EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS TO SOME SEMIPOSITONE SYSTEMS

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ABSTRACT. In this paper we use the method of upper and lower solutions combined with degree theoretic techniques to prove the existence of multiple positive solutions to semipositone superlinear systems of the form

$$-\Delta u = g_1(x,u,v)$$
$$-\Delta v = g_2(x,u,v)$$

on a smooth, bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions, under suitable conditions on g_1 and g_2. Our techniques apply generally to subcritical, superlinear problems with a certain concave-convex shape to their nonlinearity.

1. INTRODUCTION

In this paper we study the multiplicity of solutions to an elliptic problem of the form

$$\begin{cases} -\Delta u &= g_1(x,u,v) \qquad \forall x \in \Omega \\ -\Delta v &= g_2(x,u,v) \qquad \forall x \in \Omega \\ u, v &> 0 \qquad\qquad\quad\ \forall x \in \Omega \\ u = v &= 0 \qquad\qquad\quad\ \forall x \in \partial\Omega \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $g_i(x,u,v)$: $\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ are differentiable functions subject to further restrictions to be named below. We are motivated in this work by the case of semipositone nonlinearities, for which there is a positive constant γ such that for $i = 1, 2$,

(H1): $g_i(x,0,0) \leq -\gamma < 0 \; \forall x \in \overline{\Omega}$.

However, our work requires no sign condition on the nonlinearity. In addition, we assume that $g_i(x,u,v)$ satisfies suitable conditions on a

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bounded rectangle so that a positive strict lower solution pair and a positive strict upper solution pair can be constructed. These assumptions suffice to prove the existence of at least one positive solution.

For example, in Chhetri-Robinson [6], the authors prove the existence of a positive solution for a single equation analogous to (1) by constructing an ordered pair of lower and upper solutions. There, the nonlinearity satisfies (H1) and additional conditions. Further related references may be found therein.

To obtain the second solution, we assume that for $i, j = 1, 2$ there exist constants q_{ij} with $0 \leq q_{ij} < 2^* - 1 = \frac{n+2}{n-2}$ for $n > 2$ or $0 \leq q_{ij}$ for $n = 2$ so that the following holds: there are continuous functions $h_{ij}(x)$ on $\overline{\Omega}$ which are strictly positive in Ω such that

$$\textbf{(H2):}\ g_i(x, u, v) = h_{i1}(x)u^{q_{i1}} + h_{i2}(x)v^{q_{i2}} + r_i(x, u, v)$$

with $|r_i(x, u, v)| \leq C(1 + |u|^{\beta_{i1}} + |v|^{\beta_{i2}})$ and $\beta_{ij} < q_{ij}$ for $i, j = 1, 2$. (If $q_{ij} = 0$, β_{ij} may also be 0.)

Sun-Wu-Long [24] obtained multiple positive solutions for the single equation case when the nonlinearity is of the form $\lambda u^\beta + p(x)u^{-\alpha}$ with $0 < \alpha < 1 < \beta < 2^* - 1$. The interesting feature about such a nonlinearity is that it exhibits concave-convex type behavior, thus making it possible to obtain Ambrosetti-Brezis-Cerami [2] types of results. Our nonlinearities roughly exhibit this concave-convex behavior in the sense that there is a finite rectangle on which they are bounded above by an upper shelf, whereas for large values of u and v they are superlinear. Hence the two solution conclusion that we obtain, with one solution in the concave region and one solution reaching into the convex region, is not unexpected.

The novelty of this paper lies in the fact that we are able to get not just one positive solution to a superlinear problem, but also a second positive solution. To the best of our knowledge, this paper is the first to deal with multiple positive solutions of general semipositone superlinear problems, even for single equation. It is also important to note that our methods allow a large class of differential operators and nonlinear forcing terms. In particular, our proofs find both positive solutions using degree-theoretic arguments, rather than a mountain-pass argument. Thus, our results generalize to nonvariational elliptic operators. They also generalize easily to larger systems.

This paper is organized as follows: In Section 2, we state and prove a general theorem about the existence of two solutions. In section 3, we prove the existence of positive solutions to an auxiliary problem. These solutions will become the lower solutions for our main theorem. In Section 4, we state and prove an existence theorem in the weakly coupled quasimonotone nondecreasing case, and provide two examples of nonlinearities that satisfy the hypotheses of the theorem. In section 5, we state and prove a theorem that deals with a quasimonotone nondecreasing Hamiltonian (and thus strongly coupled) system and provide an example satisfying the hypotheses of the theorem.

