4.10}$$

This finishes S_1, which was the first term of S.

Now we consider S_2 from equation (4.9). Since $[y, x, V] = [x, V, y]$, we can rewrite it as

$$S_2 = \frac{\phi'(\alpha)}{\sin\alpha}\, \nabla_y \times [x, V(x), y] \quad .$$

Now apply Lemma 3.13 to obtain

$$\begin{aligned}
S_2 &= \frac{\phi'(\alpha)}{\sin\alpha}\, \left(2(x\cdot y)\, V - 2(V\cdot y)\, x\right) \\
S_2 &= 2\phi' \,\frac{\cos\alpha}{\sin\alpha}\, V - 2\phi' \,\frac{1}{\sin\alpha}\,(V\cdot y)\, x
\end{aligned}$$

Summing terms S_1 and S_2, we obtain the following expression for S:

$$\begin{aligned}
S = &\left(\phi'' + \phi' \frac{\cos\alpha}{\sin\alpha}\right) V \\
&+ \left(-\phi'' \frac{1}{\sin^2\alpha} + \phi' \frac{\cos\alpha}{\sin^3\alpha}\right)(V\cdot y)\; y \\
&+ \left(\phi'' \frac{\cos\alpha}{\sin^2\alpha} - \phi' \frac{\cos^2\alpha}{\sin^3\alpha} - 2\phi' \frac{1}{\sin\alpha}\right)(V\cdot y)\; x
\end{aligned} \tag{4.11}$$

We turn our attention to the second term T of the lemma.

$$\begin{aligned}
T &= -\nabla_y \left(V(x)\cdot \nabla_x \left(\phi\cos\alpha\right)\right) \\
T &= -\nabla_y \left(V(x)\cdot \frac{d}{d\alpha}\left(\phi\cos\alpha\right)\nabla_x\alpha\right) \\
T &= -\nabla_y \left(\left(\phi'\cos\alpha - \phi\sin\alpha\right) V(x)\cdot \frac{x\cos\alpha - y}{\sin\alpha}\right)
\end{aligned}$$

Since $V(x) \in T_x S^3$, we have that $V(x) \cdot x = 0$. Thus,

$$
\begin{aligned}
T &= -\nabla_y \left[(\phi' \cos\alpha - \phi \sin\alpha)\ V(x) \cdot \frac{-y}{\sin\alpha} \right] \\
T &= \nabla_y \left[\frac{\phi' \cos\alpha - \phi \sin\alpha}{\sin\alpha} (V \cdot y) \right] \\
T &= (V \cdot y)\ \nabla_y \left(\phi' \frac{\cos\alpha}{\sin\alpha} - \phi \right) + \left(\phi' \frac{\cos\alpha}{\sin\alpha} - \phi \right) \nabla_y (V \cdot y) \\
T &= (V \cdot y)\ \frac{d}{d\alpha} \left(\phi' \frac{\cos\alpha}{\sin\alpha} - \phi \right)\ \nabla_y \alpha + \left(\phi' \frac{\cos\alpha}{\sin\alpha} - \phi \right) \nabla_y (V \cdot y) \qquad (4.12)
\end{aligned}
$$

Now, using equation (3.7) for $\nabla_y \alpha$, the first term of equation (4.12) becomes

$$
(V \cdot y) \left(\phi'' \frac{\cos\alpha}{\sin^2\alpha} - \phi' \frac{1}{\sin^3\alpha} - \phi' \frac{1}{\sin\alpha} \right) (y \cos\alpha - x) \quad . \qquad (4.13)
$$

Equation (4.4) states that $\nabla_y (V \cdot y) = V - (V \cdot y)\, y$. Then the second term of T is

$$
\left(\phi' \frac{\cos\alpha}{\sin\alpha} - \phi \right) (V - (V \cdot y)\, y) \quad . \qquad (4.14)
$$

Finally, by combining equations (4.13) and (4.14) we obtain an expression for T:

$$
T = (V \cdot y) \left(\phi'' \frac{\cos\alpha}{\sin^2\alpha} - \phi' \frac{1}{\sin^3\alpha} + \phi' \frac{1}{\sin\alpha} \right) (y \cos\alpha - x) - \left(\phi' \frac{\cos\alpha}{\sin\alpha} - \phi \right) (V - (V \cdot y)\, y)
$$

Or, in terms of the components $\{V, y, x\}$:

$$
\begin{aligned}
T = \quad & \left(\phi' \frac{\cos\alpha}{\sin\alpha} - \phi \right) V \\
& + \left(\phi'' \frac{\cos^2\alpha}{\sin^2\alpha} - \phi' \frac{\cos\alpha}{\sin^3\alpha} - 2\phi' \frac{\cos\alpha}{\sin\alpha} + \phi \right) (V \cdot y)\, y \qquad (4.15) \\
& + \left(-\phi'' \frac{\cos\alpha}{\sin^2\alpha} + \phi' \frac{1}{\sin^3\alpha} + \phi' \frac{1}{\sin\alpha} \right) (V \cdot y)\, x
\end{aligned}
$$

Recall our formula (4.11) for S :

$$
\begin{aligned}
S = & \left(\phi'' + \phi' \frac{\cos\alpha}{\sin\alpha} \right) V \\
& + \left(-\phi'' \frac{1}{\sin^2\alpha} + \phi' \frac{\cos\alpha}{\sin^3\alpha} \right) (V \cdot y)\ y \qquad (4.11) \\
& + \left(\phi'' \frac{\cos\alpha}{\sin^2\alpha} - \phi' \frac{\cos^2\alpha}{\sin^3\alpha} - 2\phi' \frac{1}{\sin\alpha} \right) (V \cdot y)\ x
\end{aligned}
$$

Returning to the statement of Lemma 4.5, we can finally simplify the left-hand side by adding $S+T$ using equations (4.11) and (4.15); all x terms cancel, leaving

$$\begin{aligned} S+T &= \left(\phi'' + 2\phi' \frac{\cos\alpha}{\sin\alpha} - \phi\right) V \\ &\quad + \left(-\phi'' - 2\phi' \frac{\cos\alpha}{\sin\alpha} + \phi\right) (V \cdot y)\, y \\ S+T &= \left(\phi'' + 2\phi' \frac{\cos\alpha}{\sin\alpha} - \phi\right) (V - (V \cdot y)\, y) \end{aligned}$$

Recall from Table 3.1 that the Laplacian of $\phi(\alpha)$ is $\Delta\phi(\alpha) = \phi'' + 2\phi' \dfrac{\cos\alpha}{\sin\alpha}$. Thus we have proven the Key Lemma, since

$$S+T = (\Delta\phi - \phi)\,(V - (V \cdot y)\ y) \quad . \qquad \square$$

The Key Lemma has an important application with Maxwell's equations. We use it to prove Ampere's Law for the curl of BS, written in parallel translation format; see section 4.5.1.

Another application of the Key Lemma is in proving an integral formula for the linking number of two knots on S^3; refer to [14] for details.

4.5 Maxwell's equations

To reconnect the Biot-Savart operator with its physical origins, we show that Maxwell's equations hold on $\Omega \subset S^3$. We view V as a current and $BS(V)$ as the corresponding magnetic field. If the vector field V either has a nonzero divergence or is not tangent to the boundary, then it no longer represents a steady current contained in Ω. By considering a time-varying electric field in this case, the system can be considered "closed". Let $\rho(x,t) = -(\nabla \cdot V)t$ be the volume charge density in Ω and $\sigma(x,t) = (V \cdot \hat{n})t$ be the surface charge density on the boundary $\partial\Omega$. Each of these charge distributions contributes to the electric field E; the details follow later in this section.

Maxwell's four equations for an electric field E and magnetic field B due to a current V are

$$\begin{aligned} 1.\quad \nabla \cdot E &= \rho \\ 2.\quad \nabla \times E &= \frac{\partial B}{\partial t} \\ 3.\quad \nabla \cdot B &= 0 \\ 4.\quad \nabla \times B &= V + \frac{\partial E}{\partial t} \end{aligned}$$

We have chosen units so that both the permittivity of free space ϵ_0 and the permeability of free space μ_0 are identically 1.

We now show that these four equations hold when we think of $B = BS(V)$ as the magnetic field. The third equation holds because BS is divergence-free by Proposition 4.4. The fourth equation is known as Ampere's Law when V is steady. The following theorem demonstrates that it holds in our setting, where the volume charge ρ and the surface charge σ might be time dependent.

Theorem 4.6. *For Ω a compact subset of S^3 with smooth boundary, and V a smooth vector field on Ω,*

$$\begin{aligned} \nabla_y \times BS(V)(y) \quad = & \begin{pmatrix} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{pmatrix} \\ & -\nabla_y \int_\Omega \phi_0 \ (\nabla_x \cdot V(x)) \ dx \\ & +\nabla_y \int_{\partial\Omega} \phi_0 \, V(x) \cdot \hat{n} \ d(area_x) \end{aligned}$$

We give two proofs of this theorem in the next two subsections. First is a quick proof via the Key Lemma and parallel transport. Second is a direct proof, using left-translation.

Remark 4.7. When $\Omega = S^3$, the theorem above states

$$\nabla_y \times BS(V)(y) = V(y) - \nabla_y \int_\Omega \phi_0 \ (\nabla_x \cdot V(x)) \ dx \quad .$$

This is precisely a restatement of equation (4.3):

$$\nabla \times BS(V) \ = \ \nabla \times (-\nabla \times Gr(V)) \ = \ V - \nabla Gr(\nabla \cdot V) \qquad . \tag{4.3}$$

In order for Maxwell's fourth equation to hold, the last two terms in the theorem above should constitute the time derivative on an electrodynamic field E. Define two electric fields by

$$\begin{aligned} E_\rho(y,t) &= -\left(\nabla_y \int_\Omega \phi_0 \, (\nabla_x \cdot V(x)) \; dx\right) t &= \nabla_y \int_\Omega \phi_0 \, \rho \, dx \\ E_\sigma(y,t) &= \left(\nabla_y \int_{\partial\Omega} \phi_0 \, V(x) \cdot \hat{n} \; d(area_x)\right) t &= \nabla_y \int_{\partial\Omega} \phi_0 \, \sigma \; d(area_x) \end{aligned}$$

Then consider the electric field $E = E_\rho + E_\sigma$. By the theorem above, Maxwell's fourth equation does hold.

$$\nabla \times B \;=\; V + \dot{E}_\rho + \dot{E}_\sigma \;=\; V + \frac{\partial E}{\partial t}$$

We can view $\dot{E}_\rho$ as the time rate of change of the electrodynamic field due to $\partial\rho/\partial t$, the change in volume charge density; also we can view $\dot{E}_\sigma$ as the time rate of change of the electrodynamic field due to $\partial\sigma/\partial t$, the change in surface charge density. We also may view $\dot{E}_\rho$ as an electrostatic field itself due to the time-independent volume charge $\partial\rho/\partial t = \nabla \cdot V$; similarly, we may view $\dot{E}_\sigma$ as an electrostatic field due to the time-independent surface charge $\partial\rho/\partial t = V \cdot \hat{n}$. In this section we adopt the former viewpoint; in the next section we make use of the latter.

With our electric field in place, consider the first two Maxwell's equations. The divergence of E in Ω can be calculated by taking the divergence of both sides of Maxwell's fourth equation:

$$\begin{aligned} \nabla \cdot \nabla \times BS(V) &= \nabla \cdot V + \nabla \cdot \frac{\partial E}{\partial t} \\ 0 &= \nabla \cdot V + \frac{\partial}{\partial t}(\nabla \cdot E) \end{aligned}$$

This implies that $\nabla \cdot E = -(\nabla \cdot V)t = \rho$, Maxwell's first equation.

Maxwell's second equation holds trivially. The electric field E is defined by two gradients and ergo is curl-free. The magnetic field BS does not depend upon time, even when the electric field E is time-dependent; hence

$$\nabla \cdot E = \frac{\partial B}{\partial t} = 0 \quad .$$

4.5.1 Parallel transport proof

The key lemma, as perhaps its most important consequence, directly proves Maxwell's fourth equation, Theorem 4.6.

Proof. We will use the key lemma for the vector field V given in the theorem and for the function

$$\phi(\alpha) = -\frac{1}{4\pi^2}(\pi - \alpha)\csc\alpha \quad .$$

Recall that $\phi_0(\alpha) = \phi(\alpha)\cos\alpha$ and $\Delta\phi - \phi = \delta(\alpha) = \delta(x, y)$.

To begin, recall the statement of the Key Lemma.

$$\nabla_y \times \{P_{yx}V(x) \times \nabla_y \phi\} - \nabla_y \{V(x) \cdot \nabla_x (\phi \cos\alpha)\} = (\Delta\phi - \phi)(V(x) - (V \cdot y)\, y)$$

Now, integrate both sides over Ω with respect to x:

$$\int_\Omega \nabla_y \times \{P_{yx}V(x) \times \nabla_y \phi\}\; dx - \int_\Omega \nabla_y \{V(x) \cdot \nabla_x \phi_0\}\; dx = \int_\Omega \delta(x, y)(V(x) - (V \cdot y)\, y)\; dx$$

We may interchange the integral in x variables with the gradient and curl operators on the left-hand side, since they are in terms of y.

$$\nabla_y \times \int_\Omega P_{yx}V(x) \times \nabla_y \phi\; dx - \nabla_y \int_\Omega V(x) \cdot \nabla_x \phi_0\; dx \;=\; \int_\Omega \delta(x, y)(V(x) - (V \cdot y)\, y)\; dx$$

The first term on the left-hand side is simply the curl of the Biot-Savart operator,

$$\nabla_y \times BS(V)(y) = \nabla_y \times \int_\Omega P_{yx}V(x) \times \nabla_y \phi\; dx \quad .$$

Substituting that into the previous equation obtains

$$\nabla_y \times BS(V)(y) = \nabla_y \int_\Omega V(x) \cdot \nabla_x \phi_0\; dx + \int_\Omega \delta(x, y)(V(x) - (V(x) \cdot y)\, y)\; dx$$

Use now the vector identity $V \cdot \nabla\phi_0 = \nabla \cdot (\phi_0 V) - \phi_0 \nabla \cdot V$ (see Appendix A) to expand the right-hand side:

$$\begin{aligned}\nabla_y \times BS(V)(y) \;&=\; \nabla_y \int_\Omega \nabla_x \cdot (\phi_0 V(x))\; dx - \nabla_y \int_\Omega \phi_0\, \nabla_x \cdot V(x)\; dx \\ &\quad + \int_\Omega \delta(x, y)(V(x) - (V(x) \cdot y)\, y)\; dx\end{aligned}$$

Apply the Divergence Theorem:

$$\begin{aligned}\nabla_y \times BS(V)(y) &= \nabla_y \int_{\partial\Omega} \phi_0 \, V(x) \cdot \hat{n} \, d(area_x) - \nabla_y \int_\Omega \phi_0 \, \nabla_x \cdot V(x) \, dx \\ &\quad + \int_\Omega \delta(x,y) \, (V(x) - (V(x) \cdot y) \, y) \; dx\end{aligned}$$

The first two integrals on the right-hand side are as desired. The third integral takes on the value of its integrand when $x = y$, i.e.,

$$\int_\Omega \delta(x,y) \, (V(x) - (V(x) \cdot y) \, y) \; dx \; = \; V(y) - (V(y) \cdot y) \, y \quad .$$

Since $V(y)$ lies in the tangents space of y, the term $(V(y) \cdot y) = 0$. Also, $V(y)$ is assumed to be zero outside of Ω, so this third integral vanishes there. We write this fact explicitly,

$$\int_\Omega \delta(x,y) \, (V(x) - (V(x) \cdot y) \, y) \; dx \; = \; \begin{cases} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases} \quad ,$$

which completes the proof of the theorem. □

4.5.2 Left-translation proof

Proof. Write out the left-translation version of $BS(V)(y)$:

$$\begin{aligned}BS(V)(y) = \quad & \int_\Omega (L_{yx^{-1}})_* V(x) \times \nabla_y \phi_0 \, dx \\ & - \frac{1}{4\pi^2} \int_\Omega (L_{yx^{-1}})_* V(x) \, dx \\ & + 2\nabla_y \int_\Omega (L_{yx^{-1}})_* V(x) \cdot \nabla_y \phi_1 \, dx\end{aligned} \tag{4.2}$$

Recognize the first term as the vector convolution operator $B(V, \phi_0)$ as given in section 3.9. Call the other two terms I_2 and I_3 respectively.

