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Nonvariational problems with critical growth

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Abstract

In this paper, we develop new topological methods for handling nonvariational elliptic problems of critical growth. Our primary goal is to demonstrate how concentration compactness can be applied to achieve topological existence theorems in the nonvariational setting. Our methods apply to both semilinear single equations and systems whose nonlinearity is of critical type.

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1. Introduction

In this paper we study nonnegative weak solutions to the semilinear elliptic boundary value problem

$$\begin{aligned} &-\Delta u - \vec{c}\cdot\nabla u = \lambda u + g(x,\lambda)|u|^{2^*-2}u \quad \text{in } \Omega, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 3$, $\vec{c} \in \mathbb{R}^N$ is a constant, λ is a real parameter, g is a bounded continuous function, and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. We also study the related system

$$\begin{aligned} &\mathcal{L}u_1 + \alpha_{11}u_1 + \alpha_{12}u_2 - \lambda(\beta_{11}u_1 + \beta_{12}u_2) - g_1(u_1,u_2) = 0 \\ &\mathcal{L}u_2 + \alpha_{21}u_1 + \alpha_{22}u_2 - \lambda(\beta_{21}u_1 + \beta_{22}u_2) - g_2(u_1,u_2) = 0 \\ &u_1|_{\partial\Omega} = 0 = u_2|_{\partial\Omega}, \end{aligned} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is as above, $\mathcal{L}$ is an elliptic differential operator, λ is a real parameter, and $(g_1, g_2) = g(x,\lambda)\nabla F(u_1,u_2)$, where g is a bounded continuous function, and F is a smooth, strictly positive, 2^*-homogeneous function.

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Critical growth problems have been of intense interest since the notion of concentration compactness was introduced in [9]. Lions' approach was variational in nature and used the concentration-compactness principle to prove the Palais–Smale condition below a certain critical energy level, and thus allowed the application of standard variational theorems such as the Mountain Pass Lemma. Much of the literature since that time has extended this variational argument to different contexts. Eqs. (1.1) and (1.2) are nonvariational and thus cannot be analyzed with the same methods.

Our primary motivation is to investigate new and general methods that may be applied in the nonvariational case. Degree theory is a natural tool to choose in this setting, but it is important to note that the critical growth term effectively prevents the application of Leray–Schauder degree. We apply concentration compactness to show that the (S_+) (see [2] and [12]) condition is satisfied within a certain bound around the trivial branch of solutions $(\lambda, 0)$, and so topological degree is well defined. This allows the application of a well-known bifurcation theorem due to Rabinowitz [11].

The paper is organized as follows. Section 2 contains the existence theorem for the scalar case. The hypotheses in Section 2 are kept simple so that the main ideas can be clearly presented. Section 3 presents the existence theorem for the systems case, and simultaneously demonstrates how the argument in Section 2 can be extended to problems with more general differential operators and more general nonlinear terms. Section 4 is a conclusion which summarizes our work and discusses possible extensions.

2. The scalar case

2.1. Preliminaries

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We denote by $W_0^{1,2}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$. The associated norm is denoted $\|\cdot\|$.

Consider formal differential operators

$$\mathcal{L}u := -\Delta u - \sum_{i=1}^{N} c^i \partial_i u \quad \text{and} \quad \mathcal{L}^* u := -\Delta u + \sum_{i=1}^{N} \partial_i (c^i u),$$

where $\partial_i = \frac{\partial}{\partial x_i}$. These formal operators induce bounded linear operators L and L^*, respectively, acting from $W_0^{1,2}(\Omega)$ into its dual space, defined by

$$\langle Lu, v\rangle = \int_\Omega \left[\nabla u \cdot \nabla v - \sum_{i=1}^{N} c^i \partial_i u v\right] \mathrm{d}x,$$
$$\langle L^* u, v\rangle = \int_\Omega \left[\nabla u \cdot \nabla v - \sum_{i=1}^{N} c^i u \partial_i v\right] \mathrm{d}x$$

for any $u, v \in W_0^{1,2}(\Omega)$. The operator L^* is the adjoint of L. L is self-adjoint, i.e. $L = L^*$, if and only if $c^i = 0$ for all $i = 1, \dots, N$.

Consider the eigenvalue problems

$$Lu = \lambda u \qquad L^* u = \lambda u,$$

on $W_0^{1,2}(\Omega)$ with spectral parameter λ. It follows from the maximum principle [6] and the Krein–Rutman theorem [7,8] that the two spectra have a common real principle eigenvalue $\lambda_1 > 0$, which is simple. The corresponding eigenfunctions φ_1 and φ_1^* can be assumed to be positive in Ω. Moreover, we can normalize φ_1 and φ_1^* so that $\|\varphi_1\| = 1$ and $\int_\Omega \varphi_1 \varphi_1^* \mathrm{d}x = 1$. We fix real numbers $\underline{\lambda}$ and $\overline{\lambda}$ in such a way that the only eigenvalue of L in $[\underline{\lambda}, \overline{\lambda}]$ is λ_1.

For a given $\lambda \in \mathbb{R}$ we say that $u \in W_0^{1,2}(\Omega)$ is a weak solution of (1.1) if, $\forall v \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega (\vec{c} \cdot \nabla u) v \, \mathrm{d}x - \lambda \int_\Omega uv \, \mathrm{d}x - \int_\Omega g(x, \lambda)|u|^{2^*-2} uv \, \mathrm{d}x = 0. \tag{2.1}$$

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Let us define the operators $S, G_\lambda \colon W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ by

$$\langle Su, v\rangle = \int_\Omega uv\mathrm{d}x$$
$$\langle G_\lambda(u), v\rangle = \int_\Omega g(x,\lambda)|u|^{2^*-2}uv\mathrm{d}x,$$

$\forall u, v \in W_0^{1,2}(\Omega)$. Then S is a compact linear operator and G_λ is a continuous operator. These facts follow immediately from the compact embedding $W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ and the continuous embedding $W_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$, respectively. Moreover, we also have

$$\lim_{\|u\|\to 0} \frac{\|G_\lambda(u)\|}{\|u\|} = 0 \tag{2.2}$$

uniformly for $\lambda \in [\underline{\lambda}, \overline{\lambda}]$.