2. A General Two-Solution Existence Theorem

Using the method of upper and lower solutions, an a priori bound, and some degree theory, we formulate very general conditions under which two solutions will be guaranteed to exist for an elliptic superlinear problem. This is phrased in terms of a 2×2 system of equations for simplicity of exposition, but the technique also applies to single equations or larger systems. Throughout this paper we will use the vector notation $\vec{u} = (u, v)$ to indicate a pair of functions being considered in the first and second equations of the system respectively.

First we must define some terms for the general types of systems that we may encounter. The following are taken from Pao [Section 8.4, Page 402][19].

Definition 1. *The nonlinearities $g_i(x, u, v)$ are called* **quasimonotone nondecreasing** *if $\forall x, u, v$, $\frac{\partial g_1}{\partial v} \geq 0$ and $\frac{\partial g_2}{\partial u} \geq 0$. In this case, two pairs of functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are called an ordered lower-upper solution pair of (1) if $\underline{u}(x) \leq \overline{u}(x)$ and $\underline{v}(x) \leq \overline{v}(x)$ $\forall x \in \overline{\Omega}$, $\underline{u}(x) = 0$ and $\underline{v}(x) = 0$ on $\partial\Omega$, $\overline{u}(x) \geq 0$ and $\overline{v}(x) \geq 0$ on $\partial\Omega$ and*

$$\begin{cases} -\Delta \underline{u} & \leq g_1(x, \underline{u}, \underline{v}) \qquad \forall x \in \Omega \\ -\Delta \underline{v} & \leq g_2(x, \underline{u}, \underline{v}) \qquad \forall x \in \Omega \\ -\Delta \overline{u} & \leq g_1(x, \overline{u}, \overline{v}) \qquad \forall x \in \Omega \\ -\Delta \overline{v} & \leq g_2(x, \overline{u}, \overline{v}) \qquad \forall x \in \Omega \end{cases} \tag{2}$$

Definition 2. *The nonlinearities $g_i(x, u, v)$ are called* **quasimonotone nonincreasing** *if $\forall x, u, v$, $\frac{\partial g_1}{\partial v} \leq 0$ and $\frac{\partial g_2}{\partial u} \leq 0$. In this case, two pairs of functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are called an ordered lower-upper solution pair of (1) if $\underline{u}(x) \leq \overline{u}(x)$ and $\underline{v}(x) \leq \overline{v}(x)$ $\forall x \in \overline{\Omega}$, $\underline{u}(x) = 0$ and*

$\underline{v}(x) = 0$ *on* $\partial\Omega$, $\overline{u}(x) \geq 0$ *and* $\overline{v}(x) \geq 0$ *on* $\partial\Omega$ *and*

$$\left\{ \begin{array}{lll} -\Delta \underline{u} & \leq g_1(x, \underline{u}, \overline{v}) & \forall x \in \Omega \\ -\Delta \underline{v} & \leq g_2(x, \overline{u}, \underline{v}) & \forall x \in \Omega \\ -\Delta \overline{u} & \leq g_1(x, \overline{u}, \underline{v}) & \forall x \in \Omega \\ -\Delta \overline{v} & \leq g_2(x, \underline{u}, \overline{v}) & \forall x \in \Omega \end{array} \right. \tag{3}$$

Definition 3. *The nonlinearities* $g_i(x, u, v)$ *are called* **quasimonotone mixed** *if* $\forall x, u, v$, $\frac{\partial g_1}{\partial v} \leq 0$ *and* $\frac{\partial g_2}{\partial u} \geq 0$. *(If the opposite holds we switch* u *and* v*.) In this case, two pairs of functions* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *are called an ordered lower-upper solution pair of* (1) *if* $\underline{u}(x) \leq \overline{u}(x)$ *and* $\underline{v}(x) \leq \overline{v}(x)$ $\forall x \in \overline{\Omega}$, $\underline{u}(x) = 0$ *and* $\underline{v}(x) = 0$ *on* $\partial\Omega$, $\overline{u}(x) \geq 0$ *and* $\overline{v}(x) \geq 0$ *on* $\partial\Omega$ *and*

$$\left\{ \begin{array}{lll} -\Delta \underline{u} & \leq g_1(x, \underline{u}, \overline{v}) & \forall x \in \Omega \\ -\Delta \underline{v} & \leq g_2(x, \underline{u}, \underline{v}) & \forall x \in \Omega \\ -\Delta \overline{u} & \leq g_1(x, \overline{u}, \underline{v}) & \forall x \in \Omega \\ -\Delta \overline{v} & \leq g_2(x, \overline{u}, \overline{v}) & \forall x \in \Omega \end{array} \right. \tag{4}$$