$$BS(V)(y) \; = \; B(V, \phi_0) \; + \; I_2 \; + \; I_3$$

We take the curl of the three terms separately. Since I_3 is a gradient, its curl is zero.

$$\nabla_y \times I_3 \; = \; \nabla_y \times 2\nabla_y \int_\Omega (L_{yx^{-1}})_* V(x) \cdot \nabla_y \phi_1 \, dx = 0 \quad .$$

The second integral produces a vector field $I_2(y)$ that is left-invariant, since its integrand is left-translated to y. Ergo it is a curl eigenfield, so $\nabla \times I_2 = -2I_2$. Thus, two of the three terms in the curl of $BS(V)$ are complete:

$$
\begin{aligned}
\nabla_y \times BS(V)(y) &= \nabla_y \times B(V,\phi_0) - 2I_2(y) \\
\nabla_y \times BS(V)(y) &= \nabla_y \times B(V,\phi_0) + \frac{1}{2\pi^2}\int_\Omega (L_{yx^{-1}})_* V(x)\, dx \qquad (4.16)
\end{aligned}
$$

Now we calculate the curl of $B(V,\phi_0)$. The result is independent of the function ϕ_0. Earlier we referenced this result in order to calculate the curl of $B(V,\phi_1)$ in the proof of Theorem 4.3.

$$
\begin{aligned}
\nabla_y \times B(V,\phi_0)(y) &= \nabla_y \times \int_\Omega (L_{yx^{-1}})_* V(x) \times \nabla_y \phi_0 \, dx \\
&= \int_\Omega \nabla_y \times \big((L_{yx^{-1}})_* V(x) \times \nabla_y \phi_0\big)\, dx
\end{aligned}
$$

We now utilize Identity 8 from the Appendix,

$$
\nabla \times (U \times W) = \nabla_W U - \nabla_U W + (\nabla \cdot W)U - (\nabla \cdot U)W
$$

where we set $U = (L_{yx^{-1}})_* V(x)$ and $W = \nabla_y \phi_0$. After applying the identity, there are four terms to analyze.

$$
\begin{aligned}
\nabla_y \times B(V,\phi_0)(y) &= +\int_\Omega \nabla_W U \, dx & (4.17)\\
&\quad -\int_\Omega \nabla_U W \, dx & (4.18)\\
&\quad +\int_\Omega (\nabla_y \cdot \nabla_y \phi_0)\ (L_{yx^{-1}})_* V(x)\, dvol_x & (4.19)\\
&\quad -\int_\Omega \big(\nabla_y \cdot (L_{yx^{-1}})_* V(x)\big)\ \nabla_y \phi_0 \, dvol_x & (4.20)
\end{aligned}
$$

We claim that the last integral above, term (4.20) vanishes. We must calculate the divergence of $(L_{yx^{-1}})_* V(x)$ with respect to y variables. For a fixed value of x, the vector $V(x)$ is translated to a vector in the tangent space of each y this forms a left-invariant vector field. Left-invariant vector fields on S^3 are divergence-free, so $\nabla_y \cdot (L_{yx^{-1}})_* V(x) = 0$.

Let's examine the third integral, term (4.19)

$$
\int_\Omega (\nabla_y \cdot \nabla_y \phi_0)\ (L_{yx^{-1}})_* V(x)\, dx \ = \ \int_\Omega (\Delta_y \phi_0)\ (L_{yx^{-1}})_* V(x)\, dvol_x
$$

Recall that $\Delta\phi_0(\alpha) = \delta(\alpha) - 1/2\pi^2$. Then, (4.19) becomes

$$\begin{aligned}
&\int_\Omega (\nabla_y \cdot \nabla_y \phi(\alpha))\,(L_{yx^{-1}})_* V(x)\,dx \\
&\qquad = \int_\Omega \left(\delta(\alpha) - \tfrac{1}{2\pi^2}\right)(L_{yx^{-1}})_* V(x)\,dx \\
&\qquad = \int_\Omega \delta(\alpha)\,(L_{yx^{-1}})_* V(x)\,dx \;-\; \frac{1}{2\pi^2}\int_\Omega (L_{yx^{-1}})_* V(x)\,dx
\end{aligned}$$

In the equation on the line above, $\delta(\alpha)$ vanishes except when $\alpha = 0$, i.e., when $x = y$. In this case, the integral involving $\delta(\alpha)$ simply becomes the value of the vector field when $x = y$, namely

$$\int_\Omega \delta(\alpha)\,(L_{yx^{-1}})_* V(x)\,dx = \begin{cases} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases}.$$

We are finished with term (4.19). Recapturing all of our work so far,

$$\nabla_y \times B(V,\phi_0)(y) = \int_\Omega \nabla_W U\,dx \tag{4.17}$$

$$- \int_\Omega \nabla_U W\,dx \tag{4.18}$$

$$+ \begin{pmatrix} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{pmatrix} - \frac{1}{2\pi^2}\int_\Omega (L_{yx^{-1}})_* V(x)\,dx \tag{4.21}$$

In order to analyze only one covariant derivative term, we make use of Identity 4 from the appendix to write integral (4.18) in terms of integral (4.17) plus some additional terms.

$$\nabla_U W \;\;=\;\; \nabla(W \cdot U) \;-\; \nabla_W U \;-\; W \times \nabla \times U \;-\; U \times \nabla \times W$$

Then the integral (4.18) becomes

$$- \int_\Omega \nabla_U W\,dx = \int_\Omega -\nabla(W \cdot U) + \nabla_W U + (W \times \nabla_y \times U) + (U \times \nabla_y \times W)\;dx \tag{4.22}$$

The last term on the right-hand side vanishes since it is the curl of a gradient, even though $\phi_0(\alpha)$ has a singularity at $\alpha = 0$, i.e., $x = y$.

Now consider equation (4.22). The second and third terms vanish in Euclidean space but the first term does not; all three remain on the three-sphere. The first term of (4.22) becomes

$$
\begin{aligned}
\int_\Omega \nabla_y (W \cdot U)\, dx &= \nabla_y \int_\Omega (W \cdot U)\, dx \\
&= \nabla_y \int_\Omega \nabla_y \phi_0 \cdot (L_{yx^{-1}})_* V(x)\, dx
\end{aligned}
$$

The gradient of ϕ_0 changes sign when switching variables from y to x appropriately:

$$
\nabla_y \phi_0 = -(L_{yx^{-1}})_* \nabla_x \phi_0
$$

Thus we continue with the first term of (4.22):

$$
\begin{aligned}
\int_\Omega \nabla_y (W \cdot U)\, dx &= \nabla_y \int_\Omega -(L_{yx^{-1}})_* \nabla_x \phi_0 \cdot (L_{yx^{-1}})_* V(x)\, dx \\
&= -\nabla_y \int_\Omega \nabla_x \phi_0 \cdot V(x)\, dx \\
&= -\nabla_y \int_\Omega \nabla_x \cdot \phi_0 V(x) \;-\; \phi_0\ (\nabla_x \cdot V(x))\ dx \\
&= +\nabla_y \int_\Omega \phi_0\ (\nabla_x \cdot V(x))\ dx - \nabla_y \int_{\partial\Omega} \phi_0\, V(x) \cdot \hat{n}\ d(area_x)
\end{aligned}
$$

These are exactly the same terms as obtained in the Euclidean case, where we could have written $\phi_0(x, y) = -\dfrac{1}{4\pi} \dfrac{1}{|y - x|}$.

Now consider the third term on the right-hand side of (4.22); it integrates a $\nabla \times U$ term. The vector field $U = (L_{yx^{-1}})_* V(x)$ is left-invariant in terms of y variables, as discussed earlier. Thus it is a curl eigenfield, and so

$$
\begin{aligned}
\nabla_y \times U &= -2U \\
\nabla_y \times (L_{yx^{-1}})_* V(x) &= -2(L_{yx^{-1}})_* V(x)
\end{aligned}
$$

So the third term of (4.22) then can be written as

$$
\begin{aligned}
\int_\Omega W \times \nabla_y \times U\, dx &= \int_\Omega \nabla_y \phi_0 \times -2(L_{yx^{-1}})_* V(x)\, dx \\
&= 2 \int_\Omega (L_{yx^{-1}})_* V(x) \times \nabla_y \phi_0\, dx \\
&= -2B(V, \phi_0)(y) \qquad (4.23)
\end{aligned}
$$

We summarize the results of analyzing (4.22):

$$-\int_\Omega \nabla_U W\ dx \ = \ \int_\Omega -\nabla(W\cdot U) + \nabla_W U + (W\times\nabla\times U) + (U\times\nabla\times W)\ dx$$

$$\begin{aligned}-\int_\Omega \nabla_U W\ dx = -\nabla_y \int_\Omega \phi_0\ (\nabla_x\cdot V(x))\ dx + \nabla_y \int_{\partial\Omega} \phi_0\, V(x)\cdot\hat{n}\ d(area_x)\\ + \int_\Omega \nabla_W U\ dx\ -\ 2B(V,\phi_0)(y)\ +\ 0\end{aligned}$$

Let's gather our results:

$$\begin{aligned}\nabla_y\times B(V,\phi_0)(y) = &\begin{pmatrix} V(y) & \text{inside } \Omega \\ \\ 0 & \text{outside } \Omega\end{pmatrix} - \frac{1}{2\pi^2}\int_\Omega (L_{yx^{-1}})_* V(x)\ dx\\ &-\nabla_y \int_\Omega \phi_0\ (\nabla_x\cdot V(x))\ dx\ +\ \nabla_y\int_{\partial\Omega}\phi_0\, V(x)\cdot\hat{n}\ d(area_x)\\ &+\int_\Omega \nabla_W U\ dx\\ &+\ 2B(V,\phi_0)(y)\\ &+\int_\Omega \nabla_W U\ dx\end{aligned}$$

Notice that we are left with two terms that are both of the form $\int_\Omega \nabla_W U\ dx$, where $U = U(x,y) = (L_{yx^{-1}})_* V(x)$ and $W = W(x,y) = \nabla_x\phi_0$. Notice both U and W lie in $T_y\Omega$. In order to analyze this integral, we utilize Proposition 3.3, which implies $\nabla_W U = W\times U$. Thus,

$$\begin{aligned} 2\int_\Omega \nabla_W U\ dx &= 2\int_\Omega W\times U\ dx\\ 2\int_\Omega \nabla_{(\nabla_y\phi_0)}(L_{yx^{-1}})_* V(x)\ dx &= 2\int_\Omega \nabla_y\phi_0\times(L_{yx^{-1}})_* V(x)\ dx\\ &= -2B(V,\phi_0)(y)\end{aligned}$$

This term cancels the fourth term above, and we obtain the result for $\nabla\times B(V,\phi_0)$.

$$\nabla_y \times B(V,\phi_0)(y) = \begin{pmatrix} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{pmatrix} - \frac{1}{2\pi^2}\int_\Omega (L_{yx^{-1}})_* V(x)\, dx$$

$$- \nabla_y \int_\Omega \phi_0 \, (\nabla_x \cdot V(x))\, dx + \nabla_y \int_{\partial\Omega} \phi_0 \, V(x) \cdot \hat{n} \, d(area_x)$$

$$\nabla_y \times B(V,\phi_0)(y) = A(V, \Delta\phi_0) - G(V,\phi_0)$$

So we have the desired result about the curl of the vector convolution operator. We now recall equation (4.16).

$$\begin{aligned}
\nabla_y \times BS(V)(y) &= \nabla_y \times B(V,\phi_0) + \frac{1}{2\pi^2}\int_\Omega (L_{yx^{-1}})_* V(x)\, dx \\
\nabla_y \times BS(V)(y) &= \begin{pmatrix} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{pmatrix} \\
&\quad -\nabla_y \int_\Omega \phi_0 \, (\nabla_x \cdot V(x))\, dx \\
&\quad +\nabla_y \int_{\partial\Omega} \phi_0 \, V(x) \cdot \hat{n}\, d(area_x)
\end{aligned}$$

With this the theorem is complete. □

4.6 Properties of Biot-Savart on subdomains

In this section, the kernel of the Biot-Savart operator is calculated. Also, we determine precisely when curl is a left-inverse to BS. This chapter then concludes with a proof that BS is self-adjoint and a statement about its image.

4.6.1 Kernel of the Biot-Savart operator

By definition, the Biot-Savart operator on S^3 maps the subspace of gradients to zero. No fluid knot on S^3 lies in the kernel of B or else Ampere's Law would fail. Hence the kernel of Biot-Savart on the three-sphere is precisely the space of gradients. Gradients on S^3 all behave like grounded gradients found on a compact subset $\Omega \subset S^3$. There the Hodge Decomposition Theorem for vector fields is more complicated, so we ponder, how do the other subspaces behave? What is the kernel of BS on Ω?

Theorem 4.8. *Let $\Omega \subset S^3$ be as described above. The kernel of the Biot-Savart operator on Ω is precisely those gradients that are orthogonal to the boundary, i.e.,*

$$\ker BS = HG(\Omega) \oplus GG(\Omega) \quad .$$

When discussing the kernel, we must note carefully that as an operator BS maps into $VF(\Omega)$. Though we often extend BS to defining a vector field on $S^3 - \Omega$, that is not its natural target space. Therefore, a vector field V lies in the kernel if and only if $BS(V) = 0$ inside Ω; *a priori* nothing is known of its behavior on $S^3 - \Omega$. In proving Theorem 4.8, we will show that if $BS(V) = 0$ throughout Ω, then $BS(V)$ must vanish identically on the entire three-sphere.

In order to prove this theorem, we require a few preliminary results.

Lemma 4.9. *Let Ω be as above, and let $\hat{n}$ be the outward pointing normal vector on $\partial\Omega$. Consider a vector field $V \in VF(\Omega)$ and let $y \in S^3$. Then,*

$$\int_\Omega (L_{yx^{-1}})_* \nabla_x \times V(x) + 2(L_{yx^{-1}})_* V(x)\, dx = -\int_{\partial\Omega} (L_{yx^{-1}})_* \left[V(x) \times \hat{n}\right]\, d(area)$$

Proof. Begin with the divergence theorem:

$$\int_\Omega \nabla \cdot V(x)\, dx \quad = \quad \int_{\partial\Omega} V(x) \cdot \hat{n}\, d(area)$$

Let $U(x)$ be any left-invariant vector field on S^3. Now replace $V(x)$ with $V(x) \times U(x)$:

$$\int_\Omega \nabla \cdot (V(x) \times U(x))\, dx \quad = \quad \int_{\partial\Omega} (V(x) \times U(x)) \cdot \hat{n}\, d(area) \tag{4.24}$$

Now examine the right-hand side of this equation.

$$\begin{aligned}
\int_{\partial\Omega} (V(x) \times U(x)) \cdot \hat{n}\, d(area) &= -\int_{\partial\Omega} (U(x) \times V(x)) \cdot \hat{n}\, d(area) \\
&= -\int_{\partial\Omega} U(x) \cdot (V(x) \times \hat{n})\, d(area) \\
&= -\int_{\partial\Omega} (L_{yx^{-1}})_* U(x) \cdot (L_{yx^{-1}})_* \left(V(x) \times \hat{n}\right)\; d(area) \\
&= -\int_{\partial\Omega} U(y) \cdot (L_{yx^{-1}})_* \left(V(x) \times \hat{n}\right)\; d(area) \\
&= -U(y) \cdot \int_{\partial\Omega} (L_{yx^{-1}})_* \left(V(x) \times \hat{n}\right)\; d(area)
\end{aligned}$$

Now examine the left-hand side of equation (4.24).

$$\int_{\Omega} \nabla \cdot (V(x) \times U(x))\, dx \;=\; \int_{\partial\Omega} U(x) \cdot \nabla \times V(x) - V(x) \cdot \nabla \times U(x)\, dx$$

Since $U(x)$ is left-invariant, $\nabla \times U(x) = -2U(x)$.

$$\begin{aligned}
\int_{\Omega} \nabla \cdot (V(x) \times U(x))\, dx &= \int_{\partial\Omega} U(x) \cdot \nabla \times V(x) + 2V(x) \cdot U(x)\, dx \\
&= \int_{\partial\Omega} U(x) \cdot (\nabla \times V(x) + 2V(x))\; dx \\
&= \int_{\partial\Omega} (L_{yx^{-1}})_* U(x) \cdot (L_{yx^{-1}})_* \left(\nabla \times V(x) + 2V(x)\right)\; dx \\
&= \int_{\partial\Omega} U(y) \cdot (L_{yx^{-1}})_* \left(\nabla \times V(x) + 2V(x)\right)\; dx \\
&= U(y) \cdot \int_{\partial\Omega} (L_{yx^{-1}})_* \left(\nabla \times V(x) + 2V(x)\right)\; dx
\end{aligned}$$

The two sides of equation (4.24) are equal.

$$U(y) \cdot \int_{\partial\Omega} (L_{yx^{-1}})_* \left(\nabla \times V(x) + 2V(x)\right)\; dx \;=\; -U(y) \cdot \int_{\partial\Omega} (L_{yx^{-1}})_* \left(V(x) \times \hat{n}\right)\; d(area_x)$$

Since this holds for any left-invariant field U, we may conclude that the projections of the two integrals onto the space of left-invariant vector fields must be equal. Both sides are integrals of left-translated fields and hence both are in fact left-invariant vector fields. Therefore, we have the desired equality:

$$\int_{\Omega} (L_{yx^{-1}})_* \nabla_x \times V(x) + 2(L_{yx^{-1}})_* V(x)\, dx = -\int_{\partial\Omega} (L_{yx^{-1}})_* \left(V(x) \times \hat{n}\right)\; d(area_x) \qquad \square$$

We require another lemma which involves a particular energy estimate. In the section on Maxwell's equations, we defined the electrostatic field due to the time-independent surface charge $\sigma = V \cdot \hat{n}$ on $\partial\Omega$, and we called it $\dot{E}_\sigma$. (We also viewed $\dot{E}_\sigma$ as the time derivative of an electrodynamic field due to a surface charge $(V \cdot \hat{n})\, t$ but do not adopt that view in this section.) For convenience, we now drop the derivative notation and hereafter refer to this field as

$$E_\sigma(y) = \nabla_y \int_{\partial\Omega} \phi_0\, V(x) \cdot \hat{n}\; d(area_x) \quad .$$

Call ψ the potential function for the electrostatic field, $E_\sigma = -\nabla\psi$; the negative sign is in accordance with electrodynamics notation.