Finding a weak solution of (1.1) is thus equivalent to solving

$$Lu - \lambda Su - G_\lambda(u) = 0. \tag{2.3}$$

By a nontrivial solution to (1.1) we mean a pair $(\lambda, u) \in \mathbb{R}\times W_0^{1,2}(\Omega)$ satisfying (2.3), with $u \neq 0$. The trivial solution is $(\lambda, 0)$, for any $\lambda \in \mathbb{R}$. A point $(\lambda_0, 0) \in \mathbb{R} \times W_0^{1,2}(\Omega)$ is said to be a *bifurcation point* of (1.1) if there exists a sequence of nontrivial solutions $\{(\lambda_n, u_n)\}_{n=1}^\infty$, with $\lambda_n \to \lambda_0$ in $\mathbb{R}$ and $u_n \to 0$ in $W_0^{1,2}(\Omega)$.

Let $\mathcal{C}$ be a set in $\mathbb{R} \times W_0^{1,2}(\Omega)$ consisting of nontrivial solutions to (2.3) which is connected with respect to the topology induced by the norm

$$\|(\lambda, u)\|' = (|\lambda|^2 + \|u\|^2)^{\frac{1}{2}}.$$

Then $\mathcal{C}$ is called a *continuum of nontrivial solutions* of (1.1). If $\mathcal{C}$ is such a continuum and if $(\lambda_0, 0) \in \bar{\mathcal{C}}$, where the closure is taken with respect to the above topology, then $(\lambda_0, 0)$ is a bifurcation point of (2.3), and we say that $\mathcal{C}$ *bifurcates* from $(\lambda_0, 0)$.

2.2. The main theorem

Theorem 2.1. *Eq.* (1.1) *admits a locally compact continuum of nontrivial solutions* $(\lambda, u) \in \mathbb{R}\times W_0^{1,2}(\Omega)$ *bifurcating from* $(\lambda_1, 0)$ *and satisfying the asymptotic estimate*

$$\frac{\lambda - \lambda_1}{\|u\|^{2^*-2}} = -\int_\Omega g(x,\lambda_1)\varphi_1^{2^*-1}(x)\varphi_1^*(x)\mathrm{d}x + o(1) \tag{2.4}$$

as $\|u\| \to 0$. *Also,* $u > 0$ *for* $\|u\|$ *small enough.*

Remark 2.1. Assume that

$$\int_\Omega g(x,\lambda_1)\varphi_1^{2^*-1}(x)\varphi_1^*(x)\mathrm{d}x > 0\,(<0).$$

Then there exists $\lambda^\# < \lambda_1 (\lambda^\# > \lambda_1)$ such that for any λ between λ_1 and $\lambda^\#$ the Eq. (1.1) admits at least one positive solution.

In particular, we have the following result.

Example 2.1. Let us consider the Dirichlet problem

$$\begin{aligned} &-\Delta u - \sum_{i=1}^N \partial_i u = \lambda u + |u|^{2^*-2}u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.5}$$

and let φ_1 and φ_1^* be the principal eigenfunctions of $\Delta u + \sum_{i=1}^N \partial_i u$ and $\Delta u - \sum_{i=1}^N \partial_i u$ subject to the homogeneous Dirichlet boundary conditions, respectively, which correspond to the common eigenvalue λ_1. Then there exists $\lambda^\# < \lambda_1$ such that for all $\lambda \in (\lambda^\#, \lambda_1)$ problem (2.5) has at least one positive weak solution.

2.3. *Proof of Theorem 2.1*

Let $N_\lambda(u) := Lu - \lambda Su - G_\lambda(u)$, $u \in W_0^{1,2}(\Omega)$. Finding weak solutions to Eq. (1.1) is equivalent to solving $N_\lambda(u) = 0$. We show that there is a radius $\rho_0 > 0$ such that the *topological degree*, $\deg(N_\lambda, D, 0)$, is well defined for all open, nonempty sets $D \subset B_{\rho_0}(0) \subset W_0^{1,2}(\Omega)$, such that $N_\lambda(u) \neq 0$ for $u \in \partial D$. It is important to notice that the Leray–Schauder degree is not applicable here, because the operator N_λ cannot be written as a compact perturbation of the identity. For example, in the special case where $\mathcal{L} = -\Delta$ and $g(x, \lambda) \equiv 1$ we find that $L = I$ and it follows directly from the Sobolev Embedding Theorem that G_λ is continuous but not compact. Hence N_λ cannot be written in the form $I + K$ for some compact operator K. Here the degree is understood in the sense of Browder [2] or Skrypnik [12]. For this purpose we have to prove that on such sets D the operator N_λ satisfies an appropriate compactness condition, such as the (S_+) condition from [2].

Definition 2.1. The operator N_λ is said to satisfy condition (S_+) on $D \subset W_0^{1,2}(\Omega)$ if any sequence $\{u_n\}_{n=1}^\infty \subset D$ with $u_n \rightharpoonup u_0$ (weakly) in $W_0^{1,2}(\Omega)$ and

$$\limsup_{n\to\infty} \langle N_\lambda(u_n), u_n - u_0\rangle \leq 0$$

satisfies $u_n \to u$ (strongly) in $W_0^{1,2}(\Omega)$.

To prove the (S_+) condition for N_λ on $B_{\rho_0}(0)$, we take advantage of the Concentration-Compactness Principle (CCP) of Lions [9]. The form below is taken from [5] and the Brézis–Lieb Lemma (see e.g. Willem's book [14]) is applied to change the notation slightly. We denote by S^* the optimal coefficient in the critical Sobolev embedding. That is, $S^* = \inf \int_\Omega |\nabla u|^2 \mathrm{d}x$, where the infimum is taken over all $u \in W_0^{1,2}(\Omega)$ such that $\|u\|_{L^{2^*}(\Omega)} = 1$.