If any of the three cases above hold, we say that the nonlinearities are **quasimonotone**.

The following definitions are taken from de-Figueiredo [8].

Definition 4. *The system* (1) *under condition (H2) is* **weakly coupled** *if there are positive numbers* c_1 *and* c_2 *such that*

$$c_1 + 2 - c_1 q_{11} = 0, \qquad c_1 + 2 - c_2 q_{12} > 0,$$
$$c_2 + 2 - c_1 q_{21} > 0, \qquad c_2 + 2 - c_2 q_{22} = 0.$$

Definition 5. *The system* (1) *under condition (H2) is* **strongly coupled** *if there are positive numbers* c_1 *and* c_2 *such that*

$$c_1 + 2 - c_1 q_{11} > 0, \qquad c_1 + 2 - c_2 q_{12} = 0,$$
$$c_2 + 2 - c_1 q_{21} = 0, \qquad c_2 + 2 - c_2 q_{22} > 0.$$

Now we state our general two solution theorem.

Theorem 1. *Suppose that* $g_i(x, u, v)$ *for* $i = 1, 2$ *satisfy (H1) and (H2) and the following:*

(C1) $g_i(x, u, v)$ *are quasimonotone,*

(C2) *there exists a strictly positive ordered upper-lower solution pair according to the definition corresponding to the nonlinearities' quasimonotone type, and either*

(C3a) *the system is weakly coupled with* $q_{11} > 1$ *and* $q_{22} > 1$*; or*

(C3b) *the system is strongly coupled with $q_{12} > 1$ and $q_{21} > 1$, and an appropriate a priori bound holds for the system and a simple homotopy from the given system.*

Then (1) *has at least two solutions.*

We will prove this theorem via several lemmata. Using the method of upper and lower solutions we will find a first solution and show that the degree of an operator corresponding to (1) is one on a corresponding set. We will then prove (or assume) that the problem satisfies an a priori bound. This will allow us to show that the degree on a larger set is 0 and conclude that there is a second solution.

In order to establish the existence of solutions we will need to represent the boundary value problem (1) as an operator equation in the proper form and then perform a Leray-Schauder degree computation. Similar arguments for single equations can be found in many references. See Amann [1] or Shivaji [23] for details.

In order to work in the appropriate function space setting we consider the auxiliary problem

$$\begin{cases} -\Delta z &= 1 \qquad \forall x \in \Omega, \\ z &= 0 \qquad \forall x \in \partial\Omega. \end{cases} \tag{5}$$

By the Hopf maximum principle we know that z is strictly positive in Ω and that $|\frac{\partial z}{\partial \nu}| > 0$ on $\partial\Omega$ where ν represents the unit outward normal on the boundary. Let

$$C_z(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : -tz \le u \le tz \text{ in } \overline{\Omega} \text{ for some } t > 0\};$$

and let $||u||_z := \inf\{t > 0 : -tz \le u \le tz\}$. Define $X := C_z(\overline{\Omega}) \times C_z(\overline{\Omega})$. Notice that the rectangle $W := \{\vec{u} \in X : \underline{u}(x) < u(x) < \bar{u}(x), \underline{v}(x) < v(x) < \bar{v}(x)\} = (\underline{u}, \underline{v}) \times (\overline{u}, \overline{v})$ is open in the X topology.

Now let $F\vec{u} := I\vec{u} - L^{-1}N(\vec{u})$ for $\vec{u} \in X$, where I is the identity, L^{-1} is the inverse of $L := (-\Delta, -\Delta)$, and $N(\vec{u}) = (g_1(x, \vec{u}), g_2(x, \vec{u}))$. The standard arguments applied to elliptic operators and substitution operators show that F is a compact perturbation of the identity, and so it is valid to discuss Leray-Schauder degree computations for F.

