THE TAYLOR PROBLEM IN PLASMA PHYSICS

Robert Jason Parsley

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Supervisor of Thesis

Graduate Group Chairperson

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1 Abstract

The Taylor problem is defined on a compact, connected domain Ω in $\mathbb{R}^3$ as follows: among all divergence-free vector fields which are tangent to the boundary of the domain and have prescribed helicity and flux values, find the field with minimum energy. Such a minimizing field is an eigenfield of the curl operator (a *force-free field* to physicists) and depicts the end result of plasma relaxation in confined vessels. This paper establishes the existence of such a minimizing field, shows that it must necessarily have the lowest eigenvalue of all fields with the same helicity and flux, and provides an explicit solution of the Taylor problem on the solid flat torus.

2 Introduction

Let Ω be a compact, connected domain in $\mathbb{R}^3$ with smooth (C^∞) boundary, and let $VF(\Omega)$ be the space of all smooth vector fields on Ω with the L^2 inner product $\langle V, W\rangle = \int_\Omega V \cdot W \; dvol$.

Definition 2.1. *The* **helicity** *of a vector field V measures to what extent the flow lines of V wrap around each other. Helicity is defined as*

$$H(V) = \frac{1}{4\pi} \int_{\Omega\times\Omega} V(y) \cdot (V(x) \times \frac{y-x}{|y-x|^3}) \; dvol_x \; dvol_y$$

Helicity was introduced by Woltjer in 1958 and named by Moffat in 1969. Helicity is an important parameter in the study of fluid dynamics and plasma physics; see [1], [2], [5] for references.

We obtain a more convenient expression for helicity

$$H(V) = \langle V, BS(V)\rangle$$

by considering the Biot-Savart operator, $BS : VF(\Omega) \to VF(\Omega)$, defined as

$$BS(V)(y) = \frac{1}{4\pi} \int_\Omega V(x) \times \frac{y-x}{|y-x|^3} \; dvol_x \; .$$

This operator expresses the Biot-Savart law in electrodynamics relating a current distribution V to its magnetic field $BS(V)$. If $V = B$ is instead a magnetic field that is tangent to the boundary of its domain, $A = BS(B)$ is a vector potential for B.

Extend the Biot-Savart operator to $BS : \overline{VF}(\Omega) \to \overline{VF}(\Omega)$, where $\overline{VF}(\Omega)$ is the L^2 completion of the space of smooth vector fields. On this space, BS is a compact, self-adjoint operator. See [3] for a detailed treatment of the properties of the Biot-Savart operator.

Definition 2.2. *A vector field $V \in VF(\Omega)$ is a* **fluid knot** *if it is divergence-free and tangent to the boundary $\partial\Omega$.* (see the next section for more details)

Woltjer [14] showed that in a non-dissipative plasma, both energy and helicity are conserved, and an energy-minimizing field V is an eigenfield of curl: $\nabla \times V = \lambda V$ for some constant λ. Physicists refer to any V satisfying this equation, for $\lambda = \lambda(x)$ possibly dependent on position, as a *force-free field*. For magnetic fields, this equation implies the Lorentz force F_L is zero. If J is the current distribution inside Ω, then by Ampere's Law, $\mu_0 J = \nabla \times B = \lambda B$ which implies $F_L = \int J \times B \, dvol = 0$.

We can then state

The Woltjer Problem *Among all fluid knots on Ω, that is all divergence-free vector fields on Ω tangent to the boundary $\partial\Omega$ with a fixed helicity value, find the one with minimum energy.*

Cantarella, DeTurck, and Gluck showed in [3], [1] that such an energy-minimizing field exists and is necessarily a curl eigenfield (we keep this notation rather than

"force-free field"). Such fields represent steady states of plasmas in a simply connected domain. Lundquist [9] obtained an explicit solution of the Woltjer problem for the flat solid torus domain $D^2 \times S^1$. While not embeddable in $\mathbb{R}^3$, the flat solid torus is a good approximation of a large-aspect ratio torus of revolution and simplifies calculations greatly; see Tsuji [13] for explicit results on an ordinary torus of revolution.

On domains like the solid torus that are not simply-connected, solutions to the Woltjer problem are not the best approximations to the steady plasma states. By constraining one more parameter, the flux of the plasma, we transit from the Woltjer problem into the Taylor problem, whose solutions better approximate steady plasma states.

Definition 2.3. *Let $V \in VF(\Omega)$ and let Σ be a surface in the interior of Ω with $\partial\Sigma \subset \partial\Omega$ and consider $F_\Sigma(V) = \oint_\Sigma V \cdot n \, dA$, the* **(toroidal) flux** *of the vector field V through Σ.*

An important property of fluid knots is that their flux F_Σ depends only upon the relative homology class of Σ in $H_2(\Omega, \partial\Omega, \mathbb{R})$. For V a fluid knot, we can specify its overall flux F(V) by its fluxes through a collection of surfaces $\Sigma_1, ..., \Sigma_g$ that generate the relative homology group. Here g is the genus of $\partial\Omega$.

Example 2.4. Let V be a fluid knot. In the closed ball, the flux $F_\Sigma(V) = 0$ through any surface Σ. For the solid torus, the flux $F(V)$ can be computed using

any cross-sectional surface; the solid double torus requires two flux values to specify the overall flux $F(V)$.

Taylor [11], [12] concluded that both magnetic helicity and toroidal magnetic flux will be approximately conserved in a low-β plasma (one where magnetic forces dominate hydrodynamic forces) that is confined in a toroidal vessel with perfectly conducting walls. Thus we consider

The Taylor Problem *Among all fluid knots on* Ω *with fixed helicity and (toroidal) flux values, find the one with minimum energy.*

Taylor claimed that these energy minimizing fields represented the stable plasma states actually observed experimentally in tokamaks, toroidal chambers used in plasma experiments.