Lemma 2.1. *Let $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $W_0^{1,2}(\Omega)$ with $u_n \rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$. Then there exist nonnegative measures μ and ν on $\bar\Omega$ such that*

$$|\nabla u_n|^2 \rightharpoonup^* |\nabla u_0|^2 + \mu$$
$$|u_n|^{2^*} \rightharpoonup^* |u_0|^{2^*} + \nu$$

both weakly in the space of measures on $\bar\Omega$ with $\nu(\bar\Omega) \leq (S^*)^{-\frac{2^*}{2}} (\mu(\bar\Omega))^{\frac{2^*}{2}}$. In particular, if $\mu = 0$ then we have $u_n \to u_0$ both in $W_0^{1,2}(\Omega)$ and $L^{2^*}(\Omega)$.

Lemma 2.2. *There exists $\rho_0 > 0$ such that for any $\lambda \in [\underline{\lambda}, \bar\lambda]$, the operator N_λ satisfies the (S_+) condition on $B_{\rho_0}(0)$ for all $\rho \in (0, \rho_0]$.*

Proof. Let $\{u_n\}_{n=1}^\infty$ be a sequence in $B_{\rho_0}(0) \subset W_0^{1,2}(\Omega)$ with $u_n \rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$. We will choose ρ_0 later. Then $u_n \rightharpoonup u_0$ in $L^{2^*}(\Omega)$ and $u_n \to u_0$ in $L^2(\Omega)$. Moreover, by Lemma 2.1 we know that $\exists \nu, \mu$ such that $|\nabla u_n|^2 \rightharpoonup^* |\nabla u_0|^2 + \mu$ and $|u_n|^{2^*} \rightharpoonup^* |u_0|^{2^*} + \nu$.

We suppose that u_n satisfies

$$\limsup_{n\to\infty} \langle N_\lambda(u_n), u_n - u_0\rangle \leq 0 \tag{2.6}$$

and consider each piece of the left hand side separately. First,

$$\begin{aligned}\limsup_{n\to\infty} \langle L(u_n), u_n - u_0\rangle &= \limsup_{n\to\infty} \left[\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x - \int_\Omega \nabla u_n \cdot \nabla u_0 \,\mathrm{d}x\right] \\ &= \limsup_{n\to\infty} \left[\int_\Omega |\nabla u_n|^2 \mathrm{d}x - \int_\Omega |\nabla u_0|^2 \mathrm{d}x\right]\end{aligned}$$

$$= \int_\Omega |\nabla u_0|^2 \mathrm{d}x + \mu(\Omega) - \int_\Omega |\nabla u_0|^2 \mathrm{d}x$$
$$= \mu(\Omega).$$

Here we have used the weak convergence of u_n to u_0 in $W_0^{1,2}(\Omega)$ and the CCP (Lemma 2.1). It is straightforward to see that

$$\lim_{n\to\infty} \langle S(u_n), u_n - u_0\rangle = \lim_{n\to\infty} \int_\Omega u_n(u_n - u_0)\mathrm{d}x = 0,$$

because $u_n \to u_0$ in $L^2(\Omega)$. Finally,

$$\begin{aligned}\limsup_{n\to\infty} \langle G_\lambda(u_n), u_n - u_0\rangle &= \limsup_{n\to\infty} \left[\int_\Omega g(x,\lambda)|u_n|^{2^*}\mathrm{d}x - \int_\Omega g(x,\lambda)|u_n|^{2^*-2}u_n u_0 \mathrm{d}x\right] \\ &= \int_\Omega g(x,\lambda)|u_0|^{2^*}\mathrm{d}x + \int_\Omega g(x,\lambda)\mathrm{d}\nu - \int_\Omega g(x,\lambda)|u_0|^{2^*}\mathrm{d}x \\ &= \int_\Omega g(x,\lambda)\mathrm{d}\nu\end{aligned}$$

We have used the fact that $|u_n|^{2^*-2}u_n$ converges pointwise a.e. to $|u_0|^{2^*-2}u_0$, and that this sequence is bounded and, without loss of generality, weakly convergent in $L^{\frac{2N}{N+2}}(\Omega)$. These facts imply that $|u_n|^{2^*-2}u_n \rightharpoonup |u_0|^{2^*-2}u_0$ in $L^{\frac{2N}{N+2}}(\Omega)$. We have also applied Lemma 2.1.

Define $\overline{g} := \sup_{(x,\lambda)\in\Omega\times\mathbb{R}} |g(x,\lambda)|$. It follows from Eq. (2.6) that

$$\begin{aligned}\mu(\Omega) &\le \int_\Omega g(x,\lambda)\mathrm{d}\nu \\ &\le \overline{g}\nu(\Omega) \\ &\le \overline{g}(S^*)^{-\frac{2^*}{2}}(\mu(\Omega))^{\frac{2^*}{2}}.\end{aligned}$$

Thus either $\mu(\Omega) = 0$ or $\mu(\Omega) \ge (\overline{g})^{\frac{2-N}{2}}(S^*)^{\frac{N}{2}}$.

We know from the CCP that

$$\mu(\Omega) \le \limsup_{n\to\infty} \int_\Omega |\nabla u_n|^2 \mathrm{d}x \le \rho_0^2.$$

Hence, if $\{u_n\} \subset B_{\rho_0}(0)$ with $(\rho_0)^2 < (\overline{g})^{\frac{2-N}{2}}(S^*)^{\frac{N}{2}}$, then the only possibility is that $\mu(\Omega) = 0$ and we conclude the strong convergence of $\{u_n\}$ in $W_0^{1,2}(\Omega)$.