Lemma 1. *If g_1 and g_2 are quasimonotone, then problem* (1) *has a solution in the set W. Moreover, $deg(F, 0, W) = 1$.*

Proof. For simplicity, we provide full details only for the case when the nonlinearities are quasimonotone nondecreasing. We indicate how to modify the proof in the other two cases below.

The first goal is to transform the problem (1) to one with helpful monotonicity properties. Choose $t > 0$ such that $\frac{\partial g_1}{\partial u}(x,u,v) \geq -t$ and $\frac{\partial g_2}{\partial v}(x,u,v) \geq -t$ for $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ and $\underline{v}(x) \leq v(x) \leq \overline{v}(x)$. Let $p_t(x,u,v) := g_1(x,u,v) + tu$ and $q_t(x,u,v) = g_2(x,u,v) + tv$. Let $L_t := -\Delta + t$. The problem (1) can be rewritten as

$$\begin{cases} L_t u = p_t(x,u,v) & \forall x \in \Omega, \\ L_t v = q_t(x,u,v) & \forall x \in \Omega, \\ u = 0 & \forall x \in \partial\Omega, \\ v = 0 & \forall x \in \partial\Omega, \end{cases} \tag{6}$$

where L_t satisfies the standard maximum principle for linear elliptic operators, and p_t, q_t are monotone in both variables. Moreover, it is easy to check that $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are lower/upper solution pairs for (6).

The second goal is to restrict the problem so that solutions cannot occur outside of the rectangle W. For a given function $u(x)$, let

$$\tilde{u}(x) := \begin{cases} \underline{u}(x) & \text{if} \quad u(x) \leq \underline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) < u(x) < \bar{u}(x) \\ \bar{u}(x) & \text{if} \quad \bar{u}(x) \leq u(x) \end{cases}$$

Define $\tilde{v}(x)$ similarly. Define the substitution operators $\tilde{p}_t(u(x), v(x)) := p_t(x, \tilde{u}(x), \tilde{v}(x))$, and $\tilde{q}_t(u(x), v(x)) := q_t(x, \tilde{u}(x), \tilde{v}(x))$. We can now state a modified boundary value problem that has useful properties of monotonicity and boundedness.

$$\begin{cases} L_t u = \tilde{p}_t(u,v) & \forall x \in \Omega, \\ L_t v = \tilde{q}_t(u,v) & \forall x \in \Omega, \\ u = 0 & \forall x \in \partial\Omega, \\ v = 0 & \forall x \in \partial\Omega. \end{cases} \tag{7}$$

The third goal is to do a degree computation for (7), and then relate that computation back to the original boundary value problem. The modified boundary value problem can be represented as an operator equation of the form $\tilde{F}_t(\vec{u}) := I\vec{u} - \vec{L}_t^{-1}\tilde{N}_t(\vec{u}) = 0$ on the space X, where $\vec{L}_t^{-1}$ is the inverse of $\vec{L}_t := (L_t, L_t)$, and $\tilde{N}_t(\vec{u}) = (\tilde{p}(x,\vec{u}), \tilde{q}(x,\vec{u}))$. Since $\tilde{N}_t$ is bounded, and every solution of $\tilde{F}_t(\vec{u}) = 0$ satisfies $||\vec{u}|| = ||\vec{L}_t^{-1}\tilde{N}_t(\vec{u})|| \leq ||\vec{L}_t^{-1}||||\tilde{N}_t(\vec{u})||$, it is straightforward to obtain an a priori bound on solutions. If we then select any $R > 0$ larger than the a priori bound and consider the homotopy $h(\lambda, \vec{u}) = I\vec{u} - \lambda\vec{L}_t^{-1}\tilde{N}_t(\vec{u})$ for $\lambda \in [0,1]$ we see that $deg(\tilde{F}_t, 0, B_R(0)) = deg(I, 0, B_R(0)) = 1$.