In this paper we begin with background information first on the Hodge Theorem for vector fields in Ω, then on plasma physics and experiments related to the Taylor problem. We begin our analysis by summarizing the Woltjer problem on the solid flat torus. Then we show the existence of an energy-minimizing vector field for the Taylor problem using functional analysis techniques and prove that the minimizer is a principal eigenfield of the curl operator restricted to the subset of fluid knots which satisfy the helicity and flux restrictions. This leads directly into the heart

of this paper, a detailed explicit solution of the Taylor problem on the solid flat torus. Finally we conclude by comparing experimental and theoretical results and by describing unsolved problems of interest.

3 The Hodge Decomposition Theorem

We make frequent use of the Hodge Theorem applied to domains in $\mathbb{R}^3$. Cantarella, DeTurck, and Gluck provide a detailed treatment of this subject in [4].

Theorem 3.1. *There is a decomposition of $VF(\Omega)$ into five mutually orthogonal subspaces,*

$$VF(\Omega) = FK \oplus HK \oplus CG \oplus HG \oplus GG$$

FK = fluxless knots = $\{\nabla \cdot V = 0,\ V \cdot n = 0,$ all interior fluxes = $0\}$

HK = harmonic knots = $\{\nabla \cdot V = 0,\ V \cdot n = 0,\ \nabla \times V = 0\}$

CG = curly gradients = $\{V = \nabla\phi,\ \nabla \cdot V = 0,$ all boundary fluxes = $0\}$

HG = harmonic gradients = $\{V = \nabla\phi,\ \nabla \cdot V = 0,\ \phi$ locally constant on $\partial\Omega\}$

GG = grounded gradients = $\{V = \nabla\phi,\ \phi|_{\partial\Omega} = 0\}$

In addition, $HK \cong H_1(\Omega, \mathbb{R}) \cong \mathbb{R}^{\text{genus } \partial\Omega}$, and

$$
\begin{array}{lcccccccccc}
\text{ker div} & = & \text{FK} & \oplus & \text{HK} & \oplus & \text{CG} & \oplus & \text{HG} & & \\
\text{image curl} & = & \text{FK} & \oplus & \text{HK} & \oplus & \text{CG} & & & & \\
\text{ker curl} & = & & & \text{HK} & \oplus & \text{CG} & \oplus & \text{HG} & \oplus & \text{GG} \\
\text{image grad} & = & & & & & \text{CG} & \oplus & \text{HG} & \oplus & \text{GG}
\end{array}
$$

Definition 3.2. *Define* **fluid knots** *(which we will truncate to "knots") as the subspace* $K(\Omega) = FK \oplus HK$*. Similarly, define* **gradients** *as the subspace* $G(\Omega) = CG \oplus HG \oplus GG$*.*

Definition 3.3. *A vector field is said to be* **Amperian** *if it integrates to 0 around every closed curve* C *on* $\partial\Omega$ *that bounds a surface outside* Ω*.*

Clearly all gradients are Amperian. Which knots are Amperian? Restrict the Biot-Savart operator to knots and project its image onto knots; call the resulting operator $BS' : K(\Omega) \to K(\Omega)$.

Proposition 3.4. *The image of* BS' *is equal to the space of Amperian knots* $AK(\Omega)$*.*

4 Plasma Physics Background

Plasmas are conducting fluids, usually gases, consisting of charged particles. Heated sufficiently, most matter ionizes into a plasma state. We restrict ourselves to the study of low-β plasmas, ones where the electromagnetic forces have a dominant

effect when compared to hydrodynamic effects. These are characterized by a small resistivity and small viscosity.

In non-turbulent plasmas, energy and helicity typically decay due to their inherent resistivity. The dissipation rate is proportional to the resisitivity value; in a low-β plasma this time scale can be quite large. If the plasma is isolated, it will eventually evolve into a vacuum field with no current.

A common setting for experiments on plasmas is a tokamak. A tokamak is a toroidal plasma chamber surrounded by coils which generate a large magnetic field inside. When a plasma is introduced into the chamber and a toroidal current is added, the plasma flow becomes turbulent. The current heats and compresses the plasma, a result called the *pinch effect*. Experiments typically show that toroidal field near the walls is in the opposite direction of the average toroidal field in the chamber. This is known as the *reversed field pinch*.

Laboratory results demonstrate that the plasma experiences a highly turbulent phase when the current is first introduced and then quickly reaches a quiescent state; this is called *plasma relaxation*. Turbulence allows the plasma to quickly attain a particular state of minimum energy. Relaxation occurs on a much smaller time-scale than energy loss due to resistive decay. The helicity and flux do not vary significantly during this turbulent period and are essentially preserved during the relaxation process.

See [12], [15] for more information.

5 The Woltjer Problem on $D^2 \times S^1$

The solid flat torus $D^2 \times S^1$ is not actually a domain in $\mathbb{R}^3$. For convenience, consider its universal covering space $D^2 \times \mathbb{R}$, an infinite cylinder in $\mathbb{R}^3$ of radius a; we use cylindrical coordinates (r, φ, z) to describe this space. Smooth vector fields on the solid flat torus are precisely those on the infinite cylinder which are periodic in the $\hat{z}$ direction with period ℓ. If we write

$$V(r, \varphi, z) = u(r, \varphi, z)\hat{r} + v(r, \varphi, z)\hat{\varphi} + w(r, \varphi, z)\hat{z} \ ,$$

then $V(r, \varphi, z+\ell) = V(r, \varphi, z)$. Equivalently the three component functions $u(r, \varphi, z)$, $v(r, \varphi, z)$ and $w(r, \varphi, z)$ must have the same periodicity.