In fact, this argument is valid only to verify strong convergence of a subsequence. The fact that the full sequence must converge strongly is easily proved by contradiction. Suppose that for some subsequence $\{u_{n_k}\}_{k=1}^\infty \subset \{u_n\}_{n=1}^\infty$ we have $u_{n_k} \rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$ and

$$\limsup_{k\to\infty} \langle N_\lambda(u_{n_k}), u_{n_k} - u_0\rangle \le 0,$$

but $u_{n_k} \not\to u_0$ in $W_0^{1,2}(\Omega)$. Then, without loss of generality, we have a subsequence (u_{n_k}) such that $\|u_{n_k} - u_0\| \ge \delta$ for some $\delta > 0$ and for all k. Proceeding as above, we can prove that there is a strongly convergent subsequence of u_{n_k} yielding a contradiction. This completes the proof of Lemma 2.2. ■

We next use degree theory to prove the existence of a bifurcation from λ_1.

Proposition 2.1. *Eq.* (1.1) *admits a locally compact continuum $\mathcal{C}$ of nontrivial solutions $(\lambda, u) \in \mathbb{R} \times W_0^{1,2}(\Omega)$ bifurcating from $(\lambda_1, 0)$. The continuum meets the boundary of $[\underline{\lambda}, \bar{\lambda}] \times B_0(\rho_0)$.*

Proof. Let $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ and let

$$\tilde{N}_\lambda(u) := Lu - \lambda Su.$$

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It follows from Lemma 2.2 that the degree $\deg(N_\lambda, B_\rho(0), 0)$ is well defined for any $0 < \rho < \rho_0$ whenever $0 \notin N_\lambda(\partial B_\rho(0))$. It is trivial to check the (S_+) condition for $\tilde{N}$, and $\deg(\tilde{N}_\lambda, B_\rho(0), 0)$ is well defined under the same conditions as above. It follows from the index formula (see [12], Theorem 5.3.) that for $0 < \delta < \min\{\lambda_1 - \underline{\lambda}, \bar{\lambda} - \lambda_1\}$ we have $\deg(\tilde{N}_{\lambda_1-\delta}, B_\rho(0), 0) \neq \deg(\tilde{N}_{\lambda_1+\delta}, B_\rho(0), 0)$. It follows from Eq. (2.2) and the homotopy invariance property of the degree (see [12]) that for any $0 < \delta < \min\{\lambda_1 - \underline{\lambda}, \bar{\lambda} - \lambda_1\}$ there exists $\rho > 0$ such that

$$\deg(N_{\lambda_1-\delta}, B_\rho(0), 0) \neq \deg(N_{\lambda_1+\delta}, B_\rho(0), 0).$$

In other words, the index of the isolated zero of N_λ changes (by magnitude 2) when λ crosses λ_1. At this point we can follow the proof of Theorem 1.3 and Corollary 1.12 in [11], with only minor modifications, to get a continuum $\mathcal{C}$ of nontrivial solutions. In particular, since the operator G_λ is not compact the proof of the local compactness of $\mathcal{C}$ requires the following modification: Suppose that $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ is a sequence of solutions with $\{u_n\} \in B_{\rho_0}(0)$. This means that $N_{\lambda_n}(u_n) = 0$ for all n. Without loss of generality this sequence of solutions is weakly convergent and there exists $\lambda \in \mathbb{R}$ such that $\lambda_n \to \lambda$. The (S_+) condition now implies that $\{u_n\}$ converges strongly, and thus the local compactness of $\mathcal{C}$ follows. ■

We finally consider the asymptotic characterization of the nontrivial solutions near the bifurcation point. Let $(\lambda, u) \in \mathcal{C}$. Then

$$\langle Lu, v\rangle - \lambda\langle Su, v\rangle - \langle G_\lambda(u), v\rangle = 0 \tag{2.7}$$

for any $v \in W_0^{1,2}(\Omega)$. Choosing $v = \varphi_1^*$ and using the fact that

$$\langle Lu, \varphi_1^*\rangle - \lambda_1\langle Su, \varphi_1^*\rangle = \langle L^*\varphi_1^*, u\rangle - \lambda_1\langle S\varphi_1^*, u\rangle = 0,$$

we obtain from (2.7) that

$$(\lambda - \lambda_1)\int_\Omega u\varphi_1^* \mathrm{d}x = -\int_\Omega g(x,\lambda)|u|^{2^*-2}u\varphi_1^* \mathrm{d}x. \tag{2.8}$$

Now, let $w := \frac{u}{\|u\|}$ and $\|u\| \to 0$. Then $(\lambda, u) \in \mathcal{C}$ implies $\lambda \to \lambda_1$ and it follows from (2.2) and (2.7) that $w \to w_0$, and

$$\langle Lw, v\rangle - \lambda\langle Sw, v\rangle - \left\langle \frac{G_\lambda(u)}{\|u\|}, v\right\rangle \to \langle Lw_0, v\rangle - \lambda_1\langle Sw_0, v\rangle,$$

for any $v \in W_0^{1,2}(\Omega)$. Hence $Lw_0 - \lambda_1 Sw_0 = 0$ with $\|w_0\| = 1$. It follows that $w_0 = \pm\varphi_1$. Assume without the loss of generality that $w_0 = \varphi_1$. Dividing (2.8) by $\|u\|^{2^*-1}$ we get

$$\frac{\lambda - \lambda_1}{\|u\|^{2^*-2}}\int_\Omega w\varphi_1^* \mathrm{d}x = -\int_\Omega g(x,\lambda)|w|^{2^*-2}w\varphi_1^* \mathrm{d}x,$$

and letting $\|u\| \to 0$ yields (2.4).

Let $(\lambda, u) \in \mathcal{C}$ be such that $\|u\| \to 0$ and $w := \frac{u}{\|u\|} \to \varphi_1$ in $W_0^{1,2}(\Omega)$. It follows from [6, Theorem 8.12] that $\|w - \varphi_1\|_{W^{2,2}(\Omega)} \to 0$. This fact and the bootstrap argument combined with [6, Theorem 9.15, Lemma 9.17 and Corollary 9.18] yield that $\|w - \varphi_1\|_{W^{2,p}(\Omega)} \to 0$ with arbitrarily large p. Taking $p > \frac{N}{2}$, we have $w \in C^0(\bar{\Omega})$ and according to [6, Theorems 8.33 and 8.34] we get $\|w - \varphi_1\|_{C^{1,\alpha}(\bar{\Omega})} \to 0$. Since $\varphi_1 > 0$ in Ω and $\frac{\partial\varphi_1}{\partial\nu} < 0$ on $\partial\Omega$ by the strong maximum principle [6], we have $w > 0$ (and also $u > 0$) in Ω if $\|u\|$ is small enough.