It follows from the previous argument that (7) has at least one solution in $(u, v) \in B_R(0)$. Observe that

$$L_t u(x) = \tilde{p}_t(u(x), v(x)) \leq \tilde{p}_t(\bar{u}(x), \bar{v}(x)) < L_t \bar{u}(x). \tag{8}$$

By the maximum principle this implies that $u(x) < \bar{u}(x)$ in Ω. Similar arguments show that $\underline{u}(x) < u(x)$ and $\underline{v}(x) < v(x) < \bar{v}(x)$. It follows that all solutions of (7) are also solutions of (6) and thus of (1). Moreover, these solutions must lie strictly between the upper and lower solution pairs, and hence in W. We can now say that $deg(\tilde{F}_t, 0, B_R(0)) = deg(\tilde{F}_t, 0, W) = deg(F_t, 0, W) = 1$, where $N_t(\vec{u}) = (p_t(\vec{u}), q_t(\vec{u}))$ and $F_t := I - \vec{L}_t^{-1} N_t$.

Finally, we consider t to be a homotopy parameter and let $t \to 0$ so that $F_t \to F$. It is clear that the solutions to (6) in $\overline{W}$ do not change as t changes, so there are no solutions on ∂W for any t. Hence, degree is preserved along the homotopy and we get $deg(F, 0, W) = 1$.

The cases where g_1 and g_2 satisfy either Definition 2 or Definition 3 can be handled in a similar way. For example, if g_1 and g_2 are quasi-monotone nonincreasing, and if $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are lower and upper solution pairs as described in Definition 2, then we can modify g_1 exactly as before, and $p_t(x, u, v)$ will then be nondecreasing in u and nonincreasing in v. It is then straightforward to apply this monotonicity and the assumptions in Definition 2 to get the analog to (8), i.e.

$$L_t u = \tilde{p}_t(u, v) \leq \tilde{p}_t(\bar{u}(x), \underline{\mathrm{v}}(x)) < L_t \bar{u}(x).$$

Other comparisons follow similarly. $\square$

In order to obtain a second solution, we will do a second degree computation on a similar set, $(\underline{u}, T) \times (\underline{v}, T)$, where T is an a priori bound on the solutions of (1).

In the weakly coupled case, the existence of such an L^∞ a priori bound on solutions of (1) will follow directly from the blowup method first developed by Gidas-Spruck [11] for the scalar case and will depend on the condition (H2), and in particular the superlinearity and subcriticality of g_i.

In the strongly coupled case, we must simply assume that an appropriate a priori bound holds. We can follow the blow-up method exactly if an appropriate Liouville Theorem holds. That is, we must know that there are no nontrivial nonnegative solutions to the blown-up system

$$\begin{cases} -\Delta u &= h_{12}(x_0)v^{q_{12}} \qquad \forall x \in \mathbb{R}^n \\ -\Delta v &= h_{21}(x_0)u^{q_{21}} \qquad \forall x \in \mathbb{R}^n \end{cases} \tag{9}$$

for some $x_0 \in \overline{\Omega}$.

If an a priori bound holds by another technique, that may also be used.

We quote the treatment for a 2×2 system in de Figueiredo [8]:

Proposition 1 (Theorems 2.1 and 2.2 [8])**.** *Suppose that* (1) *satisfies condition* $(H2)$*. Suppose also that the system is weakly coupled, or it is strongly coupled and the only nonnegative solution to* (9) *is* $(0,0)$*. Then there exists some* $T > 0$ *such that every nonnegative solution* (u, v) *to* (1) *satisfies* $\|u\|_{L^\infty} < T$ *and* $\|v\|_{L^\infty} < T$*.*

First, let us discuss the weakly coupled case. It is clear that under our conditions $deg(F, 0, (\underline{u}, T) \times (\underline{v}, T))$ is well defined. To compute this degree we use homotopy invariance.

Let λ_1 be the first eigenvalue of $(-\Delta)$ on Ω. If the system is weakly coupled and condition $(C3a)$ holds, there exists some $R_0 > 0$ such that $\forall x \in \overline{\Omega}$

$$\begin{cases} g_1(x, u, v) > \lambda_1 u + 1 \qquad \forall v \geq 0 \text{ and } u > R_0 \\ g_2(x, u, v) > \lambda_1 v + 1 \qquad \forall u \geq 0 \text{ and } v > R_0. \end{cases}$$