The Woltjer Problem *Among all divergence-free vector fields on Ω tangent to the boundary $\partial\Omega$ with a fixed helicity value, find the one with minimum energy.*

First, we show that an energy minimizing vector field V must be an eigenfield of the modified Biot-Savart operator BS'. Notice BS' extends to a compact, self-adjoint operator on $\overline{K}(\Omega)$, the L^2 completion of the fluid knots $K(\Omega)$. By the spectral theorem, BS' is diagonalizable on $\overline{VF}(\Omega)$, i.e., for some orthonormal basis

of vector fields $\{e_\alpha\}$ in $\overline{K}(\Omega)$, we have $BS'(V) = \sum_\alpha \langle V, e_\alpha \rangle \mu_\alpha e_\alpha$ for any $V \in \overline{K}(\Omega)$. The set of eigenvalues $\{\mu_\alpha\}$ has countably many distinct elements, is bounded, and converges to 0. Let μ_1 be the eigenvalue of BS' with the largest absolute value.

Helicity is now related to energy as follows:

$$H(V) = \langle BS'(V), V \rangle = \sum_\alpha \langle V, e_\alpha \rangle^2 \mu_\alpha \leq |\mu_1| \sum_\alpha \langle V, e_\alpha \rangle^2 = |\mu_1| E(V)$$

$$E(V) \geq \frac{H(V)}{|\mu_1|}$$

So for a fixed helicity value, an energy minimizing field V must lie in the eigenspace associated to μ_1, i.e., $BS'(V) = \mu_1 V$.

Let $\lambda = \mu_1^{-1}$; λ is necessarily nonzero. Since curl is a left inverse of BS', an energy minimizing V must be an eigenfield of curl, i.e., $\nabla \times V = \lambda V$. Also, V is an eigenfield of the Laplacian, since $\Delta(V) = \nabla \times (\nabla \times V) = -\lambda^2 V$. By elliptic regularity, we are now assured that the minimizing field V is indeed smooth.

Cantarella, DeTurck, and Gluck solved the Woltjer problem on the solid flat torus by finding all curl eigenfields there that are Amperian knots and then comparing their energies to find the minimizer. A presentation of this roster of eigenfields follows; we use it for our work on the Taylor problem. The calculations below are based on a more detailed treatment in their upcoming paper [1].

To begin, consider vector fields with only radial dependence, i.e., $V = V(r)$. For any choice of λ, the field V_0 is a curl eigenfield:

$$V_0 = J_1(\lambda r)\hat{\varphi} \ + J_0(\lambda r)\hat{z} \ ,$$

where $J_m(x)$ denotes the mth order Bessel function of the first kind.

For arbitrary vector fields, we obtain four infinite families of curl eigenfields for each λ given by $V^{\pm i}_{m,k}$. They depend on the pair (m,k), where $m = 0, 1, 2, ...$, $k = 0, \dfrac{2\pi}{\ell}, \dfrac{4\pi}{\ell}, ...$, and $n = \sqrt{\lambda^2 - k^2} > 0$.

$$\begin{aligned}
V^{\pm 1}_{m,k} &= \left(-\frac{m}{r}J_m(nr) \mp \frac{mk}{\lambda r}J_m(nr) \pm \frac{nk}{\lambda}J_{m+1}(nr)\right)\sin(m\varphi \pm kz)\ \hat{r} \\
&\quad + \left(-\frac{m}{r}J_m(nr) \mp \frac{mk}{\lambda r}J_m(nr) + nJ_{m+1}(nr)\right)\cos(m\varphi \pm kz)\ \hat{\varphi} \\
&\quad + \frac{n^2}{\lambda}J_m(nr)\cos(m\varphi \pm kz)\ \hat{z} \ , \\
V^{\pm 2}_{m,k} &= \left(\frac{m}{r}J_m(nr) \pm \frac{mk}{\lambda r}J_m(nr) \mp \frac{nk}{\lambda}J_{m+1}(nr)\right)\cos(m\varphi \pm kz)\ \hat{r} \\
&\quad + \left(-\frac{m}{r}J_m(nr) \mp \frac{mk}{\lambda r}J_m(nr) + nJ_{m+1}(nr)\right)\sin(m\varphi \pm kz)\ \hat{\varphi} \\
&\quad + \frac{n^2}{\lambda}J_m(nr)\sin(m\varphi \pm kz)\ \hat{z} \ ,
\end{aligned} \tag{5.1}$$

When both m and k are nonzero, we obtain four linearly independent eigenfields for the eigenvalue $\lambda = +\sqrt{k^2 + n^2}$ and four for $\lambda = -\sqrt{k^2 + n^2}$. Otherwise, if only one of m or k is nonzero, there are only two linearly independent solutions. The field V_0 corresponds to the case where m and k are both zero, hence its notation.

To solve the Woltjer problem, we consider as candidates only those fields from above which are Amperian knots. All curl eigenfields (for $\lambda \neq 0$) are automatically divergence-free, so we need only check whether the candidate field is both tangent to the boundary and Amperian. On the flat solid torus, a knot is Amperian if and only if its $\hat{z}$ component on the boundary vanishes; we invite the reader to confirm this.

First consider a field of the form V_0: it is tangent to the boundary and ergo a knot. To be Amperian, we require $J_0(\lambda a)\hat{z} = 0$. Only a discrete collection of eigenvalues λ fulfill this requirement; the lowest positive one λ_0 satisfies $\lambda_0 a \approx 2.4048$, the first zero of $J_0(x)$.

Now consider any linear combination of eigenfields $V^{\pm i}_{m,k}$ having the same eigenvalue λ. Its $\hat{r}$ component includes terms of the form $cos(m\varphi + kz)$ and must vanish on the boundary. This can only occur if the $\hat{r}$ component vanishes at $r = a$ for each fixed pair of values (m, k). So it suffices to consider only one such pair; the $\hat{r}$ component of $V^{\pm i}_{m,k}$ is identically zero on the boundary precisely when

$$\frac{m}{a} J_m(na) = \pm \left(-\frac{mk}{\lambda a} J_m(na) + \frac{nk}{\lambda} J_{m+1}(na) \right) \tag{5.2}$$

where the sign of the right hand side is the same as the sign of $\pm i$. This requirement again forces the acceptable eigenvalues to be discrete. Careful analysis shows that the eigenvalue $\lambda_{m,k}$ obtained from $(m, k) \neq (0, 0)$ is larger than the minimal value λ_0 found above.