Lemma 4.10. *For $V = \nabla f \in VF(\Omega)$, let E_σ the electrostatic field that it generates as described above. Then the energy of E_σ is related to its potential ψ as*

$$\int_{S^3} |E_\sigma(y)|^2\; dy = \int_{\partial\Omega} \psi(y)\, \nabla f \cdot \hat{n}\; d(area_y) \quad .$$

Proof. Let $V = \nabla f$, and let $\sigma = \nabla f \cdot \hat{n}$, which we view as a surface charge on $\partial\Omega$. Then we can write

$$E_\sigma = \nabla_y \int_{\partial\Omega} \phi_0\, \sigma\; d(area) \quad .$$

To make this proof more convenient in terms of notation, view the surface charge σ on $\partial\Omega$ as a volume charge ρ on a thickened, compact neighborhood $N(\partial\Omega)$ of the boundary $\partial\Omega$. We choose ρ to be C^∞ smooth with support in $N(\partial\Omega)$. The electrostatic field E_ρ resulting from ρ approximates E_σ and is expressed as

$$E_\rho = \nabla_y \int_{N(\partial\Omega)} \phi_0\, \rho\; d(area) \quad .$$

For this situation, we can view $E_\rho = -\nabla\psi_\rho$, where

$$\psi_\rho = -\int_{N(\partial\Omega)} \phi_0\, \rho\; d(area) \quad .$$

Then the divergence of E_ρ is $\nabla \cdot E_\rho = -\Delta\psi_\rho$. We may extend the domain of integration of ψ_ρ to be all of S^3, since ρ vanishes outside of $N(\partial\Omega)$.

$$\psi_\rho = -\int_{S^3} \phi_0\, \rho\; d(area)$$

Recall that ϕ_0 is the fundamental solution to the Laplacian on S^3, meaning that $\Delta\psi_\rho = -(\rho - [\rho])$. Thus, the divergence $\nabla \cdot E_\rho = \rho - [\rho]$. Consider the integral of ψ times ρ.

$$\int_{N(\partial\Omega)} \psi\,\rho\,dx \quad = \quad \int_{N(\partial\Omega)} \psi\,\nabla\cdot E_\rho\,dx \;+\; \int_{N(\partial\Omega)} \psi\,[\rho]\,dx$$

Again, we extend the domain of integration to be S^3:

$$\begin{aligned}
\int_{N(\partial\Omega)} \psi\,\rho\,dx &= \int_{S^3} \psi\,\rho\,dx \\
\int_{N(\partial\Omega)} \psi\,\rho\,dx &= \int_{S^3} \psi\,\nabla\cdot E_\rho\,dx \;+\; \int_{S^3} \psi\,[\rho]\,dx \\
\int_{N(\partial\Omega)} \psi\,\rho\,dx &= \int_{S^3} \nabla\cdot(\psi\,E_\rho) - \nabla\psi\cdot E_\rho\,dx + [\rho]\int_{S^3} \psi\,dx \\
\int_{N(\partial\Omega)} \psi\,\rho\,dx &= 0 \;+\; \int_{S^3} |E_\rho|^2\,dx \;+\; [\rho]\,[\psi] \qquad (4.25)
\end{aligned}$$

Now by making the neighborhood $N(\partial\Omega)$ shrink and approach the boundary $\partial\Omega$, we have

$$\begin{aligned}
[\rho] &\rightarrow 0 \\
\int_{N(\partial\Omega)} \psi\,\rho\,dx &\rightarrow \int_{\partial\Omega} \psi\,\sigma\,d(area) \\
\int_{S^3} |E_\rho|^2\,dx &\rightarrow \int_{S^3} |E_\sigma|^2\,dx
\end{aligned}$$

Then, equation (4.25) converges to

$$\int_{\partial\Omega} \psi\,\sigma\,d(area) = \int_{S^3} |E_\sigma|^2\,dx \quad ,$$

our desired result. ☐

One more result, the following energy estimate, is required before beginning the proof of the kernel of BS.

Proposition 4.11. *Let V be a divergence-free vector field on $\Omega \subset S^3$, and let E_σ be its associated electrostatic field. Then,*

$$\int_{S^3} |E_\sigma|^2\,dy \le \int_\Omega |V|^2\,dy \quad .$$

Proof. When V is divergence-free and tangent to the boundary, the electrostatic field $E_\sigma = 0$. So it suffices to prove the proposition for V a divergence-free gradient, i.e., $V \in CG \oplus HG$. Adding a fluid knot component to V would only increase the energy on the right-hand side while not affecting the left-hand side.

So assume $V = \nabla f \in CG \oplus HG$; this implies f is harmonic. Start with the above lemma,

$$\int_{S^3} |E_\sigma(y)|^2 \, dy \quad = \quad \int_{\partial\Omega} \psi(y) \, \nabla f \cdot \hat{n} \; d(area_y)$$

Apply Green's first identity:

$$\begin{aligned} \int_{S^3} |E_\sigma(y)|^2 \, dy \quad &= \quad \int_{\Omega} \nabla\psi \cdot \nabla f \, + \, \psi \Delta f \, dy \\ \int_{S^3} |E_\sigma(y)|^2 \, dy \quad &= \quad \int_{\Omega} -E_\sigma \cdot \nabla f \, + \, 0 \, dy \end{aligned} \tag{4.26}$$

This is the L^2 inner product of $-E_\sigma$ and ∇f; apply the Cauchy-Schwartz inequality:

$$\begin{aligned} \int_{\Omega} -E_\sigma \cdot \nabla f \, dy \quad &\leq \quad \left[\int_{\Omega} -E_\sigma \cdot -E_\sigma \, dy \right]^{1/2} \left[\int_{\Omega} \nabla f \cdot \nabla f \, dy \right]^{1/2} \\ &\leq \quad \left[\int_{S^3} |E_\sigma|^2 \, dy \right]^{1/2} \left[\int_{\Omega} |\nabla f|^2 \, dy \right]^{1/2} \end{aligned}$$

Substitute this inequality into equation (4.26), to conclude

$$\begin{aligned} \int_{S^3} |E_\sigma(y)|^2 \, dy \quad &\leq \quad \left[\int_{S^3} |E_\sigma|^2 \, dy \right]^{1/2} \left[\int_{\Omega} |\nabla f|^2 \, dy \right]^{1/2} \\ \left[\int_{S^3} |E_\sigma|^2 \, dy \right]^{1/2} \quad &\leq \quad \left[\int_{\Omega} |\nabla f|^2 \, dy \right]^{1/2} \qquad , \end{aligned}$$

which proves the result. □

Finally we are ready to prove Theorem 4.8, that the kernel of BS is $HG(\Omega) \oplus GG(\Omega)$.

Proof. First, we show that the subspace $HG(\Omega) \oplus GG(\Omega)$ is contained in the kernel of BS. Let $V = \nabla f$ be in this subspace; then f is locally constant on each boundary component $\partial\Omega_i$. This implies that V is orthogonal to the boundary.

We compute $BS(\nabla f)$ using the left-translation version of the Biot-Savart operator.

$$\begin{aligned} BS(\nabla f)(y) &= \int_\Omega (L_{yx^{-1}})_* \nabla_x f(x) \times \nabla_y \phi_0 \; dx \\ &\quad -\frac{1}{4\pi^2}\int_\Omega (L_{yx^{-1}})_* \nabla_x f(x) \; dx \\ &\quad +2\nabla_y \int_\Omega (L_{yx^{-1}})_* \nabla_x f(x) \cdot \nabla_y \phi_1 \; dx \end{aligned}$$

Call these three integrals (I), (II), and (III); we compute them individually.

Use Lemma 4.9 to express the second integral (II) as

$$\begin{aligned} -\frac{1}{4\pi^2}\int_\Omega (L_{yx^{-1}})_* \nabla_x f(x) \; dx &= \frac{1}{8\pi^2}\int_\Omega (L_{yx^{-1}})_* (\nabla_x \times \nabla_x f(x)) \; dx \\ &\quad + \frac{1}{8\pi^2}\int_{\partial\Omega} (L_{yx^{-1}})_* (\nabla_x f(x) \times \hat{n}) \; dx \end{aligned}$$

Both terms on the right vanish since ∇f is orthogonal to the boundary. Hence, the second integral (II)$= 0$.

Now, examine the first integral (I),

$$\begin{aligned} \text{(I)} &= \int_\Omega (L_{yx^{-1}})_* \nabla_x f(x) \times \nabla_y \phi_0 \; dx \\ &= \int_\Omega (L_{yx^{-1}})_* \nabla_x f(x) \times -(L_{yx^{-1}})_* \nabla_x \phi_0 \; dx \\ &= -\int_\Omega (L_{yx^{-1}})_* \left(\nabla_x f(x) \times \nabla_x \phi_0\right) \; dx \end{aligned}$$

We can rewrite the right-hand side using a vector identity (see the Appendix):

$$\nabla f \times \nabla \phi_0 = \nabla \times f \nabla \phi_0 - f(\nabla \times \nabla \phi_0) \quad .$$

Then,

$$\text{(I)} = -\int_\Omega (L_{yx^{-1}})_* \left(\nabla_x \times f(x) \nabla_x \phi_0\right) \; dx + -\int_\Omega (L_{yx^{-1}})_* \left[f(x)\left(\nabla_x \times \nabla_x \phi_0\right)\right] \; dx$$

Though $\nabla \phi_0$ has a singularity, its curl still vanishes (as it only depends on α). We are left with

$$\text{(I)} = -\int_\Omega (L_{yx^{-1}})_* \left(\nabla_x \times f(x) \nabla_x \phi_0\right) \; dx \quad .$$

Apply Lemma 4.9,

$$\text{(I)} = 2\int_\Omega (L_{yx^{-1}})_* \left(f(x) \nabla_x \phi_0\right) \; dx + \int_{\partial\Omega} (L_{yx^{-1}})_* \left(f(x) \nabla_x \phi_0 \times \hat{n}\right) \; dx$$

Recall that f is constant on each boundary component; let f_i be its value on the component $\partial\Omega_i$. Denote Ω_i as the region "inside" $\partial\Omega_i$, where inside is determined opposite to the direction $\hat{n}_i$ points. Note that Ω_i is not necessarily a subset of Ω. Then,

$$\text{(I)} \quad = \quad -2\int_\Omega \nabla_y\phi_0 \, dx + \sum_i f_i \int_{\partial\Omega_i} (L_{yx^{-1}})_* \left(\nabla_x\phi_0 \times \hat{n}_i\right) \, dx \quad .$$

Now apply Lemma 4.9 again.

$$\begin{aligned}
\text{(I)} \quad &= \quad -2\int_\Omega \nabla_y\phi_0 \, dx - \sum_i f_i \int_{\Omega_i} (L_{yx^{-1}})_* \left(\nabla_x \times \nabla_x\phi_0\right) \, dx - 2\sum_i f_i \int_{\Omega_i} (L_{yx^{-1}})_* \nabla_x\phi_0 \, dx \\
\text{(I)} \quad &= \quad -2\int_\Omega \nabla_y\phi_0 \, dx - \ 0 \ + \ 2\sum_i f_i \int_{\Omega_i} \nabla_y\phi_0 \, dx \qquad (4.27)
\end{aligned}$$

We are now ready to compute the third integral, (III).

$$\begin{aligned}
\text{(III)} \quad &= \quad 2\nabla_y \int_\Omega (L_{yx^{-1}})_* \nabla_x f(x) \cdot \nabla_y\phi_1 \, dx \\
&= \quad -2\nabla_y \int_\Omega \nabla_x f(x) \cdot \nabla_x\phi_1 \, dx \\
&= \quad -2\nabla_y \int_\Omega \nabla_x \cdot \left(f(x)\nabla_x\phi_1\right) + f\Delta\phi_1 \, dx
\end{aligned}$$

Recall that $\Delta\phi_1 = \phi_0 + 1/8\pi^2$. Apply the divergence theorem to the first term,

$$\text{(III)} \quad = \quad -2\nabla_y \int_{\partial\Omega} f(x)\, \nabla_x\phi_1 \cdot \hat{n} \, d(area_x) \ + \ 2\nabla_y \int_\Omega f\phi_0 \, dx \ + \ 2\nabla_y \int_\Omega f(x)\tfrac{1}{8\pi^2} \, dx$$

The last integral is independent of y and vanishes upon applying the gradient ∇_y. To analyze the first term, recall that f is a constant f_i on each boundary component $\partial\Omega_i$. Then,

$$\begin{aligned}
\text{(III)} \quad &= \quad -2\nabla_y \sum_i f_i \int_{\partial\Omega_i} \nabla_x\phi_1 \cdot \hat{n}_i \, d(area_x) \ + \ 2\nabla_y \int_\Omega f\phi_0 \, dx \\
\text{(III)} \quad &= \quad -2\sum_i f_i \nabla_y \int_{\Omega_i} \nabla_x \cdot \nabla_x\phi_1 \, dx \ + \ 2\nabla_y \int_\Omega f\phi_0 \, dx \\
\text{(III)} \quad &= \quad -2\sum_i f_i \nabla_y \int_{\Omega_i} \phi_0 + \tfrac{1}{8\pi^2} \, dx \ + \ 2\nabla_y \int_\Omega f\phi_0 \, dx \\
\text{(III)} \quad &= \quad -2\sum_i f_i \int_{\Omega_i} \nabla_y\phi_0 \, dx \ + \ 2\nabla_y \int_\Omega f\phi_0 \, dx \qquad (4.28)
\end{aligned}$$

Examining equations (4.27) and (4.28), we see that (III) is the negative of (I). We conclude that $BS(\nabla f) = \text{(I)} + \text{(II)} + \text{(III)} = 0$. Therefore, the subspace $HG \oplus GG$ is indeed contained in the kernel of BS.

Now we prove that $\ker BS \subset HG \oplus GG$. Let $V \in FK \oplus HK \oplus CG$ and decompose it as $V = V_K + V_C$, where $V_K \in FK \oplus HK$ and $V \in CG$. Maxwell's fourth equation implies that

$$\begin{aligned} \nabla \times BS(V_K + V_C) &= V_K + V_C - \nabla_y \int_{\partial\Omega} \phi_0 \, V_C \cdot \hat{n} \, d(area) \\ \nabla \times BS(V_K + V_C) &= V_K + \nabla g \end{aligned}$$

If $V_K \neq 0$, then $\nabla \times BS(V) \neq 0$ and consequently $BS(V) \neq 0$. So it suffices to show that no curly gradients are in the kernel of BS.