This completes the proof of the Theorem 2.1.

2.4. *Special case of dimensions* $3 \le N \le 6$

The main results can be strengthened if we apply the bifurcation theorem of Crandall and Rabinowitz [4]. However, there is a price which has to be paid for that: the assumptions of this bifurcation theorem allow us to prove this result only for low dimensions $N = 3, 4, 5$ and 6. Indeed, under the assumptions that $3 \le N \le 6$ and $N_\lambda(u)\colon \mathbb{R} \times W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is a twice differentiable function of λ and u on some neighborhood of the point $(\lambda_1, 0)$, we also have that

(i) $N_\lambda(0)(\lambda, 0) = 0$ for all $\lambda \in (\underline{\lambda}, \bar{\lambda})$,
(ii) $\dim\ker N'_{\lambda_1}(0) = \operatorname{codim}\operatorname{Im} N'_{\lambda_1}(0) = 1$,
(iii) $N''_{\lambda_1}(0)(1, \phi_1) \notin \operatorname{Im} f'_2(\lambda_1, 0)$.

Note that (i) is obvious, (ii) follows from the Fredholm alternative and the simplicity of λ_1 and (iii) follows again from the algebraic simplicity of λ_1. Denote by $\tilde{W}_0^{1,2}(\Omega)$ the set $\{u \in W_0^{1,2}(\Omega)\colon \int_\Omega u\varphi_1 \mathrm{d}x = 0\}$. Then it follows from the Crandall–Rabinowitz theorem that there is $\eta > 0$ and a C^1-curve $(\lambda, \psi) : (\lambda_1 - \eta, \lambda_1 + \eta) \to \mathbb{R} \times \tilde{W}_0^{1,2}(\Omega)$ such that

$$\lambda(0) = \lambda_1, \qquad \psi(0) = 0, \quad N_{\lambda(t)}(t(\phi_1 + \psi(t))).$$

Moreover, there is a neighbourhood $\mathcal{U}$ of $(\lambda_1, 0)$ in $\mathbb{R} \times \tilde{W}_0^{1,2}(\Omega)$ such that

$$N_\lambda(u) = 0 \quad \text{for } (\lambda, u) \in \mathcal{U}$$

if and only if

$$\text{either } u = 0 \quad \text{or} \quad \lambda = \lambda(t), \qquad u = t(\varphi_1 + \psi(t)).$$

In other words, for $3 \le N \le 6$, the continuum of nontrivial solutions $\mathcal{C}$ from Theorem 2.1 is a curve in a sufficiently small neighborhood of the point $(\lambda_1, 0)$.

3. The systems case

In this section we consider the system (1.2) and derive a result similar to the scalar result. We allow for more general hypotheses, but the overall structure of the proof remains the same.

3.1. Preliminaries

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. We will reformulate (1.2) as an operator equation in $H := W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ with inner product

$$\langle \vec{u}, \vec{v} \rangle_H := \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle,$$

where we write $\vec{u} = (u_1, u_2)$. The corresponding norm is denoted by $\|\vec{u}\|_H$.

Define the linear operator $\mathcal{L}$ by

$$\mathcal{L}u = -\sum_{i,j=1}^N \partial_i (a^{ij} \partial_j u) + \sum_{i=1}^N b^i \partial_i u + cu,$$

where $\partial_i := \frac{\partial}{\partial x_i}$. We require the following conditions on $\mathcal{L}$:

($\mathcal{L}$1): a^{ij} is uniformly Lipshitz continuous in Ω for all i, j,
($\mathcal{L}$2): $A := [a^{ij}]$ is a strongly uniformly elliptic matrix, i.e. $\exists \alpha > 0$ such that $\forall x \in \Omega, \xi \in \mathbb{R}^N$, $A\xi \cdot \xi \ge \alpha |\xi|^2$.
($\mathcal{L}$3): b^i is bounded and measurable for all i, and
($\mathcal{L}$4): c is a positive, bounded, and measurable function.

As in the previous section $\mathcal{L}$ induces a bounded linear operator, $L_0 : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$, such that

$$\langle L_0 u, v \rangle = \int_\Omega (A\nabla u \cdot \nabla v)\mathrm{d}x + \int_\Omega \left(\sum_{i=1}^N b^i \partial_i u + cu \right) v\mathrm{d}x \quad \forall u, v \in W_0^{1,2}(\Omega).$$

Let $L : H \to H$ be defined by $L = \begin{pmatrix} L_0 & 0 \\ 0 & L_0 \end{pmatrix}$. It will become clear in the arguments below that (possibly coupled) lower order terms can be added to each equation in the system (1.2). For the sake of simplicity we do not include those additional terms.

Let $M = [\gamma_{ij}]$ such that

(M): $\gamma_{ij} \in C(\overline{\Omega})$ for all i, j, and there exist a real number s and a nonnegative irreducible 2×2 matrix M' with spectral radius $\rho(M') < s$, uniformly for $x \in \Omega$, such that $M = sI - M'$ (i.e. M is an irreducible *M-matrix*).

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We define the compact linear operator $M : H \to H$ by

$$\langle M\vec{u}, \vec{v}\rangle_H = \int_\Omega (M\vec{u}) \cdot \vec{v}\mathrm{d}x \quad \forall \vec{u}, \vec{v} \in H.$$

With the assumptions above it follows from Proposition 3.1 of [13] that the eigenvalue problems

$$L\vec{u} + M\vec{u} = \lambda\vec{u}$$
$$L^*\vec{u} + M^T\vec{u} = \lambda\vec{u}$$

have a common simple principle eigenvalue λ_0 with positive eigenvector. In all that follows we assume that $\lambda_0 > 0$. By Theorem 1.1 in [13] it follows that $L + M$ has a strictly positive inverse satisfying a standard maximum principle. Observe that $(L + M)^* = \begin{pmatrix} L_0^* & 0 \\ 0 & L_0^* \end{pmatrix} + M^T$ satisfies the same conditions as $L + M$.