Thus the solution to the Woltjer problem for positive helicity is some constant multiple of the Lundquist field

$$V_0 = J_1(\lambda_0 r)\hat{\varphi} \ + J_0(\lambda_0 r)\hat{z} \ .$$

corresponding to $\lambda_0 \approx \frac{2.4048}{a}$. The helicity determines which constant multiple.

The Woltjer problem for negative helicity requires negative eigenvalues. Consider the transformation $(r, \phi, z) \to (r, -\phi, z)$ which maps the Lundquist field to

$$\tilde{V_0} = -J_1(\lambda_0 r)\hat{\varphi} \ + J_0(\lambda_0 r)\hat{z}$$

which has eigenvalue $-\lambda_0$. By the arguments above, $-\lambda_0$ is the largest negative eigenvalue meeting our requirements. Thus, some constant multiple of $\tilde{V_0}$ solves the Woltjer problem for negative helicity.

6 Existence of an Energy Minimizer

[I am indebted to Ilya Elson for suggesting many of the results and some of the methods in this section.]

For the Woltjer problem, we proved the existence of an energy minimizer by establishing that such a field was a principal eigenfield of the modified Biot-Savart operator BS', which is compact and self-adjoint. For the Taylor problem, the minimizer is not necessarily an eigenfield of BS' (although it remains an eigenfield of

curl), so existence requires a different argument. The following theorem will establish that a solution to the Taylor problem must exist.

Theorem 6.1. *Let B be a continuous, compact operator on a Hilbert space H. Let Φ be a continuous linear functional on H; let $h, f \in \mathbb{R}$. Suppose that the set*

$$S = \{v \in H \ : \ \langle Bv, v\rangle = h, \ \Phi(v) = f\}$$

is non-empty. Then there exists $v \in S$ of minimal norm.

Consider the L^2 completion of smooth knots on Ω as $\overline{K}(\Omega)$, a Hilbert space. The flux F is a continuous linear functional on $K(\Omega)$ and thus on its completion. The modified Biot-Savart operator BS' extends to a continuous, compact operator on $\overline{K}(\Omega)$. Provided that all pairs of prescribed helicity and flux values are physically attainable, the above theorem guarantees that a field of minimum energy exists for any helicity and flux prescription. In the next section we will show that any energy minimizer V must be an eigenfield of the curl operator. Since V is divergence-free, it is also an eigenfield of the Laplacian operator. Then by elliptic regularity, V must be smooth, so it lies in $VF(\Omega)$.

Are all pairs of helicity and flux values attainable? On the solid flat torus, the answer is yes; our solution in section 8 demonstrates a solution for any values. On arbitrary domains we expect the answer is also yes, but this is an unsolved problem.

Before beginning the proof of this theorem, we exhibit some facts from functional

analysis. Recall that the dual space X^* to X is the collection of all continuous linear maps $X \to \mathbb{R}$.

Definition 6.2. *Let X be a Banach space. The* weak topology *on X is given by the collection of sets $f^{-1}((a,b))$ for $f \in X^*$. A sequence (or net) $\{x_n\}$* converges weakly *to x if $\forall f \in X^*$ the sequence $f(x_n)$ converges to $f(x)$. The usual topology on X given by the norm is also called the* strong topology*.*

Theorem 6.3. (Banach-Steinhaus) *Let X and Y be Banach spaces and let $\{A_n\}$ be a sequence of bounded transformations from X to Y. If for all $x \in X$ there exists $y \in Y$ such that $\|A_n x - y\| \to 0$, then there is a bounded transformation A such that $\|A_n x - Ax\| \to 0$ for all $x \in X$ and $\sup \|A_n\| < \infty$.*

Lemma 6.4. *A weakly convergent sequence is bounded.*

Proof. Consider the canonical embedding $i : X \hookrightarrow X^{**}, i : x \mapsto \hat{x}$, where $\hat{x}(f)$ is defined to be $f(x)$ for $f \in X^*$. It follows that $\|\hat{x}\| = \|x\|$.

Let $x_n \to x$ weakly. Then $\hat{x}_n(f) = f(x_n)$ converges to $f(x)$ for all $f \in X^*$, and so $\sup \hat{x}_n(f) < \infty$. By the Banach-Steinhaus Theorem, $\sup \|\hat{x}_n\| < \infty$, so the sequence $\{x_n\}$ is bounded. □

Definition 6.5. *Let X, Y be Banach spaces, and let $A : X \to Y$ be a bounded operator. Call A* completely continuous *if for any sequence $\{x_n\}$ that converges weakly to x, then $\{Ax_n\}$ converges strongly to Ax.*

Lemma 6.6. *If $A : X \to Y$ is compact, then A is completely continuous.*

The converse holds if X is reflexive, i.e., $X = X^{**}$. For a proof of both the lemma and converse, see [7], Proposition 6.3.3.

Lemma 6.7. *If $\{x_n\}$ weakly converges to x and $\{y_n\}$ strongly converges to y, then $\langle x_n, y_n\rangle$ weakly converges to $\langle x, y\rangle$.*

Proof. By Lemma 6.4, $\{x_n\}$ is bounded by some $M < \infty$.