Assume $V = \nabla f$ is a curly gradient that is in the kernel of BS. We will show then that $V = 0$. Since curly gradients are divergence-free, Maxwell's fourth equation states, for $y \in \Omega$, that $\nabla \times BS(V)(y) = V - E_\sigma$, where E_σ the electrostatic field defined in the previous section. This must be zero since $BS(V) = 0$ on Ω. Thus $V = E_\sigma$ on Ω. By Proposition 4.11, the energy of V on Ω is no less than the energy of E_σ throughout the three-sphere. Since E_σ equals V inside Ω, it has no available energy left on the complement $S^3 - \Omega$, and so E_σ must be identically zero there. This implies that $BS(V) = 0$ on all of the three-sphere, a fact which we could not conclude *a priori*.

Since E_σ equals a gradient $-\nabla\psi$, the potential function ψ must be locally constant on $S^3 \bar{-} \Omega$. In particular ψ is constant on each boundary component. Any gradient like $-\nabla\psi$ whose potential function is locally constant on $\partial\Omega$ must lie in $HG \oplus GG(\Omega)$. Now V was equal to $-\nabla\psi$ on Ω, so V lies in $HG \oplus GG$. This subspace is orthogonal to the curly gradients, hence V must be trivial.

Thus we have shown that the kernel of BS cannot contain any curly gradients or fluid knots. Thus it must be included in $HG \oplus GG$; by the first half of the proof, the kernel is precisely that subspace. $\square$

Indeed, in proving Theorem 4.8, we have shown an even stronger result, namely that no curly gradient can lie in the kernel of $\nabla \times BS$. No fluid knot lies in this kernel due to Ampere's Law. Thus, the kernel of $\nabla \times BS$ is exactly the kernel of BS.

Theorem 4.12. *The kernel of $\nabla \times BS$ is precisely $HG(\Omega) \oplus GG(\Omega)$.*

4.6.2 Curl of the Biot-Savart operator

The Biot-Savart operator is quite useful because, for certain vector fields, it describes an inverse operator to curl. The inverse to the curl operator is quite useful in energy minimization problems, which often are solved as eigenvalue problems for curl; see for example [1, 10]. Let us state then precisely when curl inverts the Biot-Savart operator.

Theorem 4.13. *(1) The equation $\nabla \times BS(V) = V$ holds on $\Omega \subset S^3$ if and only if V is a divergence-free field tangent to the boundary $\partial\Omega$, i.e., $V \in FK(\Omega) \oplus HK(\Omega)$.*

(2) The equation $\nabla \times BS(V) = 0$ holds on $S^3 - \Omega$ if and only if $V \in FK(\Omega) \oplus HK(\Omega) \oplus HG(\Omega) \oplus GG(\Omega)$.

Proof. The first statement is far more complicated to prove than the second. One inclusion is immediate from Ampere's Law guarantees that $\nabla \times BS(V) = V$ holds on $\Omega \subset S^3$ for a fluid knot V.

Now for the other inclusion. Assume $\nabla \times BS(V) = V$ holds. Then V lies in the image of curl, and so must be orthogonal to $HG \oplus GG$. The equation $\nabla \times BS(V) = V$ holds for any fluid knot, and so it suffices to show that it cannot hold for a curly gradient.

Let $V = \nabla f \in CG(\Omega)$. Maxwell's equation implies that $\nabla \times BS(V) = V + E_\sigma$ on Ω, where

$$E_\sigma(y) = -\nabla_y \int_{\partial\Omega} \phi_0 \, V(x) \cdot \hat{n} \; d(area_x)$$

is the electrostatic field resulting from the surface charge $\sigma = V \cdot \hat{n}$ on $\partial\Omega$. Suppose that $\nabla \times BS(V) = V$, which implies $E_\sigma = 0$ on Ω. We show that this also implies V is trivial.

Our first step is to show that $E_\sigma = 0$ outside Ω. As before, express $E_\sigma = -\nabla\psi$. Since $E_\sigma = 0$ on Ω, the potential function ψ is locally constant on Ω; in particular ψ is a constant ψ_i on each boundary component $\partial\Omega_i$.

An exercise in Euclidean electrodynamics shows that any electrostatic field derived from charge on closed surface will have a jump discontinuity across the surface, but will be divergence free on the interior and exterior of the surface. The same result follows from our definitions on the three-sphere. Then

$$\begin{aligned}
\nabla \cdot \psi E_\sigma &= \nabla\psi \cdot E_\sigma \;+\; \psi(\nabla \cdot E_\sigma) \\
\nabla \cdot \psi E_\sigma &= -|E_\sigma|^2 \;+\; 0
\end{aligned}$$

Integrate over the complement of Ω.

$$\begin{aligned}
\int_{S^3-\Omega} -|E_\sigma|^2 \, dx &= \int_{S^3-\Omega} \nabla \cdot \psi E_\sigma \, dx \\
&= \int_{\partial\Omega} \psi \, E_\sigma \cdot \hat{n} \; d(area) \\
&= \sum_i \psi_i \int_{\partial\Omega_i} E_\sigma \cdot \hat{n}_i \; d(area) \quad , \qquad (4.29)
\end{aligned}$$

where $\hat{n}_i$ is the normal vector on $\partial\Omega_i$ pointing out of $S^3 - \Omega$.

In Euclidean space, *Gauss's law* states that the flux of an electrostatic field like E_σ over a closed surface equals the total charge enclosed by this surface. An analogous result holds on S^3:

$$\int_{\partial\Omega_i} E_\sigma \cdot \hat{n}_i \; d(area) \;=\; \text{charge } Q \text{ enclosed by } \partial\Omega_i$$

Above when we write that charge is enclosed by $\partial\Omega_i$, we intend that the charge lies in $S^3 - \partial\Omega_i$. The only possible charge in the region $S^3 - \partial\Omega_i$ lies on other boundary components $\partial\Omega_j$ that lie outside Ω_i. Thus,

$$\int_{\partial\Omega_i} E_\sigma \cdot \hat{n}_i \; d(area) \;=\; \sum_j (\pm 1) \int_{\partial\Omega_j} \sigma(x) \; d(area_x)$$

The sign on the right-hand side is determined according to whether the orientations of n_i and n_j agree.

$$\begin{aligned}
\int_{\partial\Omega_i} E_\sigma \cdot \hat{n}_i \; d(area) &= \sum_j (\pm 1) \int_{\partial\Omega_j} \sigma(x) \; d(area_x) \\
&= \sum_j (\pm 1) \int_{\partial\Omega_j} V(x) \cdot \hat{n}_j \; d(area_x) \\
&= 0
\end{aligned}$$

Since V is a curly gradient, it has zero flux over any boundary component. By equation (4.29) we conclude that

$$\int_{S^3-\Omega} -|E_\sigma|^2 \, dx = 0 \quad ,$$

and so $E_\sigma = 0$ on $S^3 - \Omega$.

We now have that $E_\sigma = 0$ on the interiors of Ω and $S^3 - \Omega$. Now we apply a *pillbox argument*, a standard tool in electrodynamics, to a point $x \in \partial\Omega$. Take a neighborhood of x in $\partial\Omega$ and extend this into a "pillbox" P of small height ϵ. Then by Gauss's law:

$$\int_{\partial P} E_\sigma \cdot \hat{n} \, d(area) \;=\; \text{charge } Q_{\text{enclosed}} \;=\; \int_{P \cup \partial\Omega} \sigma \, d(area) \quad .$$

But $E_\sigma = 0$ on $\partial P - \partial\Omega$, so the left-hand side above is zero. Thus, for any choice of pillbox, $\int_{P\cup\partial\Omega} \sigma \, d(area) = 0$. Thus σ is identically zero on $\partial\Omega$. That implies that V is tangent to the boundary, so it cannot be a curly gradient unless it is trivial.

Thus we have shown that the only gradients for which $\nabla \times BS(V) = V$ holds are trivial; the proof of the first statement is now complete.

The second statement is far easier to prove. For a fluid knot V, Ampere's Law guarantees that $\nabla \times BS(V) = 0$ holds on $S^3 - \Omega$. The other subspaces $HG \oplus GG$ lie in the kernel of $\nabla \times BS$ by Theorem 4.12. Thus $FK \oplus HK \oplus HG \oplus GG \subset \ker BS$.

To prove the reverse implication, it suffices to show, for V a curly gradient, that $\nabla \times BS(V)$ is nonzero. In proving Theorem 4.8, we showed that if $V \in CG(\Omega)$, then having $\nabla \times BS(V) = 0$ outside Ω implied that $V = 0$. Thus the second statement holds. $\square$

4.6.3 Self-adjointness and image of the Biot-Savart operator

In this section we show that the Biot-Savart operator is self-adjoint, whether it is defined on a subdomain or on all of S^3. As a corollary, we learn something about its image.

Theorem 4.14. *The Biot-Savart operator is self-adjoint.*

Proof. Let $V, W \in VF(\Omega)$. We use the parallel transport version of Biot-Savart to show

$$\langle BS(V), W\rangle = \langle V, BS(W)\rangle \quad .$$

$$\begin{aligned} \langle BS(V)(y), W(y)\rangle &= \int_{\Omega\times\Omega} P_{yx}V(x) \times \nabla_y\phi \cdot W(y)\, dx\, dy \\ \langle BS(V), W\rangle &= -\int_{\Omega\times\Omega} W(y) \times \nabla_y\phi \cdot P_{yx}V(x)\, dx\, dy \quad , \end{aligned}$$

where we have used identity 1 in Appendix A to rearrange vectors. Since $\nabla_y\phi = -P_{yx}\nabla_x\phi$, we have

$$\langle BS(V), W\rangle = \int_{\Omega\times\Omega} W(y) \times P_{yx}\nabla_x\phi \cdot P_{yx}V(x)\, dx\, dy$$

Now parallel translate all vectors from y to x. For $y \neq -x$, clearly $P_{xy}P_{yx} = 1$.

$$\begin{aligned} \langle BS(V), W\rangle &= \int_{\Omega\times\Omega} P_{xy}W(y) \times P_{xy}P_{yx}\nabla_x\phi \cdot P_{xy}P_{yx}V(x)\, dx\, dy \\ \langle BS(V), W\rangle &= \int_\Omega \left[\int_\Omega P_{xy}W(y) \times \nabla_x\phi\, dy\right] \cdot V(x)\, dx \\ \langle BS(V), W\rangle &= \int_\Omega BS(W)(x) \cdot V(x)\, dx \\ \langle BS(V), W\rangle &= \langle BS(W)(x), V(x)\rangle \end{aligned}$$

$\square$

Corollary 4.15. *For $\Omega \subsetneq S^3$, the image of Biot-Savart lies in $FK \oplus HK \oplus CG$.*

For comparison, the image of BS on all of S^3 is precisely the space of fluid knots $K(S^3)$.

Proof. Since BS is self-adjoint, its image must be orthogonal to its kernel. In more detail, let W be the component of $BS(V)$ that lies in $HG \oplus GG$ the kernel of BS. Then $BS(W) = 0$ and self-adjointness implies that $\langle BS(V), W\rangle = \langle V, BS(W)\rangle = 0$. But, $\langle BS(V), W\rangle\langle W, W\rangle$. So W must be trivial, and thus $HG \oplus GG$ is not in the image of BS. $\square$

Chapter 5

Helicity

In this section, we establish upper bounds for helicity and for the Biot-Savart operator. From Theorem 5.6,

$$
\begin{aligned}
\|BS(V)\| &\leq N(R)\, \|V\| \\
|H(V)| &\leq N(R)\, \langle V, V\rangle
\end{aligned}
$$

where R is the radius of a spherical ball with the same volume as the domain where V is defined, and

$$
N(R) = \tfrac{1}{\pi}\left(2(1-\cos R) + (\pi - R)\sin R\right) \quad .
$$

These bounds are not sharp, but we provide examples which show that they are the right order of magnitude.

Definition 5.1. *The* **helicity** *of a vector field $V \in VF(\Omega)$ is defined to be*

$$
\begin{aligned}
H(V) &= \langle V, BS(V)\rangle \\
&= \int_{\Omega} V(y) \cdot BS(V)(y)\, dy \\
&= \int_{\Omega\times\Omega} V(y) \cdot P_{yx} V(x) \times \nabla_y \phi(x,y)\, dx\, dy
\end{aligned}
$$

Helicity measures the extent to which the flow lines of a vector field wrap and coil around each other. It was discovered in $\mathbb{R}^3$ by the astrophysicist Woltjer and named ten years later by Moffatt. For divergence-free fields, helicity is the same as Arnold's asymptotic Hopf invariant [1].

5.1 Calculating upper bounds

Upper bounds for the helicity of a vector field in Euclidean space were described in section 2.3.1. We seek to now bound helicity on subdomains of S^3. We begin by showing that the Biot-Savart operator is bounded in the L^2 norm, which follows a standard Young's inequality proof from functional analysis.

Proposition 5.2. *Let $\psi(\alpha(x,y)) : \Omega \times \Omega \to \mathbb{R}$ be a function that depends only upon the distance $\alpha(x,y)$ between x and y and that is defined such that*

$$N_\Omega(\psi) := \max_{y\in\Omega} \int_\Omega |\psi(\alpha)|\, dx < \infty \quad .$$

Then the operator $T_\psi : VF(\Omega) \to VF(\Omega)$, defined as

$$T_\psi(V)(y) = \int_\Omega P_{yx}V(x) \times \psi(\alpha)\nabla_y \alpha \, dx$$

is bounded with respect to the L^2 norm. Specifically,

$$\|T_\psi(V)(y)\| \le N_\Omega(\psi)\, \|V\| \quad .$$

Proof. Begin by estimating the length of $T_\psi(V)$:

$$|T_\psi(V)(y)| \le \int_\Omega |P_{yx}V(x)|\, |\psi(\alpha)|\, |\nabla_y \alpha|\, dx$$

Note that $|\nabla_y \alpha| = 1$ by equation (3.7). We rewrite the inequality above as

$$\begin{aligned} |T_\psi(V)| &\le \int_\Omega |V|\, |\psi|^{1/2}\, |\psi^{1/2}|\, dx \\ &= \left\langle |V|\, |\psi|^{1/2},\, |\psi|^{1/2} \right\rangle \end{aligned}$$

Now apply the Cauchy-Schwartz inequality,

$$\begin{aligned}|T_\psi(V)| &\leq \left[\int_\Omega |V|^2\,|\psi|\,dx\right]^{1/2}\left[\int_\Omega |\psi|\,dx\right]^{1/2}\\ &\leq \left[\int_\Omega |V|^2\,|\psi|\,dx\right]^{1/2} N_\Omega(\psi)^{1/2}\end{aligned}$$

Now square both sides and integrate with respect to y.

$$\begin{aligned}|T_\psi(V)|^2 &\leq N_\Omega(\psi)\int_\Omega |V|^2\,|\psi| dx\\ \int_\Omega |T_\psi(V)(y)|^2\,dy &\leq \int_\Omega N_\Omega(\psi)\left\{\int_\Omega |V(x)|^2\,|\psi|\,dx\right\}dy\\ \langle T_\psi(V),T_\psi(V)\rangle &\leq N_\Omega(\psi)\int_{\Omega\times\Omega} |V(x)|^2\,|\psi|\,dx\,dy\\ \langle T_\psi(V),T_\psi(V)\rangle &\leq N_\Omega(\psi)\int_\Omega |V(x)|^2\left\{\int_\Omega |\psi|\,dy\right\}dx\\ \langle T_\psi(V),T_\psi(V)\rangle &\leq N_\Omega(\psi)\int_\Omega |V(x)|^2\,N_\Omega(\psi)\,dx\\ \langle T_\psi(V),T_\psi(V)\rangle &\leq N_\Omega(\psi)^2\,\langle V,V\rangle\end{aligned}$$

Finally, take the square root of each side to obtain the desired inequality,

$$\|T_\psi(V)(y)\| \leq N_\Omega(\psi)\,\|V\| \quad .$$

□

Consider this proposition with the operator $BS(V)$ in place of T_ψ and with $\psi = \phi'(\alpha)$:

$$BS(V) = \int_\Omega P_{yx}V(x)\times\phi'(\alpha)\nabla_y\alpha\,dx \quad ,$$

where $\phi(\alpha)$ and its first two derivatives are, as shown in chapter 4,

$$\begin{aligned}\phi(\alpha) &= -\frac{1}{4\pi^2}(\pi-\alpha)\csc\alpha\\ \phi'(\alpha) &= \frac{1}{4\pi^2}\left(\csc\alpha+(\pi-\alpha)\csc\alpha\cot\alpha\right)\\ \phi''(\alpha) &= -\frac{\csc\alpha}{4\pi^2}\left(2\cot\alpha+(\pi-\alpha)(\cot^2\alpha+\csc^2\alpha)\right)\end{aligned}$$

The proposition guarantees $BS(V)$ is bounded:

$$\|BS(V)\| \leq N_\Omega(\phi')\,\|V\| \quad . \tag{5.1}$$

Consequently, helicity is also bounded, since

$$H(V) = \langle V, BS(V) \rangle \leq \|BS(V)\| \, \|V\|$$

by the Cauchy-Schwartz inequality. This estimate implies

$$H(V) \leq N_\Omega(\phi') \, \langle V, V \rangle \quad . \tag{5.2}$$

Thus we have shown

Corollary 5.3. *The Biot-Savart operator $BS : VF(\Omega) \to VF(\Omega)$ is bounded in the L^2 norm. Consequently, helicity $H : VF(\Omega) \to \mathbb{R}$ is a bounded functional.*

Now we hunt for an upper bound for $N_\Omega(\phi')$. We will need two lemmas before establishing any estimates.