Let $B = [\beta_{ij}]$ such that

(B) B is a nonnegative nontrivial matrix with $\beta_{ij} \in C(\overline{\Omega})$ for all i, j.

We define the compact linear operator $B : H \to H$ by

$$\langle B\vec{u}, \vec{v}\rangle_H = \int_\Omega (B\vec{u}) \cdot \vec{v}\mathrm{d}x \quad \forall \vec{u}, \vec{v} \in H.$$

With the assumptions above it follows from Theorem 5.1 in [10], which depends on the Krein–Rutman Theorem, that there is a simple positive principle eigenvalue, λ_1, for the weighted eigenvalue problem

$$L\vec{u} + M\vec{u} = \lambda B\vec{u} \quad \vec{u} \in H,$$

where the associated eigenvector, $\vec{\phi}_1$, has positive components that are bounded above and below by positive multiples of the principle eigenfunction associated with L_0. Finally, notice that the adjoint eigenvalue problem

$$L^*\vec{u} + M^T\vec{u} = \lambda B^T\vec{u} \quad \vec{u} \in H,$$

has the same principle eigenvalue, by the Krein–Rutman theorem, and this problem also satisfies $(\mathcal{L}_1)$–$(\mathcal{L}_4)$, (M), (B), so the corresponding eigenfunction $\vec{\phi}_1^*$ has positive components with all of the same properties as $\vec{\phi}_1$. Moreover, the positivity of $B, \vec{\phi}_1$, and $\vec{\phi}_1^*$ implies that $\langle B\vec{\phi}_1, \vec{\phi}_1^*\rangle_H > 0$. In all that follows we assume that the eigenfunction pairs have been normalized so that $\langle\vec{\phi}_1, \vec{\phi}_1\rangle = \langle B\vec{\phi}_1, \vec{\phi}_1^*\rangle_H = 1$.

Finally, we assume

(G): $g(x, \lambda)$ is a bounded, continuous function on $\overline{\Omega} \times \mathbb{R}$, and there is a smooth, strictly positive, 2^*-homogeneous function $F(\vec{u})$ with $|\nabla_u F(\vec{u})| \le |\vec{u}|^{2^*-1}$ such that $g_1(x, \lambda, \vec{u}) = g(x, \lambda)F_1(\vec{u})$, and $g_2(x, \lambda, \vec{u}) = g(x, \lambda)F_2(\vec{u})$, where $F_1(\vec{u})$ and $F_2(\vec{u})$ represent the partial derivatives of F with respect to u_1 and u_2 respectively.

For the purposes of this paper, we could also freely add a strictly subcritical nonlinearity of general type. However, since we are primarily interested in the behavior of the critical nonlinearity we have omitted these extra terms for the sake of brevity. We define the continuous nonlinear operator

$$G_\lambda : H \to H \text{ by } \langle G_\lambda(\vec{u}), \vec{v}\rangle_H = \int_\Omega g(x, \lambda)\nabla F(\vec{u}) \cdot \vec{v}\mathrm{d}x \ \forall \vec{u}, \vec{v} \in H.$$

A weak solution pair $(\lambda, \vec{u}) \in \mathbb{R} \times H$ of (1.2) must satisfy

$$L\vec{u} + M\vec{u} = \lambda B\vec{u} + G_\lambda(\vec{u}),$$

where this statement is equivalent to the usual integral definition. We use the terms *continua* and *bifurcation* exactly as in the scalar case.

3.2. The main theorem

Theorem 3.1. *Assume* $(\mathcal{L}1)$–$(\mathcal{L}4)$, (M), (B), *and* (G). *Also assume that the principle eigenvalue of* $L + M$ *is positive. Then Eq.* (1.2) *admits a locally compact continuum* $\mathcal{C}$ *of nontrivial solutions* $(\lambda, \vec{u}) \in \mathbb{R} \times H$ *bifurcating from* $(\lambda_1, \vec{0})$

and satisfying the asymptotic estimate

$$\frac{\lambda-\lambda_1}{\|\vec{u}\|_H^{2^*-2}} = -\int_\Omega g(x,\lambda_1)\nabla F(\vec{\phi}_1)\cdot\vec{\phi}_1^*(x)\mathrm{d}x + o(1) \tag{3.1}$$

as $\|\vec{u}\|_H \to 0$ *and* $\vec{u} > 0$ *for* $\|\vec{u}\|_H$ *small enough.*

We define the operator

$$N_\lambda(\vec{u}) = L\vec{u} + M\vec{u} - \lambda B\vec{u} - G_\lambda(\vec{u}), \quad \vec{u}\in H.$$

Clearly, for any λ the pair $(\lambda,\vec{0})\in\mathbb{R}\times H$ solves $N_\lambda(\vec{u})=0$. We will use degree theory, as in the scalar case, to find a continuum of solutions that bifurcates from $(\lambda_1,\vec{0})$. In order to prove the (S_+) condition for N_λ we need a variant of the CCP of P.L. Lions for systems. Note that it follows from [1] that an embedding exists from H to the space of functions with norm

$$\left(\int_\Omega F(\vec{u})\mathrm{d}x\right)^{\frac{1}{2^*}}$$

and minimizers exist for this modified critical Sobolev embedding. We define

$$S_F = \inf\int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2)\mathrm{d}x,$$

where the infimum is taken over all $\vec{u}\in H$ such that $\int_\Omega F(\vec{u})\mathrm{d}x = 1$.