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &\leq |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle| \\ &\leq \|x_n\|\|y_n - y\| + |\langle x_n - x, y\rangle| \\ &< M\frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

if n is chosen large enough such that both $\|y_n - y\| < \frac{\epsilon}{2M}$ and $|\langle x_n - x, y\rangle| < \frac{\epsilon}{2}$, which is possible by weak convergence since $\langle \cdot, y\rangle$ is a linear functional. □

Lemma 6.8. *If $\{x_n\}$ weakly converges to x, then $\|x\| \leq \liminf_n \|x_n\|$ (i.e., the norm is a weakly lower semicontinuous function).*

Proof. Begin with $\langle x_n, x\rangle \leq \|x_n\|\|x\|$ and take the lim inf of each side:

$\liminf_n |\langle x_n, x\rangle| \leq \|x\| \liminf_n \|x_n\|$. But $\lim_{n\to\infty}\langle x_n, x\rangle = \|x\|^2$,

so $\|x\| \leq \liminf_n \|x_n\|$. □

Now we are ready to prove the existence theorem:

Proof. Some closed ball of radius ρ about the origin must intersect the set S, which was assumed non-empty. Let r be the infimum of all such ρ; let $\{v_k\} \in S$ be a sequence such that $\|v_k\| \to r$. The sequence $\{v_k\}$ is thus bounded. Now the Banach-Alaoglu Theorem in Hilbert spaces states that the unit ball is weakly compact, so $\{v_k\}$ has some weakly convergent subsequence (we keep the same notation $\{v_k\}$ for it) to some $v \in H$. We claim $v \in S$.

One condition follows easily: $\Phi(v) = \lim_{n\to\infty} \Phi(v_k)$. But $\Phi(v_k) = f \forall k$, so $\Phi(v) = f$. As for the second condition, $H(v_k) = \langle v_k, Bv_k \rangle = h$. Now, Bv_k is strongly convergent since B is compact and thus completely continuous. Then Lemma 6.7 implies $H(v_k) \to H(v)$.

So $v \in S$; it is minimal since by Lemma 6.8, $\|v\| \leq \liminf_n \|v_k\| = r$. □

7 The Energy Minimizer Has the Smallest Eigenvalue

In this section, we show that the energy-minimizing solution of the Taylor problem corresponds to the principal eigenvalue of the curl operator on all fluid knots with the prescribed helicity and flux. This useful result reduces the Taylor problem into an eigenvalue problem. Throughout this section, we assume all vector fields V are fluid knots on Ω.

Theorem 7.1. *A solution to the Taylor problem is an eigenfield of the curl operator.*

Proof. Choose a set of surfaces $\{\Sigma_i\}$ which generate the relative homology group $H_2(\Omega, \partial\Omega, \mathbb{R})$. Let $F_i(V) = \oint_{\Sigma_i} V \cdot n_i \, dA$ be the flux of V through Σ_i. As discussed in section 2, the overall flux $F(V)$ is defined by the values $F_i(V)$. To each surface Σ_i, there is associated a unique vector field W_i which measures flux through the surface, in the sense that $F_i(V) = \langle V, W_i \rangle$. The W_i must be orthogonal to fluxless knots and therefore lie in the kernel of curl.

We use a Lagrange multipliers approach. Define

$$\Phi(V) = E(V) - \lambda H(V) - 2\sum_i \mu_i F_i(V)$$

$$\Phi(V) = E(V) - \lambda \langle V, BS'(V) \rangle - 2\sum_i \mu_i \langle V, W_i \rangle$$

Any energy minimizer V must satisfy, for any $\tilde{V} \in K(\Omega)$,

$$\begin{aligned} \delta\Phi_V(\tilde{V}) &= \frac{d}{dt}\big|_{t=0} \Phi(V + t\tilde{V}) = 0 \\ &= 2\langle V, \tilde{V} \rangle - \lambda \langle BS'(V), \tilde{V} \rangle - \lambda \langle BS'(\tilde{V}), V \rangle - 2\sum_i \mu_i \langle \tilde{V}, W_i \rangle = 0 \end{aligned}$$

Since BS' is self-adjoint, we have

$$\delta\Phi_V(\tilde{V}) = 2\langle \left(V - \lambda BS'(V) - \sum_i \mu_i W_i \right), \tilde{V} \rangle = 0 \, .$$

Since this holds for any $\tilde{V}$, we have that

$$V - \lambda BS'(V) - \sum_i \mu_i W_i = 0 \, .$$

Now take the curl of both sides and find that $\nabla \times V - \lambda V = 0$. □

Definition 7.2. *For a closed curve $C \subset \partial\Omega$, the* **period** $p_C(V)$ *of the vector field V is its integral around C, $p_C(V) = \oint_C V \cdot ds$.*

Consider loops on the boundary $\partial\Omega$ of the domain. Denote a loop which bounds some surface inside the domain as M, and a loop on $\partial\Omega$ which bounds only surfaces outside the domain as L. We can choose a collection of loops $\{L_j\}$ that form a minimal set of generators for the homology $H_1(\Omega, \mathbb{R})$ and also a collection of loops $\{M_j\}$ such that the corresponding surfaces Σ_j form a minimal set of generators for the homology $H_2(\Omega, \partial\Omega, \mathbb{R})$. Many vector fields, including all gradients, harmonic knots, and curl eigenfields, have periods around these loops that do not depend on the choice of L_j and M_j within their respective homology classes.

Lemma 7.3. *Let $V \in K(\Omega)$ be any eigenfield of the curl operator with eigenvalue $\lambda \neq 0$. Let the vector fields $W_1, ..., W_n$ be a set of generators for the harmonic knots such that $p_{L_j}(W_i) = \delta_{ij}$. Then any Amperian vector potential B of V, i.e., $\nabla \times B = V$, has the form*

$$B = \frac{1}{\lambda} V + \nabla\eta - \sum \frac{p_{L_i}(V)}{\lambda} W_i \ ,$$

for a suitable choice of η.

Proof. The existence of the $\{W_i\}$ follows from the one-to-one correspondence between harmonic knots $HK(\Omega)$ and homology $H_1(\Omega, \mathbb{R})$ (see the Hodge Theorem section).

Clearly a vector potential B must equal $\frac{1}{\lambda}V$ plus some term from the kernel of curl. So

$$B = \frac{1}{\lambda}V + \nabla\eta + \sum c_i W_i.$$

To find the constants c_i, we use the Amperian condition:

$$0 = p_{L_i}(B) = \oint_{L_i} B \cdot ds = \sum_i \frac{1}{\lambda} p_{L_i}(V) + c_i$$

$$c_i = -\frac{1}{\lambda} p_{L_i}(V)$$

□

Corollary 7.4. $BS'(V) = \frac{1}{\lambda}V - \sum \frac{p_{L_i}(V)}{\lambda} W_i$

Example 7.5. On the solid ball B^3, there are no nontrivial harmonic knots since $H_1(B^3, \mathbb{R}) = 0$, so $B = \frac{1}{\lambda}V + \nabla\eta$.