Lemma 5.4. *The function $\phi'(\alpha)$ is a strictly decreasing function of α on $(0, \pi]$.*

Figure 5.1 depicts a graph of the function ϕ and its first two derivatives which clearly shows that $\phi'(\alpha)$ is decreasing on $(0, \pi)$.

Proof. We show two facts about $\phi''(\alpha)$ which are sufficient to prove the lemma:

1. $\phi''(\alpha) < 0 \;$ on $(0, \pi)$
2. $\lim_{\alpha \to \pi} \phi''(\alpha) = -\dfrac{1}{12\pi^2}$

On the interval $0 < \alpha < \pi$, the function $\csc(\alpha)$ is positive; thus to show the first fact, $\phi''(\alpha) < 0$, it suffices to show that

$$\big(2 \cot \alpha + (\pi - \alpha)(\cot^2 \alpha + \csc^2 \alpha)\big) > 0 \quad .$$

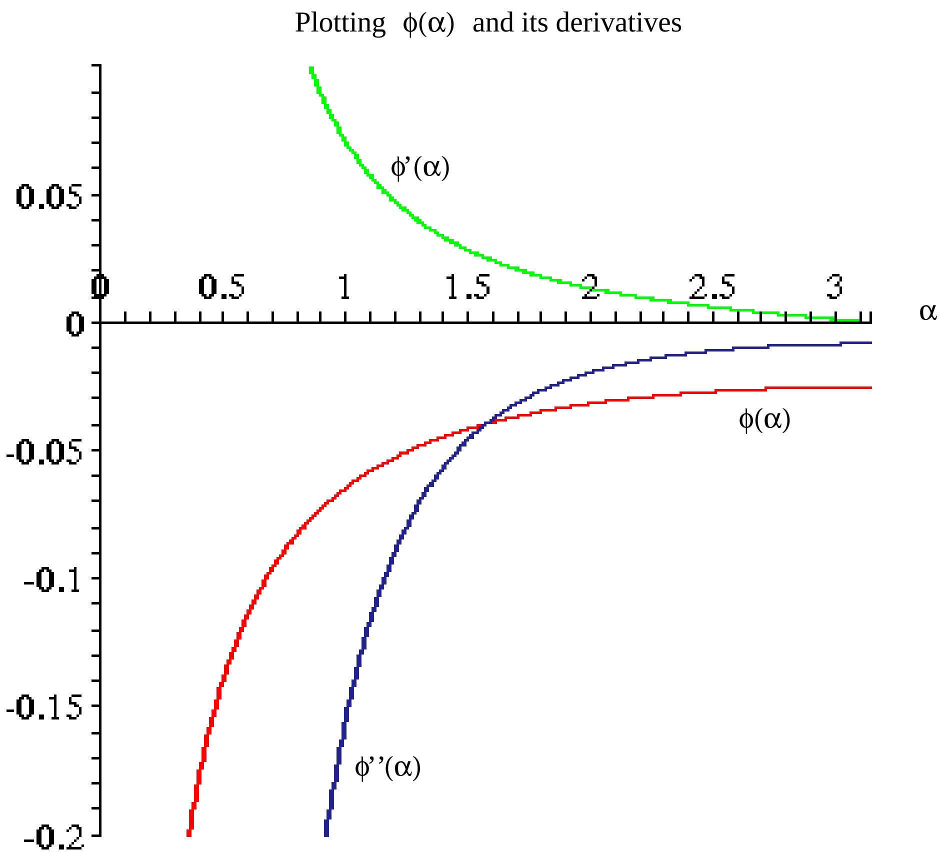

Figure 5.1: Graph of potential functions $\phi(\alpha)$, $\phi'(\alpha)$, and $\phi''(\alpha)$

First note that $(\pi - \alpha) > \sin\alpha$ on $(0, \pi)$. Then,

$$\begin{aligned}
2\cot\alpha + (\pi-\alpha)(\cot^2\alpha + \csc^2\alpha) &> 2\cot\alpha + \sin\alpha(\cot^2\alpha + \csc^2\alpha) \\
&= \frac{2\cos\alpha}{\sin\alpha} + \frac{\cos^2\alpha + 1}{\sin\alpha} \\
&= \frac{\cos^2\alpha + 2\cos\alpha + 1}{\sin\alpha} \\
&= \frac{(\cos\alpha + 1)^2}{\sin\alpha} \\
&\geq 0
\end{aligned}$$

Thus we have shown $\phi''(\alpha) < 0$, which implies $\phi'(\alpha)$ is decreasing on $(0, \pi)$. To show that ϕ' is decreasing at $\alpha = \pi$, we find the limit of ϕ'' as $\alpha \to \pi$.

$$\begin{aligned}
\lim_{\alpha\to\pi} \phi''(\alpha) &= \lim_{\alpha\to\pi} -\frac{\csc\alpha}{4\pi^2}\left(2\cot\alpha + (\pi-\alpha)(\cot^2\alpha + \csc^2\alpha)\right) \\
&= -\frac{1}{4\pi^2}\lim_{\alpha\to\pi}\frac{(\pi-\alpha)(\cos^2\alpha + 1) + 2\sin\alpha\cos\alpha}{\sin^3\alpha}
\end{aligned}$$

Now apply l'Hôpital's Rule:

$$\begin{aligned}
\lim_{\alpha\to\pi} \phi''(\alpha) &= -\frac{1}{4\pi^2}\lim_{\alpha\to\pi}\frac{-(\cos^2\alpha + 1) - 2(\pi-\alpha)\sin\alpha\cos\alpha + 2\cos^2\alpha - 2\sin^2\alpha}{3\sin^2\alpha\cos\alpha} \\
&= -\frac{1}{4\pi^2}\lim_{\alpha\to\pi}\frac{-2(\pi-\alpha)\sin\alpha\cos\alpha - 3\sin^2\alpha}{3\sin^2\alpha\cos\alpha} \\
&= -\frac{1}{4\pi^2}\left(\lim_{\alpha\to\pi}\frac{-2(\pi-\alpha)}{3\sin\alpha} + \lim_{\alpha\to\pi}\frac{-1}{\cos\alpha}\right) \\
&= -\frac{1}{4\pi^2}\left(-\frac{2}{3} + 1\right) \\
&= -\frac{1}{12\pi^2}
\end{aligned}$$

Thus, we have shown that $\phi'' < 0$ on $(0, \pi)$ and has a well-defined limit, which is less than zero, at the endpoint $\alpha = \pi$. This allows us to conclude that ϕ' is decreasing on $(0, \pi]$. $\square$

Now return to the bound $N_\Omega(\phi') = \max_{y\in\Omega}\int_\Omega |\phi'(\alpha(x,y))|\,dx$. Since $\phi'(\alpha)$ blows up at the origin and is always strictly decreasing, the maximum of the integral will occur at a point $y \in \Omega$ which is closest in some sense to all other points of Ω. Precisely, choose y to be the point where $\{\delta \,|\, \delta \geq d(x,y)\ \forall x \in \Omega\}$ is minimized. This δ can be thought of as the smallest radius possible for a ball $B(y, \delta)$ containing Ω.

Let v be the volume of Ω. Define R to be the radius of a ball $B = B(y, R) \subset S^3$ having the same volume as Ω, i.e., R satisfies

$$v = vol(B) = \pi\,(2R - \sin 2R) \quad .$$

Lemma 5.5. *For Ω and B as above, $N_\Omega(\phi') \leq N_B(\phi')$. Equality holds if and only if $\Omega = B$ up to a set of (Lebesgue) measure zero.*

Proof. For both norms, we are taking the maximum of an integral; we have chosen $y \in \Omega \cap B$ so that both integrals achieve their maximum there. So we need to show that

$$\int_\Omega |\phi'(\alpha(x,y))|\; dx \leq \int_B |\phi'(\alpha(x,y))|\; dx \quad .$$

The two integrals agree on the set $\Omega \cap B$; after subtracting this set from the domains of integration, it suffices to show, assuming the sets $(\Omega - B)$ and $(B - \Omega)$ have positive Lebesgue measure, that

$$\int_{\Omega - B} |\phi'(\alpha(x,y))|\; dx \leq \int_{B - \Omega} |\phi'(\alpha(x,y))|\; dx \quad . \tag{5.3}$$

The sets $\Omega - B$ and $B - \Omega$ have the same volume, equal to $v - vol(\Omega \cap B)$. All points x in the set $\Omega - B$, have $\alpha(x, y) \geq R$. Since ϕ' is a decreasing function of α,

$$\int_{\Omega - B} |\phi'(\alpha(x,y))|\; dx \;\leq\; \int_{\Omega - B} |\phi'(R)|\; dx \;=\; |\phi'(R)|\, vol(\Omega - B) \quad .$$

However, all points x in the set $B - \Omega$ have $\alpha(x, y) < R$. Thus

$$\int_{B - \Omega} |\phi'(\alpha(x,y))|\; dx \;>\; \int_{B - \Omega} |\phi'(R)|\; dx \;=\; |\phi'(R)|\, vol(B - \Omega) \quad .$$

Since the volumes are equal, this proves that the inequality (5.3) holds.

If the sets $(\Omega - B)$ and $(B - \Omega)$ have measure zero, then the only contribution to the integrals for both norms, $N_\Omega(\phi')$ and $N_B(\phi')$, must come from the set $(\Omega \cap B)$. Therefore the two norms must be the same in this case. □

Now calculate $N_B(\phi')$. Assume the ball is centered at y=0, and for notational ease write $N(R) = N_{B(0,R)}(\phi')(R)$.

$$\begin{aligned}
N(R) &= \int_B |\phi'(x,0)|\,dx \\
N(R) &= \int_{\alpha=0}^{R}\int_{\beta=0}^{\pi}\int_{\gamma=0}^{2\pi} \phi'(\alpha)\sin^2\alpha\sin\beta\,d\gamma\,d\beta\,d\alpha \\
N(R) &= \frac{1}{\pi}\int_{\alpha=0}^{R} \sin\alpha + (\pi-\alpha)\cos\alpha\,d\alpha \\
N(R) &= \frac{1}{\pi}\left(2(1-\cos R) + (\pi - R)\sin R\right) \qquad (5.4)
\end{aligned}$$

With this estimate in hand, we are now ready to state our upper bounds on helicity. By Lemma 5.5, we have that $N_\Omega(\phi') \leq N(R)$. By inequalities (5.1) and (5.2), we have shown the following theorem.

Theorem 5.6. *Let R be the radius of a ball in S^3 with the same volume as Ω. Then for any vector field $V \in VF(\Omega)$, we have bounds on $BS(V)$ and the helicity of V as follows:*

$$\begin{aligned}
\|BS(V)\| &\leq N(R)\,\|V\| \\
|H(V)| &\leq N(R)\,\langle V, V\rangle \quad ,
\end{aligned}$$

where $N(R) = \frac{1}{\pi}\left(2(1-\cos R) + (\pi - R)\sin R\right)$.

To get a sense of the behavior of this bound, note that $N(R) \leq R$ (equality holds only at $R = 0$).

Since the three-sphere is compact, it is also possible for unit length vector fields to construct bounds on helicity of the form

$$H(V) \leq a_k\, vol(\Omega)^k \quad ,$$

where $k > 1$ and typically $a_k < 1$. In particular, we have the following two bounds.

Proposition 5.7. *Let V be a unit vector field in $VF(\Omega)$. Then*

$$\begin{aligned}
H(V) &\leq \tfrac{2}{3}\, vol(\Omega)^{4/3} \\
H(V) &\leq \tfrac{3}{4}\, vol(\Omega)^{6/5}
\end{aligned}$$

Proof. The volume of a ball in S^3 with radius R is $\pi(2R - \sin 2R)$. Consider the function ρ : $[0, 2\pi^2] \to [0, \pi]$, defined such that $\rho(v)$ is the radius of a ball in S^3 with volume v; i.e.,

$$vol\,(B(0, \rho(v))) = v = \pi\,[\,2\rho(v) - \sin(2\rho(v))] \quad .$$

For a unit vector field V, its helicity is

$$H(V) \quad \leq \quad N(R)\,\langle V, V\rangle = N(R)\,vol(\Omega)$$

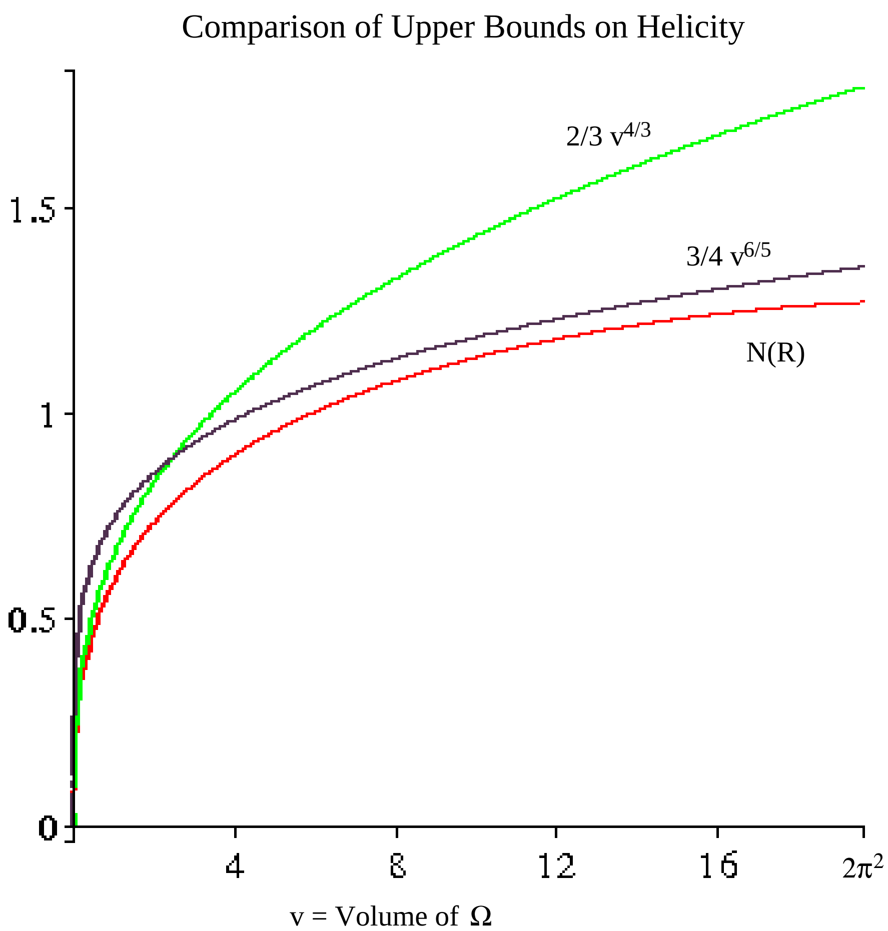

Figure 5.2: The upper bound on helicity $N(R)$ is less than the bounds given in Proposition 5.7.

We can establish the following bounds numerically for $v = vol(\Omega)$, see Figure 5.2:

$$
\begin{aligned}
N(\rho(v)) &\leq \tfrac{2}{3}\, v^{1/3} \\
N(\rho(v)) &\leq \tfrac{3}{4}\, v^{1/5}
\end{aligned}
$$

These estimates prove the proposition. $\square$

5.2 Examples

Now we turn our attention to some examples in order to measure how sharp the bounds are.