Lemma 3.1. *Let* $\{\vec{u}_n\}_{n=1}^\infty$ *be a bounded sequence in* H *with* $\vec{u}_n \rightharpoonup \vec{u}_0$ *in* H. *Then there exist nonnegative measures* μ *and* ν *on* $\bar{\Omega}$ *such that*

$$\left(\sum_{i=1}^2 A\nabla u_{ni}\cdot\nabla u_{ni}\right)\mathrm{d}x \rightharpoonup^* \left(\sum_{i=1}^2 A\nabla u_{0i}\cdot\nabla u_{0i}\right)\mathrm{d}x + \mu, \quad \textit{and}$$
$$F(\vec{u}_n)\mathrm{d}x \rightharpoonup^* F(\vec{u}_0)\mathrm{d}x + \nu$$

weakly in the space of measures on $\bar{\Omega}$. *Moreover,* $\nu(\bar{\Omega}) \le (\alpha S_F)^{-\frac{2^*}{2}}(\mu(\bar{\Omega}))^{\frac{2^*}{2}}$. *Here* α *is the ellipticity constant in condition* ($\mathcal{L}2$). *In particular, if* $\mu=0$ *then we have* $\vec{u}_n\to\vec{u}_0$ *both in* H *and in* $L^{2^*}(\Omega)\times L^{2^*}(\Omega)$.

Proof. Let

$$\nu_n := (F(\vec{u}_n) - F(\vec{u}_0))\mathrm{d}x, \quad \text{and} \quad \mu_n := \left(\sum_{i=1}^2 A\nabla u_{ni}\cdot\nabla u_{ni} - \sum_{i=1}^2 A\nabla u_{0i}\cdot\nabla u_{0i}\right)\mathrm{d}x.$$

It is straightforward to show that for an appropriate subsequence there exists ν such that $\nu_n \rightharpoonup \nu$ and there exists μ such that $\mu_n \rightharpoonup \mu$. Using weak convergence it is easy to check that

$$\mu_n = \sum_{i=1}^2 A\nabla(u_{ni}-u_{0i})\cdot\nabla(u_{ni}-u_{0i})\mathrm{d}x + o(1).$$

By the Brezis–Lieb Lemma, we know that $\nu_n = F(\vec{u}_n - \vec{u}_0)\mathrm{d}x + o(1)$. A proof of the Brezis–Lieb Lemma in this case is given in [1]. For convenience we let $\vec{e}_n := \vec{u}_n - \vec{u}_0$, so

$$\nu_n = F(\vec{e}_n)\mathrm{d}x + o(1), \quad \text{and} \quad \mu_n = \left(\sum_{i=1}^2 A\nabla e_{ni}\cdot\nabla e_{ni}\right)\mathrm{d}x + o(1).$$

Let $\xi\in C_0^\infty(\Omega)$. Then

$$\begin{aligned}\int_\Omega |\xi|^{2^*}\mathrm{d}\nu &= \lim_{n\to\infty}\int_\Omega |\xi|^{2^*}\mathrm{d}\nu_n\\ &= \lim_{n\to\infty}\int_\Omega |\xi|^{2^*}F(\vec{e}_n)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int_\Omega F(|\xi|\vec{e}_n)\mathrm{d}x \\
&\le S_F^{-\frac{2^*}{2}} \liminf_{n\to\infty} \left(\int_\Omega (|\nabla(|\xi|e_{n1})|^2 + |\nabla(|\xi|e_{n2})|^2)\mathrm{d}x \right)^{\frac{2^*}{2}} \\
&= S_F^{-\frac{2^*}{2}} \liminf_{n\to\infty} \left(\int_\Omega (|\xi|^2(|\nabla e_{n1}|^2 + |\nabla e_{n2}|^2)\mathrm{d}x) + o(1) \right)^{\frac{2^*}{2}} \\
&\le S_F^{-\frac{2^*}{2}} \liminf_{n\to\infty} \left(\alpha^{-1} \int_\Omega |\xi|^2 \sum_{i=1}^{2} (A\nabla e_{ni} \cdot \nabla e_{ni})\mathrm{d}x \right)^{\frac{2^*}{2}} \\
&\le S_F^{-\frac{2^*}{2}} \left(\alpha^{-1} \int_\Omega |\xi|^2 \mathrm{d}\mu \right)^{\frac{2^*}{2}}.
\end{aligned}
$$

Here we have used the following facts: F is homogeneous of degree 2^*; the usual critical Sobolev Embedding Theorem applies for functions of degree 2^* meeting condition (G); e_{ni} converges strongly in $L^2(\Omega)$ for $i = 1, 2$ and weakly in $W_0^{1,2}(\Omega)$ (to eliminate lower order terms in the gradient of $|\xi|e_{ni}$); and finally, $A = [a^{ij}]$ is uniformly strongly elliptic with ellipticity constant α.

An appropriate choice of the sequence $\{\xi_n\}$ substituted into the previous inequality leads to $\nu(\bar{\Omega}) \le (\alpha S_F)^{-\frac{2^*}{2}} (\mu(\bar{\Omega}))^{\frac{2^*}{2}}$.

If $\mu = 0$, then $\|\vec{u}_n - \vec{u}_0\|_H^2 \le \frac{1}{\alpha}[\mu_n + o(1)]$, so $\|\vec{u}_n - \vec{u}_0\|_H^2 \to 0$, and we get strong convergence in both H and, by continuous embedding, $L^{2^*}(\Omega) \times L^{2^*}(\Omega)$. ■

Once again, define $\underline{\lambda}$ and $\overline{\lambda}$ such that λ_1 is the only eigenvalue of $L + M$ weighted with respect to B in the interval $[\underline{\lambda}, \overline{\lambda}]$. We now prove that, there exists $\rho_0 > 0$ such that, $\forall\lambda \in [\underline{\lambda}, \overline{\lambda}]$, the operator $N_\lambda = L + M - \lambda B - G_\lambda$ satisfies the (S_+) condition on $B_{\rho_0}(0)$.