On the solid flat torus $D^2 \times S^1$, the harmonic knots form a space of dimension 1 since $H_1(D^2 \times S^1, \mathbb{R}) = \mathbb{R}$. The vector field $W = \frac{1}{\ell}\hat{z}$ generates the harmonic knots and

$$B = \frac{1}{\lambda}V + \nabla\eta - \frac{p_L(V)}{\lambda\ell}\hat{z}\ .$$

On the usual solid torus of revolution embedded in $\mathbb{R}^3$, the vector field $W = \frac{1}{2\pi r}\hat{\varphi}$ generates the harmonic knots and

$$B = \frac{1}{\lambda}V + \nabla\eta - \frac{p_L(V)}{2\pi r\lambda}\hat{\varphi}\ .$$

By Proposition 3.4, $BS'(V)$ is Amperian. So $BS'(V)$ satisfies Lemma 7.3, and B and $BS'(V)$ can only differ by a gradient; thus $B = BS'(V) + \nabla\gamma$.

Lemma 7.6. *With V and B as above, $H(V) = \langle V, B\rangle$*

Proof. $H(V) = \langle V, BS'(V)\rangle = \langle V, B\rangle - \langle V, \nabla\gamma\rangle$. But V is a knot and therefore orthogonal to gradients. □

Proposition 7.7. *Let $V_1, V_2 \in K(\Omega)$ be curl eigenfields on Ω with equal fluxes. Then there exist corresponding vector potentials B_1 and B_2 which agree on the boundary $\partial\Omega$.*

Proof. Let λ_1 and λ_2 be the respective eigenvalues of V_1 and V_2. From Lemma 7.3, write

$$B_1 = \frac{1}{\lambda_1}V_1 + \nabla\eta_1 - \sum \frac{p_{L_i}(V_1)}{\lambda}W_i \,.$$
$$B_2 = \frac{1}{\lambda_2}V_2 + \nabla\eta_2 - \sum \frac{p_{L_i}(V_2)}{\lambda}W_i \,.$$

Then

$$B_1 - B_2 = \frac{1}{\lambda_1}V_1 - \frac{1}{\lambda_2}V_2 - \sum \frac{p_{L_i}(V_1 - V_2)}{\lambda}W_i - \nabla(\eta_2 - \eta_1) \,.$$

Let $U = B_1 - B_2 + \nabla(\eta_2 - \eta_1) = \frac{1}{\lambda_1}V_1 - \frac{1}{\lambda_2}V_2 - \sum \frac{p_{L_i}(V_1 - V_2)}{\lambda}W_i$. Then it suffices to show that $U = \nabla\psi$ on $\partial\Omega$. Notice that U is a fluid knot. All of its periods are zero:

$$p_{L_i}(U) = p_{L_i}(B_1) - p_{L_i}(B_2) = 0 - 0 = 0$$
$$p_{M_i}(U) = \oint_{M_i} (\frac{1}{\lambda_1}V_1 + \frac{1}{\lambda_2}V_2) \cdot ds = \int_{\Sigma_i} (V_1 + V_2) \cdot \hat{n_i}\, dA = F_i(V_1) - F_i(V_2) = 0,$$

since V_1 and V_2 are assumed to have the same flux values.

Since all periods are zero, the field $U|_{\partial\Omega}$ is conservative. On the boundary it must be a gradient $\nabla|_{\partial\Omega}\psi$, where the subscript denotes that this is a two-dimensional gradient taken on the surface $\partial\Omega$. Extend ψ on a neighborhood of the boundary so that $\dfrac{\partial\psi}{\partial n} = 0$ on $\partial\Omega$, and then extend it to the rest of the domain in a smooth manner. Then $U = \nabla\psi$ on the boundary. So we have shown that by choosing the gradient terms appropriately, B_1 and B_2 will agree on the boundary. □

Proposition 7.8. *Suppose we have two curl eigenfields V_1 and V_2 with corresponding eigenvalues λ_1 and λ_2 that have the same helicity H and flux F. Let B_1 and B_2 be Amperian vector fields such that $\nabla \times B_i = V_i$ that agree on $\partial\Omega$. Then, the energy of their difference is given by*

$$E(V_2 - V_1) = (\lambda_2 + \lambda_1) \int_\Omega (B_2 - B_1) \cdot (\nabla \times B_2)\, dvol$$

Proof. Start with $E(V_2 - V_1) = \langle \nabla \times (B_2 - B_1), \nabla \times (B_2 - B_1) \rangle$.

We use the following identity from vector calculus,

$$C \cdot (\nabla \times D) = D \cdot (\nabla \times C) + \nabla \cdot (D \times C) \tag{7.1}$$

Then,

$$\begin{aligned} E(V_2 - V_1) &= \int_\Omega \nabla \cdot ((B_2 - B_1) \times (\nabla \times (B_2 - B_1)))\, dvol \\ &\qquad + \int_\Omega (B_2 - B_1) \cdot [\nabla \times (\nabla \times (B_2 - B_1))]\, dvol \\ &= \int_{\partial\Omega} ((B_2 - B_1) \times [\nabla \times (B_2 - B_1)]) \cdot \hat{n}\, dA + \langle (B_2 - B_1), (\lambda_2 V_2 - \lambda_1 V_1) \rangle \,. \end{aligned}$$

Since B_1 and B_2 agree on the boundary, the first term vanishes. Then,

$$\begin{aligned} E(V_2 - V_1) &= 0 + \langle (B_2 - B_1), (\lambda_2 V_2 - \lambda_1 V_1) \rangle \\ &= \lambda_2 \langle B_2, V_2 \rangle + \lambda_1 \langle B_1, V_1 \rangle - \lambda_2 \langle B_2, V_1 \rangle - \lambda_1 \langle B_1, V_2 \rangle \ . \end{aligned} \tag{7.2}$$