Example 5.8. Let $\Omega = S^3$, and let U be a left-invariant field. Then $BS(U) = -\frac{1}{2}U$, and its helicity is $H(U) = \int_{S^3} U \cdot BS(U)\, dx = -\frac{1}{2}\langle U, U\rangle = -\frac{1}{2}\, vol(S^3) = -\pi^2$. For W a right-invariant field, these values change sign: $BS(W) = \frac{1}{2}W$, and its helicity is $H(W) = \pi^2$.

Our bound for helicity on the entire 3-sphere is given by $|H(V)| \leq N(\pi)\,\langle V, V\rangle$, where the bound $N(\pi) = 4/\pi \approx 1.27$. So, while not sharp, the bound on the entire three-sphere is the right order of magnitude, roughly 2.5 times an attained value.

Example 5.9. As in Example 3.8, let Ω be a tubular neighborhood of the circle $x^2 + y^2 = 1$ in $S^3 = \{(x, y, u, v) | x^2 + y^2 + u^2 + v^2 = 1\}$ defined using toroidal coordinates as $\Omega = \{(\sigma, \theta, \phi) : 0 \leq \sigma \leq \sigma_a\}$. Set $a = \sin \sigma_a$. The boundary of Ω is a torus defined by the circles $u^2 + v^2 = a^2$ and $x^2 + y^2 = 1 - a^2$, or simply by the toroidal coordinate $\sigma = \arcsin a$.

The volume of Ω is

$$vol(\Omega) = \int_0^{2\pi} \int_0^{2\pi} \int_{\sigma=0}^{\sigma_a} \sin\sigma \cos\sigma \; d\sigma\, d\theta\, d\phi = 2\pi^2 a^2 \quad .$$

The Hopf vector field $\hat{u}_1 = -y\,\hat{x} \,+\, x\,\hat{y} \,+\, v\,\hat{u} \,-\, u\,\hat{v}$ has an orbit along the core circle of Ω and is tangent to the boundary torus $\partial\Omega$. So it lies in $K(\Omega)$, because it is of course divergence-free.

Calculate $BS(\hat{u}_1)$ using the left-translation formula:

$$\begin{aligned} BS(\hat{u}_1)(y) &= \int_\Omega (L_{yx^{-1}})_* \hat{u}_1(x) \times \nabla_y \phi_0(x,y)\, dx \\ &\quad - \frac{1}{4\pi^2} \int_\Omega (L_{yx^{-1}})_* \hat{u}_1(x)\, dx \\ &\quad + 2\nabla_y \int_\Omega (L_{yx^{-1}})_* \hat{u}_1(x) \cdot \nabla_y \phi_1(x,y)\, dx \end{aligned}$$

Start with the second integral, (II). Since $\hat{u}_1$ is left-invariant, $(L_{yx^{-1}})_* \hat{u}_1(x) = \hat{u}_1(y)$.

$$\begin{aligned} \text{(II)} &= -\frac{1}{4\pi^2} \int_\Omega \hat{u}_1(y)\, dx \\ &= -\frac{1}{4\pi^2}\, vol(\Omega)\, \hat{u}_1(y) \\ &= -\frac{a^2}{2}\, \hat{u}_1(y) \end{aligned}$$

By Remark 3.18, $\nabla_y \phi_1(\alpha) = -(L_{yx^{-1}})_* \nabla_x \phi_1(\alpha)$; substitute this into the third integral, (III).

$$\begin{aligned} \text{III} &= -2\nabla_y \int_\Omega (L_{yx^{-1}})_* \hat{u}_1(x) \cdot (L_{yx^{-1}})_* \nabla_x \phi_1(x,y)\, dx \\ &= -2\nabla_y \int_\Omega \hat{u}_1(x) \cdot \nabla_x \phi_1(x,y)\, dx \\ &= -2\nabla_y \int_\Omega \nabla_x \cdot \phi_1 \hat{u}_1(x)\, dx + 2\nabla_y \int_\Omega \phi_1 \nabla_x \cdot \hat{u}_1(x)\, dx \\ &= -2\nabla_y \int_{\partial\Omega} \phi_1 \hat{u}_1(x) \cdot \hat{n}\, d(area_x) + 2\nabla_y \int_\Omega \phi_1 \nabla_x \cdot \hat{u}_1(x)\, dx \\ &= 0 \end{aligned}$$

since $\hat{u}_1$ is both divergence-free and tangent to the boundary.

Now, only the calculation of the first integral, (I), remains.

$$\begin{aligned} \text{(I)} &= \int_\Omega (L_{yx^{-1}})_* \hat{u}_1(x) \times \nabla_y \phi_0(\alpha(x,y))\, dx \\ &= \int_\Omega \hat{u}_1(y) \times \nabla_y \phi_0(\alpha(x,y))\, dx \\ &= \hat{u}_1(y) \times \nabla_y \int_\Omega \phi_0(\alpha(x,y))\, dx \end{aligned}$$

We will use the symmetry of the domain to interpret the integral above. Considered as a function of x, the integrand ϕ_0 only depends on the distance $\alpha(x,y)$ between x and y. Because our domain is rotationally symmetric in both the $\hat{\theta}$ and $\hat{\phi}$ directions, the integral cannot depend upon those coordinates. Thus we can conclude that the integral only depends upon σ_y, the coordinate of y in the direction normal to the concentric tori comprising Ω. Let $f(\sigma_y)$ be the function so that

$$f(\sigma_y) = \int_\Omega \phi_0(\alpha(x, y))\, dx \quad .$$

Now, the first integral is

$$\begin{aligned}
(\mathrm{I}) &= \hat{u}_1(y) \times \nabla_y f(\sigma) \\
&= \left(\cos\sigma\hat{\theta} - \sin\sigma\hat{\phi}\right) \times f'(\sigma)\hat{\sigma} \\
&= f'(\sigma)\sin\sigma\,\hat{\theta} + f'(\sigma)\cos\sigma\,\hat{\phi}
\end{aligned}$$

(n.b., $(\hat{\sigma}, \hat{\theta}, \hat{\phi})$ is a left-handed frame; we must incorporate this when evaluating cross products in toroidal coordinates.)

Notice that integral (I) is orthogonal to $\hat{u}_1$ at every point in the domain. We summarize our calculations so far:

$$\begin{aligned}
BS(\hat{u}_1) &= (\mathrm{I}) + -\frac{vol(\Omega)}{4\pi^2}\,\hat{u}_1(y) \\
BS(\hat{u}_1) &= \left(-\frac{vol(\Omega)}{4\pi^2}\cos\sigma + f'(\sigma)\sin\sigma\right)\hat{\theta} + \left(\frac{vol(\Omega)}{4\pi^2}\sin\sigma + f'(\sigma)\cos\sigma\right)\hat{\phi} \qquad (5.5)
\end{aligned}$$

We will later solve explicitly for $f'(\sigma)$, but for now leave it undetermined. The helicity of $\hat{u}_1$ is independent of this uncertainty:

$$\begin{aligned}
H(\hat{u}_1) &= \langle \hat{u}_1, (\mathrm{I}) - \frac{vol(\Omega)}{4\pi^2}\,\hat{u}_1(y)\rangle \\
H(\hat{u}_1) &= -\frac{vol(\Omega)}{4\pi^2}\,\langle \hat{u}_1, \hat{u}_1\rangle \\
H(\hat{u}_1) &= -\frac{a^2}{4\pi^2}\,\langle \hat{u}_1, \hat{u}_1\rangle \\
H(\hat{u}_1) &= -\pi^2 a^4
\end{aligned}$$

This value is certainly less than our estimate in Theorem 5.6. When Ω contains 20% or more of the volume of the three-sphere, our bound is within an order of magnitude of the actual helicity. However for thin tubes, our bound is not nearly so sharp. See Figure 5.3 for a graph of the actual helicity, normalized by $\langle \hat{u}_1, \hat{u}_1\rangle$, and the bound $N(R)$ versus the tubular radius a.

Now we reexamine $BS(\hat{u}_1)$. Notice that both the domain Ω and the vector field $\hat{u}_1$ are invariant under the torus action which rotates them in the $\hat{\theta}$ and $\hat{\phi}$ directions independently. Therefore, $BS(\hat{u}_1)$ must also remain invariant under this torus action.

Suppose $BS(\hat{u}_1)$ has a nonzero component normal to the boundary $\partial\Omega$ at any point. Then it must have the same component at every point of the boundary. But then $BS(\hat{u}_1)$ would have nonzero flux through $\partial\Omega$, which means it would have a component in $HG \oplus GG$. But this is a contradiction, for this subspace is orthogonal to the image of $BS \subset FK \oplus HK \oplus CG$, as proved in Corollary 4.15. Thus $BS(\hat{u}_1)$ must be tangent to the boundary. It is also divergence-free, which implies $BS(\hat{u}_1) \in FK \oplus HK$.

Since $\hat{u}_1$ is a fluid knot, Ampere's Law implies that $\nabla \times BS(\hat{u}_1) = \hat{u}_1$. Because $\hat{u}_1$ is an eigenfield of curl, we know that

$$BS(\hat{u}_1) \quad = \quad -\tfrac{1}{2}\hat{u}_1 + cW \quad ,$$

where $W = \dfrac{1}{\cos\sigma}\,\hat{\theta}$ is a generator of $HK(\Omega)$ and c is a constant. Expand this into coordinates as

$$BS(\hat{u}_1) \quad = \quad \left(-\tfrac{1}{2}\cos\sigma + \frac{c}{\cos\sigma}\right)\hat{\theta} \;+\; \tfrac{1}{2}\sin\sigma\,\hat{\phi} \tag{5.6}$$

Set the two equations, (5.5) and (5.6), for $BS(\hat{u}_1)$ equal to each other, and solve for $f'(\sigma)$ and c:

$$\begin{aligned} f'(\sigma) &= \frac{1-a^2}{2}\tan\sigma \\ c &= \frac{1-a^2}{2} \end{aligned}$$

Finally, we obtain an explicit formula for $BS(\hat{u}_1)$ on a tube of (Euclidean) radius a, valid inside the tube $\Omega = (0 \leq \sigma \leq \arcsin(a))$:

$$BS(\hat{u}_1) = \frac{\sin^2\sigma - a^2}{2\cos\sigma}\,\hat{\theta} + \tfrac{1}{2}\sin\sigma\,\hat{\phi} \quad .$$

We now verify Ampere's Law, which implies that the circulation around a loop in the $\hat{\theta}$ direction on $\partial\Omega$ should be 0, since the loop bounds a surface outside Ω. On $\partial\Omega$, the coordinate σ is constant, equal to $\sigma_a = \arcsin(a)$. Thus, the $\hat{\theta}$ component of $BS(\hat{u}_1)$ vanishes on $\partial\Omega$, where we have

$BS(\hat{u}_1) = \dfrac{a}{2}\,\hat{\phi}$. The circulation of $BS(\hat{u}_1)$ clearly vanishes along such a loop, and Ampere's Law is satisfied.

Now we calculate $BS(\hat{u}_1)$ on the complement of Ω. This vector field must lie in the space $FK \oplus HK(S^3 - \Omega)$, by the same torus action argument as on Ω. Maxwell's equations imply $\nabla \times BS(\hat{u}_1) = 0$. Thus $BS(\hat{u}_1) \in HK(S^3 - \Omega)$, as it is both curl-free and divergence-free

The harmonic knots on $S^3 - \Omega$ are generated by $W_2 = \dfrac{1}{\sin\sigma}\,\hat{\phi}$. So $BS(\hat{u}_1) = \dfrac{c_2}{\sin\sigma}\,\hat{\phi}$ on $S^3 - \Omega$. The BS operator is continuous across the boundary of a domain, which implies $c_2 = a^2/2$. We conclude

$$BS(\hat{u}_1)(y) = \begin{cases} \dfrac{\sin^2\sigma - a^2}{2\cos\sigma}\,\hat{\theta} \;+\; \frac{1}{2}\sin\sigma\,\hat{\phi} & y \in \Omega \\[2ex] \dfrac{a^2}{2\sin\sigma}\,\hat{\phi} & y \notin \Omega \end{cases}$$

As a final exercise, we calculate the norms of $BS(\hat{u}_1)$, considered as a vector field first in Ω then in S^3.

$$\begin{aligned} \|BS(\hat{u}_1)(y)\|_\Omega &= \left[\frac{2a^2-1}{4} - \frac{(1-a^2)^2}{4a^2}\ln(1-a^2)\right]^{1/2} \|\hat{u}_1\| \\ \|BS(\hat{u}_1)(y)\|_{S^3} &= \left[\frac{2a^2-1}{4} - \frac{(1-a^2)^2}{4a^2}\ln(1-a^2) - \frac{a^2}{2}\ln(a)\right]^{1/2} \|\hat{u}_1\| \end{aligned}$$

Both the Biot-Savart operator and helicity of $\hat{u}_1$ respect our calculated bounds on the solid torus Ω. In fact, for all such domains,

$$\frac{|H(\hat{u}_1)|}{\langle \hat{u}_1, \hat{u}_1\rangle} \leq \frac{|BS(\hat{u}_1)|_\Omega}{\|\hat{u}_1\|} \leq \frac{|BS(\hat{u}_1)|_{S^3}}{\|\hat{u}_1\|} < N(R) \quad .$$

Equality occurs if and only if $\Omega = \emptyset$ or $\Omega = S^3$. Figure 5.3 shows a graph of these normalized quantities and the calculated bound $N(R)$ as a function of the tubular radius a of Ω.

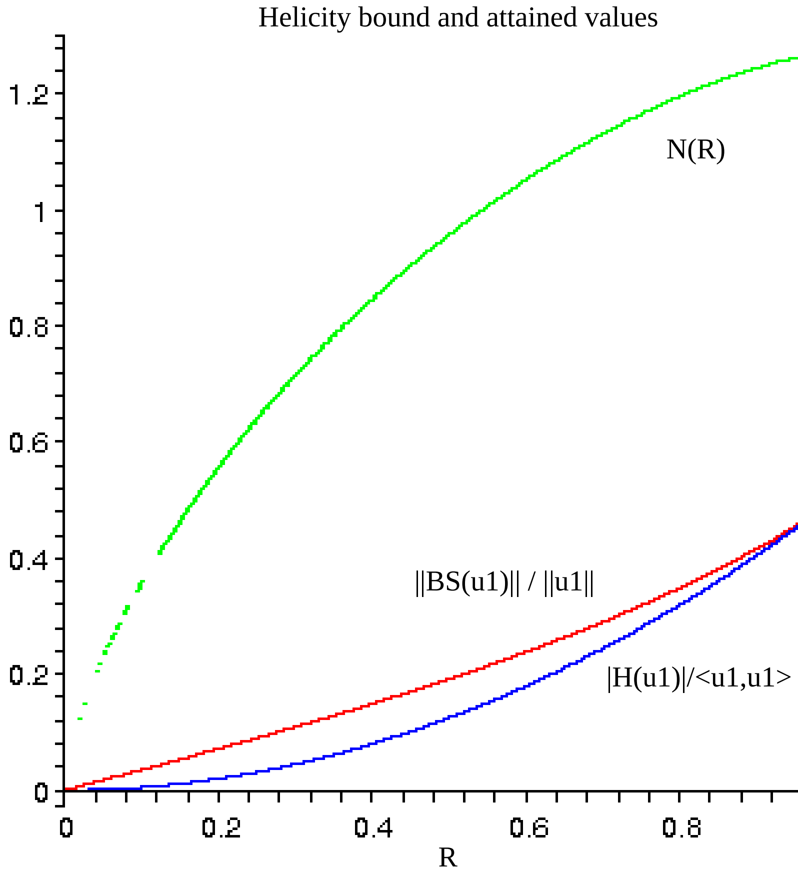

Figure 5.3: The upper bound on helicity $N(R)$ is greater than the attained values of $BS(\hat{u}_1)/\|\hat{u}_1\|$ and $H(\hat{u}_1)/\langle \hat{u}_1, \hat{u}_1 \rangle$.

Chapter 6

Future study

The geometrical setup of electrodynamics on $\mathbb{R}^3$ has many interesting applications in both pure and applied mathematics. We described several of these applications in chapter 2. Many of these remain as open problems in curved geometries.

Solar physics is the setting for one application of our work. The problem is to analytically model the observed metastable plasma states which persist in the sun's atmosphere before they erupt into coronal mass ejections and flares. Ideally one could apply energy minimization results from the Woltjer problem (see section 2.3.2) to this problem. However, the natural domain is the complement of the sun in the universe; this region is not compact, and on it there are no finite energy curl eigenfields. If we model the universe as a three-sphere with very large radius, then the complement of the sun is compact, and our techniques using curl (and Biot-Savart) eigenfields are applicable.