Lemma 3.2. *There exists $\rho_0 > 0$ such that $\forall\lambda \in [\underline{\lambda}, \overline{\lambda}]$ the operator N_λ as defined above satisfies the (S_+) condition as an operator from $B_{\rho_0}(0) \subset H$ to H.*

Proof. Let $\{\vec{u}_n\}$ be a sequence in $B_{\rho_0}(0)$, with ρ_0 to be chosen later. Suppose that $\vec{u}_n \rightharpoonup \vec{u}_0$ in H and that

$$\limsup_{m\to\infty} \langle N_\lambda(\vec{u}_n), \vec{u}_n - \vec{u}_0\rangle_H \le 0.$$

By passing to an appropriate subsequence we may assume that $\vec{u}_n \to \vec{u}_0$ in $L^2(\Omega) \times L^2(\Omega)$, with convergence pointwise a.e. in Ω, and $\vec{u}_n \rightharpoonup \vec{u}_0$ in $L^{2^*}(\Omega) \times L^{2^*}(\Omega)$. We need to show that $\vec{u}_n \to \vec{u}_0$ in H. Using the L^2 convergence mentioned above it is clear that the terms $\langle M\vec{u}_n, \vec{u}_n - \vec{u}_0\rangle_H$ and $\langle B\vec{u}_n, \vec{u}_n - \vec{u}_0\rangle_H$ vanish in the limit. Using weak convergence, it is clear that the lower order terms in $\langle L\vec{u}_n, \vec{u}_n - \vec{u}_0\rangle_H$ vanish in the limit leaving

$$\langle L\vec{u}_n, \vec{u}_n - \vec{u}_0\rangle_H = \int_\Omega \sum_{i=1}^{2} (A\nabla u_{ni} \cdot \nabla u_{ni} - A\nabla u_{0i} \cdot \nabla u_{0i})\,\mathrm{d}x + o(1).$$

Notice that

$$
\begin{aligned}
\langle G_\lambda(\vec{u}_n), \vec{u}_n - \vec{u}_0\rangle &= \int_\Omega g(x,\lambda)\nabla F(\vec{u}_n) \cdot (\vec{u}_n - \vec{u}_0)\mathrm{d}x \\
&= \int_\Omega g(x,\lambda)(\nabla F(\vec{u}_n) \cdot \vec{u}_n - \nabla F(\vec{u}_n) \cdot \vec{u}_0)\mathrm{d}x, \\
&= \int_\Omega g(x,\lambda)(\nabla F(\vec{u}_n) \cdot \vec{u}_n - \nabla F(\vec{u}_0) \cdot \vec{u}_0)\mathrm{d}x + o(1)
\end{aligned}
$$

because $\nabla F(\vec{u}_n)$ converges weakly to $\nabla F(\vec{u}_0)$ in $L^{2^{*\prime}} \times L^{2^{*\prime}}$ by the weak convergence of $\vec{u}_n$ in $L^{2^*} \times L^{2^*}$, the bound on the derivatives of F, and pointwise-almost-everywhere convergence of $\vec{u}_n$ to $\vec{u}_0$. Also, by the 2^*-homogeneity of

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F, $\nabla F(\vec{u})\vec{u} = 2^* F(\vec{u})$. This is the only place in the proof where we are using the fact that our nonlinearity is of gradient type. Hence

$$\langle G_\lambda(\vec{u}_n), \vec{u}_n - \vec{u}_0 \rangle = 2^* \int_\Omega g(x,\lambda)(F(\vec{u}_n) - F(\vec{u}_0))\mathrm{d}x + o(1).$$

Now, define ν_n and μ_n as in Lemma 3.1. We then find that there exists ν such that $\nu_n \rightharpoonup \nu$ and there exists μ such that $\mu_n \rightharpoonup \mu$, and

$$\lim_{n\to\infty} \langle L\vec{u}_n, \vec{u}_n - \vec{u}_0 \rangle_H = \mu(\Omega),$$

and

$$\lim_{n\to\infty} \langle G_\lambda(\vec{u}_n), \vec{u}_n - \vec{u}_0 \rangle_H = \int_\Omega g(x,\lambda)\mathrm{d}\nu.$$

The remainder of the proof follows precisely as in the scalar case with (αS_F) in place of S^*. ■

Now that an appropriate concentration-compactness result is in place, and the (S_+) condition has been established, the remainder of the existence proof and the proof of the asymptotic estimate follows precisely as in the scalar case. Note that the normalization $\langle B\vec{\varphi}_1, \vec{\varphi}_1^* \rangle = 1$ has been chosen because this quantity appears in the asymptotic estimate where $\langle \varphi_1, \varphi_1^* \rangle$ appears in the scalar case. Therefore this choice of normalization produces the asymptotic estimate as claimed.

4. Conclusion

The main purpose of this work has been to demonstrate that the method of concentration compactness can be applied to nonvariational problems, through the lens of topological degree. Although topological degree has been generally used to obtain existence of similar problems in Hölder spaces, these spaces require Schauder estimates to hold on the linear operator and they require strong smoothness of the function $g(x,\lambda)$ in the nonlinearity. Our methods prove existence of solutions to a wide range of nonvariational problems with rough coefficients and critical growth that were previously untreatable. In this direction, note also that the regularity we have required on (a_{ij}) is necessary only to obtain pointwise positivity of our solutions; existence and the asymptotic estimate hold for merely bounded, measurable coefficients.

Several generalizations of our results are immediate:

1. Lower order terms, both subcritical nonlinearities and first-order linear terms may be added to the equation without difficulty. These terms do not affect whether the operator satisfies the (S_+) condition. These lower order terms may be quite general in form.
2. Systems of n equations with the same conditions on M and B can be treated as in Section 3.
3. Other kinds of matrices can be handled so long as the corresponding linear problem satisfies the conditions of the Krein–Rutman Theorem, with positive first eigenvalue and vector of eigenfunctions. The class of matrices mentioned above is a large class meeting this description, but it is certainly not comprehensive.

The authors intend to continue this study by examining the case of nongradient nonlinearities and quasilinear operators, although the scalar equation involving the p-Laplacian was recently studied in [3]. Each of these generalizations poses its own difficulties, and will require the development of additional tools.

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