By substituting the identity (7.1) into the inner product and applying the divergence theorem, we obtain

$$\begin{aligned} \langle B_2, V_1 \rangle &= \int_\Omega B_1 \cdot (\nabla \times B_2) \ dvol + \int_\Omega \nabla \cdot (B_1 \times B_2) \ dvol \\ &= \langle B_2, V_1 \rangle + \int_{\partial\Omega} (B_1 \times B_2) \cdot \hat{n} \ dA \ . \end{aligned}$$

But the second term vanishes since B_1 and B_2 agree on the boundary. Therefore we have $\langle B_2, V_1 \rangle = \langle B_1, V_2 \rangle$.

Proceeding, equation (7.2) becomes

$$E(V_2 - V_1) = (\lambda_2 + \lambda_1) H - (\lambda_2 + \lambda_1) \langle B_1, V_2 \rangle \ .$$

Finally we conclude

$$E(V_2 - V_1) = (\lambda_2 + \lambda_1) \langle (B_2 - B_1), V_2 \rangle \ .$$

□

Remark 7.9. Notice by symmetry that $\langle (B_2 - B_1), V_2 \rangle = \langle (B_1 - B_2), V_1 \rangle$. We will use this fact in proving the next statement.

Proposition 7.10. *With the same assumptions as above, the difference of the energies of V_1 and V_2 is given by*

$$E(V_2) - E(V_1) = (\lambda_2 - \lambda_1) \int_\Omega (B_2 - B_1) \cdot \nabla \times B_2 \ dvol$$

on the torus that winds once around the short way and bounds a surface inside the torus as M (for *meridional*), and a loop on the torus that winds once around the long way and bounds only surfaces outside the domain as L (for *latitudinal*). Recall that for a large class of vector fields, including curl eigenfields, harmonic knots, and gradients, their meridional and latitudinal periods are independent of the choice of loops L and M. So we make specific choices on the flat torus; let M be the loop in the $\hat{\varphi}$ direction for fixed $r = a, z = z_0$ and let L as the loop from $z = 0$ to $z = \ell$ (these two points are identified) for fixed $r = a, \varphi = \varphi_0$.

Proposition 8.1. *Let V be a fluid knot and a curl eigenfield. Then $p_M(V) = \lambda F(V)$.*

Proof. $p_M(V) = \oint_M V \cdot ds = \int_{\Sigma_M} (\nabla \times V) \cdot n \, dA = \int_{\Sigma_M} \lambda V \cdot n \, dA = \lambda F(V)$ □

We begin our analysis by examining the zero eigenvalue case. Let V be a fluid knot which is a curl eigenfield with eigenvalue $\lambda = 0$, i.e., $\nabla \times V = 0$. Then V must be a harmonic knot since it is in the kernel of curl; on the flat solid torus, $V = c\hat{z}$. Then the flux $F(V) = \pi a^2 c$. Recall that helicity measures the coiling and wrapping of field lines; by this definition, the helicity of V is necessarily 0. One can also obtain this by noting that $BS(V)$ must be orthogonal to $\hat{z}$ from its integral formula, so $H(V) = \langle BS(V), V \rangle = 0$.

Theorem 8.2. *For the case of zero helicity, the solution to the Taylor problem corresponds to eigenfields with eigenvalue 0.*

Proof. Let $V = c\hat{z}$ be the eigenfield given above with eigenvalue $\lambda = 0$. Its flux is $F(V) = \pi a^2 c$ and its energy is $E(V) = \pi a^2 \ell c^2$. Suppose a vector field $W \in K(\Omega)$ has flux $F(W) = \pi a^2 c$ and zero helicity. Use the Hodge Theorem to decompose W into a fluxless knot and a harmonic part, $W = W_{FK} + W_{HK}$. Now $F(W) = F(W_{HK})$, which implies $W_{HK} = V$. So

$$E(V) = E(W_{HK}) \leq E(W) = E(W_{FK}) + E(W_{HK}) \ .$$

□

Proposition 8.3. $E(V) = \lambda H(V) + p_L(V)F(V)$.

Proof. This holds for $\lambda = 0$ by the above calculations. For $\lambda \neq 0$, recall Corollary 7.4, $BS'(V) = \dfrac{1}{\lambda}V - \dfrac{p_L(V)}{\lambda\ell}\hat{z}$. Then the helicity

$$H(V) = \langle V, BS'(V)\rangle = \frac{1}{\lambda}\langle V, V\rangle - \frac{p_L(V)}{\lambda}\langle V, \frac{1}{\ell}\hat{z}\rangle = \frac{1}{\lambda}E(V) - \frac{p_L(V)}{\lambda}F(V),$$

since the vector field $\dfrac{1}{\ell}\hat{z}$ is the unique flux-measuring field on $D^2 \times S^1$, i.e., $F(V) = \langle V, \frac{1}{\ell}\hat{z}\rangle$. □

Now we may assume $\lambda \neq 0$. In solving the Woltjer problem, we found all curl eigenfields with $\lambda \neq 0$ that are fluid knots: the Lundquist fields V_0 and fields of the form $V_{m,k}^{\pm i}$ (see eq. 5.1) whose $\hat{r}$ component vanished on the boundary.

$$V_0 = J_1(\lambda r)\hat{\varphi} \ + J_0(\lambda r)\hat{z}$$

$$
\begin{aligned}
V^{+1}_{m,k} = &\left(-\frac{m}{r}J_m(nr) - \frac{mk}{\lambda r}J_m(nr) + \frac{nk}{\lambda}J_{m+1}(nr)\right)\sin(m\varphi + kz)\ \hat{r} \\
&+ \left(-\frac{m}{r}J_m(nr) - \frac{mk}{\lambda r}J_m(nr) + nJ_{m+1}(nr)\right)\cos(m\varphi + kz)\ \hat{\varphi} \\
&+ \frac{n^2}{\lambda}J_m(nr)\cos(m\varphi + kz)\ \hat{z}
\end{aligned} \tag{5.1}
$$

Here $m = 0, 1, 2, ...,\ k = 0, \dfrac{2\pi}{\ell}, \dfrac{4\pi}{\ell}, ...,$ and $n^2 = \lambda^2 - k^2 > 0$. Define the integer $\tilde{k}$ so that $k = \dfrac{2\pi}{\ell}\tilde{k}$. We adopt the convention of using $\tilde{k}$ in all references to the fields $V_{m,k}$.