In Corollary 5.3, we showed that the norm of the Biot-Savart operator is bounded. For certain vector fields curl acts as a left-inverse to Biot-Savart. We anticipate lower bounds on the eigenvalues on the curl operator will arise from our work.

Helicity leads to several future areas of research. We aim to show that helicity will remain bounded under volume-preserving diffeomorphisms of a domain Ω. Also, does Berger and Field's formula

$$H(V) \;=\; Flux(V)^2 \; Wr(K)$$

hold for tubular neighborhoods of knots on S^3?

In section 2.3.2, we described three energy-minimization problems which arise from helicity, namely the Woltjer problem, the Taylor problem, and the optimal domains problem. A natural extension of this work would be to solve these problems on different subdomains of the three-sphere. We anticipate that the compactness of S^3 will make the optimal domains problem easier to solve on the three-sphere than in 3-space, where it is still open.

Beyond that, another research avenue suggested via this dissertation is to understand the effect of negative curvature on this program: what is the Biot-Savart operator for subdomains of H^3? What are its properties? DeTurck and Gluck in [14] already have examined linking integrals there; together we have goals of understanding this vast story on as many as possible of the eight different geometries available for three-manifolds.

Appendix A

Vector identities on Riemannian 3-manifolds

A.1 Vector identities

The formulas for Euclidean vector identities involving the gradient, divergence, and curl operators are well-known and used throughout the sciences, especially in physics. Many electrodynamics books (e.g., Griffiths [18]) conveniently list these identities. These identities are similarly useful when studying vector fields on other 3-manifolds. In this section, we prove the appropriate generalizations of 11 such identities, listed below.

Let M^3 be an orientable, Riemannian, three-dimensional manifold, with ∂M smooth if it exists. Let A, B, C be smooth vector fields on M; let $f, h \in C^\infty(M)$. Denote A acting on the function f as $A(f)$, where $A(f) = \nabla f \cdot A$. Let $L(V)$ denote the vector Laplacian of V, as in section 3.7. Then the identities given in Table A.1 hold.

Table A.1: List of Vector Identities

1. $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$

2. $A \times (B \times C) = (A \cdot C)\, B - (A \cdot B)\, C$

3. $\nabla(fh) = f\,(\nabla h) + h\,(\nabla f)$

4. $\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + \nabla_A B + \nabla_B A$

5. $\nabla \cdot (fA) = f\,(\nabla \cdot A) + A(f)$

   $\nabla \cdot (fA) = f\,(\nabla \cdot A) + A \cdot \nabla f$

6. $\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - (\nabla \times B) \cdot A$

7. $\nabla \times (fA) = f\,(\nabla \times A) + \nabla f \times A$

8. $\nabla \times (A \times B) = (\nabla \cdot B)\, A - (\nabla \cdot A)\, B + \nabla_B A - \nabla_A B$

   $\nabla \times (A \times B) = (\nabla \cdot B)\, A - (\nabla \cdot A)\, B + [B, A]$

9. $\nabla \cdot (\nabla \times A) = 0$

10. $\nabla \times (\nabla f) = 0$

11. $L(fA) = (\Delta f)\, A + 2\nabla_{\nabla f} A + f L(A)$

A.2 Notation

We will make extensive use of tensor notation and will sum over all repeated indices. For local coordinates x_i, write vector fields as $A = a^i \frac{\partial}{\partial x_i}$, etc.

Let σ_{rsk} be the sign of the permutation (rsk) in the permutation group S_3; e.g., $\sigma_{123} = 1$ and $\sigma_{132} = -1$. If (rsk) is not a permutation of $\{1, 2, 3\}$, by convention define $\sigma_{rsk} = 0$.

Establish local coordinates $\{x_i\}$ on M^3. Let $g = (g_{ij})$ be the Riemannian metric on M. Let $G^2 = det(g_{ij})$ and describe the inverse matrix of the metric as $g^{-1} = \left(g^{ij}\right)$. Below are descriptions of the vector operations in coordinates. Let $A = a^i \frac{\partial}{\partial x_i}$ and $B = b^j \frac{\partial}{\partial x_j}$.

Table A.2: Vector Operations in Local Coordinates

- *Inner product*: $\langle A, B\rangle = \langle a^i \dfrac{\partial}{\partial x_i}, b^j \dfrac{\partial}{\partial x_j}\rangle = g_{ij} a^i b^j$
- *Cross product*: $A \times B = \sigma_{rsk} \dfrac{1}{G} g_{ri} g_{sj} a^i b^j \dfrac{\partial}{\partial x_k}$
- *Gradient*: $\nabla f = g^{ij} \dfrac{\partial f}{\partial x_j} \dfrac{\partial}{\partial x_i}$
- *Divergence*: $\nabla \cdot A = \dfrac{1}{G} \dfrac{\partial}{\partial x_i} \left(G a^i\right)$
- *Laplacian*: $\Delta f = \dfrac{1}{G} \dfrac{\partial}{\partial x_i} \left(G g^{ij} \dfrac{\partial f}{\partial x_j} \dfrac{\partial}{\partial x_i}\right)$
- *Curl*: $\nabla \times A = \sigma_{ijk} \dfrac{1}{G} \dfrac{\partial}{\partial x_j} \left(g_{km} a^m\right) \dfrac{\partial}{\partial x_i}$
- *Covariant derivative*: $\nabla_A B = \left[a^i b^j \, \Gamma^k_{ij} + a^i \dfrac{\partial}{\partial x_i} \left(b^k\right)\right] \dfrac{\partial}{\partial x_k}$

  Here, Γ^k_{ij} denote the Christoffel symbols of the metric.

Throughout this section, we use Einstein's repeated index summation convention, unless otherwise noted. We also attempt to maintain proper tensor notation: superscript indices depict vector-like quantities and subscript indices depict differential quantities.

A.3 Useful lemmas

Here are two lemmas which appear numerous times in the upcoming proofs. The first concerns permutations of metric coefficients and the second involves calculating products of permutation symbols such as $\sigma_{ijk}\sigma_{rsk}$.

Lemma A.1. *For σ_{rsk}, the sign of the permutation (rsk), and metric coefficients g_{ij} as defined above, we have*

$$\begin{aligned}\sum_{r,s,k=1}^{3} \sigma_{rsk}\, g_{ri}g_{sj}g_{kn} &= G^2\sigma_{ijn} \\ &= \sigma_{ijn}\det(g)\end{aligned}$$

Proof. There are 27 possible 3-tuples $(i,j,n) \in \{1,2,3\}^3$. Consider first the case when (i,j,n) is in the permutation group S_3; this occurs for 6 of the 27 possible 3-tuples. Then by linear algebra,

$$\sum_{r,s,k=1}^{3} \sigma_{rsk}\, g_{ri}g_{sj}g_{kn} = \sigma_{ijn}\det(g) \quad .$$

We now consider the remaining cases where (i,j,n) is not a permutation. There are three possible terms of the form (i,i,i). Then $\sum_{r,s,k=1}^{3} \sigma_{rsk}g_{ri}g_{si}g_{ki}$ vanishes, since the permutations (abk) and (bak) provide identical summands with opposite signs.

The last case involves 3-tuples (i,j,n) with one repeated index; there are 18 such 3-tuples. Without loss of generality, assume $j=i$ so that the 3-tuple is (i,i,n). Then $\sum_{r,s,k=1}^{3} \sigma_{rsk}\, g_{ri}g_{si}g_{kn}$ also vanishes, since the permutations (a,b,k) and (b,a,k) provide identical summands with opposite signs. □

Lemma A.2. *Let (ijk) and (rsk) be two permutations in S_3. The product of their two permutation symbols is given by*

$$\sigma_{ijk}\sigma_{rsk} = \begin{cases} 1 & \text{when} \quad i=r,\ j=s \\ -1 & \text{when} \quad i=s,\ j=r \end{cases} \quad .$$

Proof. If $i = r,\ j = s$, then $\sigma_{ijk}\sigma_{rsk} = \sigma_{ijk}^2 = 1$.

If $i = s,\ j = r$, then $\sigma_{ijk}\sigma_{rsk} = \sigma_{ijk}\sigma_{jik} = -\sigma_{ijk}^2 = -1$. □

A.4 Proofs of identities

Identity 1. $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$

Proof. Using Table A.2 and Lemma A.1, we can write

$$\begin{aligned} A \cdot (B \times C) = &\quad g_{lk} a^l \sigma_{rsk} \tfrac{1}{G}\, g_{ri} g_{sj} b^i c^j = &\quad \tfrac{1}{G} G^2\, \sigma_{lij}\, a^l b^i c^j \\ B \cdot (C \times A) = &\quad g_{lk} b^l \sigma_{rsk} \tfrac{1}{G} g_{ri} g_{sj} c^i a^j = &\quad \tfrac{1}{G} G^2\, \sigma_{jli}\, a^j b^l c^i \\ C \cdot (A \times B) = &\quad g_{lk} c^l \sigma_{rsk} \tfrac{1}{G} g_{ri} g_{sj} a^i b^j = &\quad \tfrac{1}{G} G^2\, \sigma_{ijl}\, a^i b^j c^l \end{aligned}$$

Since $\sigma_{ijl} = \sigma_{jli} = \sigma_{lij}$, we conclude that all three formulas are thus the same. □

Identity 2. $A \times (B \times C) = (A \cdot C)\, B - (A \cdot B)\, C$

Proof. Begin with the left-hand side:

$$\begin{aligned} B \times C &= \sigma_{rsl} \frac{1}{G}\, g_{ri} g_{sj} b^i c^j \frac{\partial}{\partial x_l} \\ A \times (B \times C) &= \sigma_{tuk} \frac{1}{G^2}\, g_{tm} g_{ul} a^m\, \sigma_{rsl}\, g_{ri} g_{sj} b^i c^j \frac{\partial}{\partial x_k} \end{aligned}$$

Rewrite $g_{lu} = g_{ul}$. Using Lemma A.1, we see

$$\sigma_{rsl}\, g_{ri} g_{sj} g_{lu} = G^2 \sigma_{iju} \quad .$$

Then $A \times (B \times C)$ simplifies as

$$\begin{aligned} A \times (B \times C) &= \sigma_{tuk} \frac{1}{G^2}\, g_{tm} a^m\, G^2 \sigma_{iju}\, b^i c^j \frac{\partial}{\partial x_k} \\ &= \sigma_{tuk} \sigma_{iju}\, g_{tm} a^m b^i c^j \frac{\partial}{\partial x_k} \end{aligned}$$

Now, use Lemma A.2 to say that

$$\sigma_{tuk}\sigma_{iju} = \sigma_{ktu}\sigma_{iju} = \begin{cases} +1 & i = k, j = t \\ -1 & i = t, j = k \end{cases} \quad .$$

So,

$$\begin{aligned} A \times (B \times C) &= \left(g_{jm} a^m c^j\right) b^i \, \frac{\partial}{\partial x_i} \, - \, \left(g_{im} a^m b^i\right) c^j \, \frac{\partial}{\partial x_j} \\ &= (A \cdot C) B \, - \, (A \cdot B) C \qquad \square \end{aligned}$$

Identity 3. $\nabla(fh) = f\,(\nabla h) + h\,(\nabla f)$

Proof. We expand the left-hand side:

$$\begin{aligned} \nabla(fh) &= g^{ij} \frac{\partial}{\partial x_j} \, [fh] \, \frac{\partial}{\partial x_i} \\ &= g^{ij} \frac{\partial f}{\partial x_j} h \frac{\partial}{\partial x_i} + g^{ij} f \frac{\partial h}{\partial x_j} \frac{\partial}{\partial x_i} \\ &= h \left(g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \right) + f \left(g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \right) \\ &= h\,(\nabla f) + f\,(\nabla h) \quad , \end{aligned}$$

which concludes the proof. $\square$

Identity 4. $\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + \nabla_A B + \nabla_B A$

Proof. Out of all 11 identities, this one is by far the most complicated to prove. We will expand both sides of the identity and obtain three terms on the left and nine terms on the right. The nine terms combine nicely and equate to the three terms on the left.

We begin by expanding the left-hand side; we denote the resulting three terms $(L1)$, $(L2)$, $(L3)$.

$$\begin{aligned} \nabla(A \cdot B) &= \nabla(g_{ij} a^i b^j) \\ &= g^{kl} \frac{\partial}{\partial x_l} \left(g_{ij} a^i b^j \right) \frac{\partial}{\partial x_k} \\ &= g^{kl} g_{ij} a^i \frac{\partial}{\partial x_l} (b^j) \frac{\partial}{\partial x_k} \quad + \quad g^{kl} g_{ij} b^j \frac{\partial}{\partial x_l} (a^i) \frac{\partial}{\partial x_k} \quad + \quad g^{kl} a^i b^j \frac{\partial}{\partial x_l} (g_{ij}) \frac{\partial}{\partial x_k} \\ &= \quad (L1) \quad + \quad (L2) \quad + \quad (L3) \end{aligned}$$

On the right-hand side, first consider the covariant derivative terms:

$$\begin{aligned} \nabla_A B &= a^i b^j \, \Gamma^k_{ij} \frac{\partial}{\partial x_k} + a^i \frac{\partial}{\partial x_i} (b^j) \frac{\partial}{\partial x_j} \\ \nabla_B A &= a^i b^j \, \Gamma^k_{ji} \frac{\partial}{\partial x_k} + b^j \frac{\partial}{\partial x_j} (a^i) \frac{\partial}{\partial x_i} \end{aligned}$$

Since Christoffel symbols Γ^k_{ij} are symmetric in i and j, we obtain three terms: (I), (II), and (III).

$$\begin{aligned}
\nabla_A B + \nabla_B A &= 2a^i b^j \,\Gamma^k_{ij} \frac{\partial}{\partial x_k} + a^i \frac{\partial}{\partial x_i}(b^j) \frac{\partial}{\partial x_j} + b^j \frac{\partial}{\partial x_j}(a^i) \frac{\partial}{\partial x_i} \\
\nabla_A B + \nabla_B A &= \quad \text{(I)} \quad + \text{(II)} \quad + \quad \text{(III)}
\end{aligned}$$

Using the formula for a Christoffel symbol,

$$\Gamma^k_{ij} = \frac{1}{2} g^{mk} \left(\frac{\partial}{\partial x_i}(g_{jm}) + \frac{\partial}{\partial x_j}(g_{im}) - \frac{\partial}{\partial x_m}(g_{ij}) \right) \quad ,$$

we can break term (I) apart into three pieces

$$\begin{aligned}
(I) &= a^i b^j g^{mk} \frac{\partial}{\partial x_i}(g_{jm}) \frac{\partial}{\partial x_k} + a^i b^j g^{mk} \frac{\partial}{\partial x_j}(g_{im}) \frac{\partial}{\partial x_k} - a^i b^j g^{mk} \frac{\partial}{\partial x_m}(g_{ij}) \frac{\partial}{\partial x_k} \\
(I) &= (Ia) + (Ib) + (Ic)
\end{aligned}$$

Notice that term (Ic) has appeared before as $-(L3)$.

The next term to consider is

$$\begin{aligned}
A \times (\nabla \times B) &= A \times \frac{1}{G} \sigma_{pqk} \frac{\partial}{\partial x_q} \left(g_{kj} b^j \right) \frac{\partial}{\partial x_p} \\
&= \frac{1}{G^2} \sigma_{rsl} \sigma_{pqk} \, g_{ri} g_{sp} \, a^i \frac{\partial}{\partial x_q} \left(g_{kj} b^j \right) \frac{\partial}{\partial x_l}
\end{aligned}$$

We use the product rule to break this into two terms, enumerated (IV) and (V).

$$\begin{aligned}
A \times (\nabla \times B) &= (IV) + (V) \\
(IV) &= \frac{1}{G^2} \sigma_{rsl} \sigma_{pqk} \, g_{ri} g_{sp} \, a^i b^j \frac{\partial}{\partial x_q} \left(g_{kj} \right) \frac{\partial}{\partial x_l} \\
(V) &= \frac{1}{G^2} \sigma_{rsl} \sigma_{pqk} \, g_{ri} g_{sp} g_{kj} \, a^i \frac{\partial}{\partial x_q} \left(b^j \right) \frac{\partial}{\partial x_l}
\end{aligned}$$

Similarly, $B \times (\nabla \times A)$ can be expressed as the sum of two analogous terms, which we will denote (VI) and (VII).

$$\begin{aligned}
B \times (\nabla \times A) &= (VI) + (VII) \\
(VI) &= \frac{1}{G^2} \sigma_{rsl} \sigma_{pqk} \, g_{rj} g_{sp} \, a^i b^j \frac{\partial}{\partial x_q} \left(g_{ki} \right) \frac{\partial}{\partial x_l} \\
(V) &= \frac{1}{G^2} \sigma_{rsl} \sigma_{pqk} \, g_{rj} g_{sp} g_{ki} \, b^j \frac{\partial}{\partial x_q} \left(a^i \right) \frac{\partial}{\partial x_l}
\end{aligned}$$

Now that we have listed all terms on the right-hand side, we are ready to describe the proof. We claim that the following four equations hold

$$(IV) = (L3) - (Ia) \tag{A.1}$$

$$(V) = (L1) - (II) \tag{A.2}$$

$$(VI) = (L3) - (Ib) \tag{A.3}$$

$$(VII) = (L2) - (III) \tag{A.4}$$

Using equations (A.1) - (A.4), the proof is immediate. The right-hand side becomes

$$\begin{aligned} RHS &= (I) + (II) + (III) + (IV) + (V) + (VI) + (VII) \\ &= (Ia) + (Ib) + (Ic) + (L1) + (L2) + 2(L3) - (Ia) - (Ib) \\ &= (L1) + (L2) + 2(L3) + (Ic) \\ &= (L1) + (L2) + 2(L3) - (L3) \\ &= LHS \end{aligned}$$

We start with a proof of equation (A.1). In order to use Lemma A.1, we insert a factor of $g_{ly}g^{ly}$ into terms (IV) and (V). Since g^{ly} represents the inverse matrix of g_{ly}, this factor is precisely 1. Then (IV) is

$$(IV) = \frac{1}{G^2}\sigma_{rsl}\sigma_{pqk}\, g_{ri}g_{sp}\, g_{ly}g^{ly}\, a^i b^j \frac{\partial}{\partial x_q}(g_{kj})\frac{\partial}{\partial x_l} \quad .$$

Now Lemma A.1 states that

$$\frac{\sigma_{rsl}\, g_{ri}g_{sp}g_{ly}}{G^2} = \sigma_{pyi} \quad .$$

Thus, term (IV) is

$$(IV) = \sigma_{pyi}\sigma_{pqk}\, g^{ly}a^i b^j \frac{\partial}{\partial x_q}(g_{kj})\frac{\partial}{\partial x_l} \quad .$$

Whenever two permutation signs share an index, we may use Lemma A.2 to simplify the expression.

$$(IV) \quad = \quad g^{lq}a^i b^j \frac{\partial}{\partial x_q}(g_{ij})\frac{\partial}{\partial x_l} - g^{lk}a^i b^j \frac{\partial}{\partial x_q}(g_{kj})\frac{\partial}{\partial x_l}$$

After suitably changing variable names, term (IV) is recognized as (L3) - (Ia); thus we have shown Equation A.1. Equation A.3 follows via the same proof.

We now turn to Equation A.2. After inserting the factor $g_{lx}g^{lx}$ and applying Lemma A.1, we obtain

$$(V) = \sigma_{pyi}\sigma_{pqk}\, g^{ly} g_{kj} a^i \frac{\partial}{\partial x_q}\left(b^j\right)\frac{\partial}{\partial x_l} \quad .$$

Now apply Lemma A.2 and we see that

$$\begin{aligned} (V) &= g^{lq} g_{ij} a^i \frac{\partial}{\partial x_q}\left(b^j\right)\frac{\partial}{\partial x_l} - g^{li} g_{kj} a^i \frac{\partial}{\partial x_i}\left(b^j\right)\frac{\partial}{\partial x_l} \\ (V) &= (L1) - (II) \end{aligned}$$

This proves equation (A.2). Equation (A.4) follows via the same argument. □

Identity 5. $\nabla \cdot (fA) = f\,(\nabla \cdot A) + A(f)$

Proof. This identity results from writing the left-hand side in coordinates. Note that $A(f) = a^i \frac{\partial f}{\partial x_i}$.

$$\begin{aligned} \nabla \cdot fA &= \frac{1}{G}\frac{\partial}{\partial x_i}\left[Gfa^i\right] \\ &= f\frac{1}{G}\frac{\partial}{\partial x_i}\left[Ga^i\right] + \frac{1}{G}Ga^i\frac{\partial f}{\partial x_i} \\ &= f\,(\nabla \cdot A) + A(f) \end{aligned}$$

□

Identity 6. $\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - (\nabla \times B) \cdot A$

Proof. Recall $A \times B = \sigma_{rsk}\frac{1}{G} g_{ri} g_{sj} a^i b^j \frac{\partial}{\partial x_k}$.

The left-hand side is written in coordinates as

$$\begin{aligned} \nabla \cdot (A \times B) =& \frac{1}{G}\frac{\partial}{\partial x_k}\left[\sigma_{rsk} g_{ri} g_{sj} a^i b^j\right] \\ =& \frac{1}{G}\sigma_{rsk}\left(g_{ri}a^i \frac{\partial}{\partial x_k}\left[g_{sj}b^j\right] + g_{sj}b^j\frac{\partial}{\partial x_k}\left[g_{ri}a^i\right]\right) \end{aligned} \tag{A.5}$$

The first term on the right-hand side is

$$\begin{aligned}(\nabla \times A) \cdot B =& \frac{1}{G}\sigma_{nkl} \frac{\partial}{\partial x_k} \left(g_{lm} a^m\right) \frac{\partial}{\partial x_n} \cdot b^j \frac{\partial}{\partial x_j} \\ =& \frac{1}{G}\sigma_{nkl}\, g_{nj} b^j \frac{\partial}{\partial x_k} \left(g_{lm} a^m\right)\end{aligned}$$

By renaming indices ($l = r$, $m = i$, $n = s$), this becomes

$$(\nabla \times A) \cdot B = \frac{1}{G}\sigma_{skr}\, g_{sj} b^j \frac{\partial}{\partial x_k} \left[g_{ri} a^i\right] \tag{A.6}$$

Similarly, the second term on the right-hand side is

$$(\nabla \times B) \cdot A = \frac{1}{G}\sigma_{rks}\, g_{ri} a^i \frac{\partial}{\partial x_k} \left[g_{sj} b^j\right] \tag{A.7}$$

Since $\sigma_{skr} = \sigma_{rsk}$ but $\sigma_{rks} = -\sigma_{rsk}$, we combine Equations A.6 and A.7 to report the right-hand side as

$$(\nabla \times A) \cdot B - (\nabla \times B) \cdot A = \frac{1}{G}\sigma_{rsk}\, g_{sj} b^j \frac{\partial}{\partial x_k} \left[g_{ri} a^i\right] + \frac{1}{G}\sigma_{rsk}\, g_{ri} a^i \frac{\partial}{\partial x_k} \left[g_{sj} b^j\right] \quad ,$$

which is precisely the left-hand side as described in (A.5). □

Identity 7. $\nabla \times (fA) = f\,(\nabla \times A) + \nabla f \times A$

Proof. Begin by examining the last term, $\nabla f \times A$. The formula for the cross product of a gradient and a vector field is

$$\nabla f \times A = \sigma_{rsq} \frac{1}{G}\, g_{ri} g_{sp} g^{il} \frac{\partial f}{\partial x_l}\, a^p \frac{\partial}{\partial x_q} \quad .$$

Recall that $g_{ri} g^{il} = \delta_r^l$, so we have

$$\begin{aligned}\nabla f \times A =& \sigma_{rsq} \frac{1}{G} \delta_r^l\, g_{sp} \frac{\partial f}{\partial x_l} a^p \frac{\partial}{\partial x_q} \\ \nabla f \times A =& \sigma_{rsq} \frac{1}{G} g_{sp} a^p \frac{\partial f}{\partial x_r} \frac{\partial}{\partial x_q} \\ \nabla f \times A =& \sigma_{jki} \frac{1}{G} g_{km} a^m \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}\end{aligned} \tag{A.8}$$

In the last line, variables (p, q, r, s) have been changed to (m, i, j, k) respectively.

The curl of fA is apparent from the earlier curl formula:

$$\nabla \times (fA) = \sigma_{ijk} \frac{1}{G} \frac{\partial}{\partial x_j} \left[g_{km} f a^m \right] \frac{\partial}{\partial x_i} \quad .$$

By expanding the formula, we obtain

$$\nabla \times (fA) = f \left(\sigma_{ijk} \frac{1}{G} \frac{\partial}{\partial x_j} \left[g_{km} a^m \right] \right) \frac{\partial}{\partial x_i} + \sigma_{ijk} \frac{1}{G} g_{km} a^m \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$$

The first term on the right is $f(\nabla \times A)$. Since $\sigma_{jki} = \sigma_{ijk}$, the second term on the right is precisely $\nabla f \times A$ by Equation A.8. Thus we conclude $\nabla \times (fA) = f(\nabla \times A) + \nabla f \times A$. $\square$

Identity 8. $\nabla \times (A \times B) = \nabla_B A - \nabla_A B + [B, A]$

Proof. We start by expanding the left-hand side:

$$\begin{aligned} \nabla \times (A \times B) &= \nabla \times \left(\sigma_{rsk} \frac{1}{G} g_{ri} g_{sj} a^i b^j \frac{\partial}{\partial x_k} \right) \\ &= \sigma_{lmn} \frac{1}{G} \frac{\partial}{\partial x_m} \left[\sigma_{rsk} \frac{1}{G} g_{nk} g_{ri} g_{sj} a^i b^j \right] \frac{\partial}{\partial x_l} \end{aligned}$$

Lemma A.1 states that $\sigma_{rsk}\, g_{nk} g_{ri} g_{sj} = G^2 \sigma_{ijn}$. So, our formula above simplifies to

$$\nabla \times (A \times B) = \sigma_{lmn} \sigma_{ijn} \frac{1}{G} \frac{\partial}{\partial x_m} \left[G\, a^i b^j \right] \frac{\partial}{\partial x_l} \quad .$$

Lemma A.2 determines the sign of $\sigma_{lmn}\sigma_{ijn}$: it is $+1$ if $l = i$ and $m = j$; it is -1 if $l = j$ and $m = i$. Thus,

$$\begin{aligned} \nabla \times (A \times B) &= \frac{1}{G} \frac{\partial}{\partial x_j} \left[G a^i b^j \right] \frac{\partial}{\partial x_i} - \frac{1}{G} \frac{\partial}{\partial x_i} \left[G a^i b^j \right] \frac{\partial}{\partial x_j} \\ \nabla \times (A \times B) &= \frac{1}{G} a^i \tfrac{\partial}{\partial x_j} \left[G b^j \right] \frac{\partial}{\partial x_i} + b^j \frac{\partial a^i}{\partial x_j} \frac{\partial}{\partial x_i} - \frac{1}{G} b^j \frac{\partial}{\partial x_i} \left[G a^i \right] \frac{\partial}{\partial x_j} - a^i \frac{\partial b^j}{\partial x_i} \frac{\partial}{\partial x_j} \\ \nabla \times (A \times B) &= (\nabla \cdot B) A + b^j \frac{\partial a^i}{\partial x_j} \frac{\partial}{\partial x_i} - (\nabla \cdot A) B - a^i \frac{\partial b^j}{\partial x_i} \frac{\partial}{\partial x_j} \\ \nabla \times (A \times B) &= (\nabla \cdot B) A - (\nabla \cdot A) B + b^j \frac{\partial a^i}{\partial x_j} \frac{\partial}{\partial x_i} - a^i \frac{\partial b^j}{\partial x_i} \frac{\partial}{\partial x_j} \\ \nabla \times (A \times B) &= (\nabla \cdot B) A - (\nabla \cdot A) B + [B, A] \end{aligned}$$

$\square$

Identity 9. $\nabla \cdot (\nabla \times A) = 0$

Proof. This identity is directly calculated.

$$\begin{aligned}
\nabla \times A &= \sigma_{ijk} \frac{1}{G} \frac{\partial}{\partial x_j} [g_{km} a^m] \frac{\partial}{\partial x_i} \\
\nabla \cdot (\nabla \times A) &= \frac{1}{G} \frac{\partial}{\partial x_i} \left[\sigma_{ijk} \frac{1}{G} \frac{\partial}{\partial x_j} [g_{km} a^m] \right] \\
\nabla \cdot (\nabla \times A) &= \frac{1}{G} \sigma_{ijk} \frac{\partial^2}{\partial x_i \, \partial x_j} [g_{km} a^m] \\
\nabla \cdot (\nabla \times A) &= 0 \ ,
\end{aligned}$$

since permuting i and j changes the sign of σ_{ijk} while keeping the mixed partial derivatives the same. □

Identity 10. $\nabla \times (\nabla f) = 0$

Proof. This identity also follows directly.

$$\nabla \times (\nabla f) = \sigma_{ijk} \frac{1}{G} \frac{\partial}{\partial x_j} \left[g_{km} g^{mn} \frac{\partial f}{\partial x_n} \right] \frac{\partial}{\partial x_i}$$

Since $g_{km} g^{mn} = \delta_k^n$, rewrite the above equation:

$$\begin{aligned}
\nabla \times (\nabla f) &= \sigma_{ijk} \frac{1}{G} \frac{\partial}{\partial x_j} \left[\delta_k^n \frac{\partial f}{\partial x_n} \right] \frac{\partial}{\partial x_i} \\
\nabla \times (\nabla f) &= \sigma_{ijk} \frac{1}{G} \frac{\partial}{\partial x_j} \left[\frac{\partial f}{\partial x_k} \right] \frac{\partial}{\partial x_i} \\
\nabla \times (\nabla f) &= \sigma_{ijk} \frac{1}{G} \frac{\partial^2 f}{\partial x_j \, \partial x_k} \frac{\partial}{\partial x_i} \\
\nabla \times (\nabla f) &= 0 \ ,
\end{aligned}$$

since permuting j and k changes the sign of σ_{ijk} while keeping the mixed partial derivatives the same. □

Identity 11. $L(fA) = (\Delta f)\, A + 2\nabla_{\nabla f} A + f L(A)$

Proof. By definition,

$$L(fA) = \nabla \left(\nabla \cdot (fA) \right) - \nabla \times (\nabla \times fA) \tag{A.9}$$

We analyze the two terms of equation (A.9) separately. We will employ vector identities 3, 4, 5, 7, and 8 in Table A.1.

$$\begin{aligned}
\nabla(\nabla\cdot(fA)) &= \nabla(f(\nabla\cdot A)+\nabla f\cdot A) \\
&= (\nabla\cdot A)\nabla f + f\nabla(\nabla\cdot A) + \nabla(\nabla f\cdot A) \\
&= (\nabla\cdot A)\nabla f + f\nabla(\nabla\cdot A) + \nabla f\times(\nabla\times A) + A\times(\nabla\times\nabla f) + \nabla_{\nabla f}A + \nabla_A\nabla f \\
&= (\nabla\cdot A)\nabla f + \nabla f\times(\nabla\times A) + \nabla_A\nabla f + \nabla_{\nabla f}A + f\nabla(\nabla\cdot A)
\end{aligned}$$

Now for the second term of (A.9).

$$\begin{aligned}
\nabla\times(\nabla\times fA) &= \nabla\times(f(\nabla\times A)) + \nabla\times(\nabla f\times A) \\
&= f\nabla\times(\nabla\times A) + \nabla f\times(\nabla\times A) + (\nabla\cdot A)\nabla f - (\nabla\cdot\nabla f)A + \nabla_A\nabla f - \nabla_{\nabla f}A \\
&= (\nabla\cdot A)\nabla f + \nabla f\times(\nabla\times A) + \nabla_A\nabla f - \nabla_{\nabla f}A + f\nabla\times(\nabla\times A) - (\Delta f)A
\end{aligned}$$

The difference of these two terms is precisely $L(fA)$. The first three quantities of both terms are precisely the same; they cancel upon taking the difference of the terms. The fourth terms have different signs but are otherwise identical, and so they combine nicely in the difference of the terms. We are left with

$$\begin{aligned}
L(fA) &= 2\nabla_{\nabla f}A + f\nabla(\nabla\cdot A) - f\nabla\times(\nabla\times A) + (\Delta f)A \\
L(fA) &= fL(A) + 2\nabla_{\nabla f}A + (\Delta f)A \quad ,
\end{aligned}$$

which is precisely what was desired. $\square$

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