In order for $V^{\pm i}_{m,k}$ to be tangent at the boundary, the following equation must hold:

$$
\frac{m}{a}J_m(na) = \pm\left(-\frac{mk}{\lambda a}J_m(na) + \frac{nk}{\lambda}J_{m+1}(na)\right)\ . \tag{5.2}
$$

Proposition 8.4. *For the Lundquist fields V_0, all pertinent information for the Taylor problem is easily calculated.*

$$
\begin{aligned}
p_L(V_0) &= \ell J_0(\lambda a) \\
F(V_0) &= \frac{2\pi a}{\lambda}J_1(\lambda a) \\
p_M(V_0) &= 2\pi a J_1(\lambda a) \\
E(V_0) &= 2\pi\ell\left(a^2J_0^2(\lambda a) + a^2J_1^2(\lambda a) - \frac{a}{\lambda}J_0(\lambda a)J_1(\lambda a)\right) \\
H(V_0) &= \frac{2\pi\ell}{\lambda}\left(a^2J_0^2(\lambda a) + a^2J_1^2(\lambda a) - \frac{2a}{\lambda}J_0(\lambda a)J_1(\lambda a)\right)
\end{aligned}
$$

Proposition 8.5. *For the fields $V^i_{m,k}$, for all i, $p_L(V_{m,k}) = 0,\ F(V_{m,k}) = 0$.*

Proposition 8.9. *For fixed m, k values, if $V^{+i}_{m,k}$ has an eigenvalue λ, then $V^{-j}_{m,k}$ has an eigenvalue $-\lambda$, where $i, j \in \{1, 2\}$. The lowest positive eigenvalue λ^+ of $V^{+i}_{m,k}$ is less than the norm of its greatest negative eigenvalue λ^-.*

Proof. By rewriting the boundary condition (5.2) the first statement is obvious

$$\frac{m}{a}J_m(na) = \pm\frac{1}{\lambda}\left(-\frac{mk}{a}J_m(na) + nkJ_{m+1}(na)\right) . \tag{8.2}$$

We obtained λ^+ in Step 3 by defining the function $g_m(x) \equiv \dfrac{J_{m+1}(x)}{mJ_m(x)}$ where $x = na$ and solving $g_m(x) = \dfrac{\lambda^+ + k}{kx} > 0$. This gave us an n^+ value, dependent on (m, k); when $m = 1$, we obtained a minimal n^+ value which was less than $3.832/a$. For λ^-, we are instead solving $g_m(x) = \dfrac{k + \lambda^-}{kx} < 0$. The function $g_m(x)$ does not attain negative values until $x > \zeta_m$, the first zero of $J_m(x)$. Thus the minimal n^- value for this case is larger than $3.832/a$. Since $n^+ > n^-$, we see $\lambda^+ > \lambda^-$. □

We seek linear combinations of fields with the same eigenvalues. Since both λ and $-\lambda$ cannot both be eigenvalues, we can only combine two fields, e.g., $V^1_{m,k}$ and $V^2_{m,k}$. Taking a linear combination of the two only produces a phase angle shift φ_0 from considering one alone, and clearly these fields have the same energy. So we are justified in considering just $V^1_{m,k}$ in our analysis.

Step 5: Compare Eigenvalues. Determine which of the two fields V_0 and W_0, given in Steps 1-3, has the lower eigenvalue. According to Taylor [12](section III),

the eigenvalue for W_0 is approximately $3.11/a$. Then the solution to the Taylor problem is:

Theorem 8.10. (Taylor Problem Solution) *For nonzero helicity, calculate the eigenvalue λ_0 as in Step 1. If $\lambda_0 < 3.11/a$, then the vector field cV_0 is the solution, where the flux can be used to determine the constant c. If $\lambda_0 > 3.11/a$, then we have a "mixed" solution $W = \alpha V_0 + \beta V_{1,k}$, where the constants α and β are determined from the helicity and flux values, and k depends on the radius a of the domain.*

In the case of zero helicity, the Taylor solution is given by the harmonic knot $V = c\hat{z}$, where the flux determines the constant c.

9 Conclusions

First, some bookkeeping. Two results appear in this paper without proof. First, in section 6 we asked is every prescription of helicity and flux values physically realized by some vector field? Proposition 8.6 guarantees this on the solid flat torus. We hypothesize that it is true for arbitrary domains.

Second, in Proposition 8.1, we quoted Taylor's claim that the lowest eigenvalue $\lambda_{m,k}$ has $k \approx 1.25/a$; this identity remains to be proven. Taylor in [12] references his own internal unpublished memo for this result. In the same paper, Taylor also claims that the mixed solution W_0 has eigenvalue $\lambda_0 \approx 3.11/a$. Both values agree with our numerically obtained results.

The vector fields predicted by the solution do indeed exhibit the reversed field pinch found experimentally; see Taylor's paper [12] for certain experimental results. Reversed field pinch occurs in practice for eigenvalues slightly greater than those predicted by theory. Taylor attributes this result to increased plasma resistivity near the boundary walls.

In a literature search, we found no explicit solutions of the Taylor problem on other domains nor any generalizations to arbitrary domains. This question regarding arbitrary domains is likely of interest to both mathematicians and physicists, and it is the obvious continuation of this work.

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