Let

$$m_1(x, u, v) := \begin{cases} g_1(x, u, v) & u \geq R_0 \\ \max\{\lambda_1 u + 1, g_1(x, u, v)\} & 0 \leq u < R_0, \end{cases}$$

and similarly

$$m_2(x, u, v) := \begin{cases} g_2(x, u, v) & v \geq R_0 \\ \max\{\lambda v + 1, g_2(x, u, v)\} & 0 \leq v < R_0, \end{cases}$$

and $\vec{m}(x, \vec{u}) := (m_1(x, u, v), m_2(x, u, v))$. Let

$$\vec{p_t}(x, \vec{u}) = (1 - t)\vec{g}(x, \vec{u}) + t\vec{m}(x, \vec{u}),$$

where $\vec{g} = (g_1, g_2)$. We proceed to study the homotopy class of problems

$$\begin{cases} -\Delta \vec{u} &= \vec{p_t}(x, \vec{u}) \qquad \forall x \in \Omega \\ \vec{u} &= \vec{0} \qquad \forall x \in \partial\Omega. \end{cases} \tag{10}$$

Observe that p_t increases as t increases, so

$$-\Delta \underline{u} \leq g_1(x, \underline{u}, \underline{v}) \leq p_t(x, \underline{u}, \underline{v})$$

for each t.

Lemma 2. *There is a $T > 0$ such that any solution triple (t, u, v) of (10) satisfies $\|u\|_{L^\infty} + \|v\|_{L^\infty} < T$.*

Proof. To prove this lemma, we will apply Proposition 1. Notice that, for all $t \in [0, 1]$, $m_1(x, u, v) = g_1(x, u, v)$ for u sufficiently large, and $m_2(x, u, v) = g_2(x, u, v)$ for v sufficiently large. Therefore, $\vec{p_t}(x, \vec{u})$ satisfies condition $(H2)$ with the same q_{ij} and $h_{ij}(x)$. However, we may now have a modified remainder term $\tilde{r}_i$ with $\tilde{r}_i(x, u, v) \leq r_i(x, u, v) + \lambda R_0 + 1$. Since this is a bounded change in the nonlinearity, it clearly does not affect the result after the blow up method is performed. Therefore, to obtain the a priori bound uniformly in t, one assumes the contrary and takes a sequence of solutions $\vec{u}_{n,t_n}$ such that $\|\vec{u}_{n,t_n}\|_{L^\infty} \to \infty$ as $n \to \infty$, with t_n allowed to vary in $[0, 1]$ and $\vec{u}_{n,t_n}$ solving (10) with $t = t_n$. Repeating the argument in [8] with this sequence, one easily obtains an a priori bound uniformly across the entire homotopy class of problems. $\square$

Combining the fact that $(\underline{u}, \underline{v})$ is a strict lower solution to (10) for any $t \in [0, 1]$ and that T is a strict a priori bound, it is clear that $(\underline{u}, T) \times (\underline{v}, T) \subset X$ is an open set and that (10) has no solutions on its boundary. If we let $F' = I\vec{u} - L^{-1}\vec{m}(x, \vec{u}) = 0$, then it follows from homotopy invariance that $deg(F, 0, (\underline{u}, T) \times (\underline{v}, T)) = deg(F', 0, (\underline{u}, T) \times (\underline{v}, T))$.

Lemma 3. *In the weakly coupled case, the BVP*

$$\begin{cases} -\Delta u & = m_1(x, u, v) \qquad \forall x \in \Omega \\ -\Delta v & = m_2(x, u, v) \qquad \forall x \in \Omega \\ u = v & = 0 \qquad\qquad\quad\;\; \forall x \in \partial\Omega \end{cases} \tag{11}$$

has no nonnegative solution.

Proof. The proof is by contradiction. Suppose (u, v) is a nonnegative solution of (11). We only need to consider one of the two equations, say the equation for $-Deltau$. Since, $\forall x, u, v$, $m_1(x, u, v) \geq \lambda_1 u + 1$, we have $-\Delta u = m_1(x, u, v) \geq \lambda_1 u + 1$. Multiplying both sides by the positive eigenfunction, ϕ_1, of the laplacian corresponding to λ_1 and integrating by parts, we get

$$\begin{aligned}
\lambda_1 \int_\Omega u\phi_1 \, dx &= \int_\Omega u(-\Delta\Phi_1 \, dx \\
&= -\int_\Omega (\Delta u)\phi_1 \, dx \\
&\geq \int_\Omega (\lambda_1 u + 1)\phi_1 \, dx \\
&= \lambda_1 \int_\Omega u\phi_1 \, dx + \int_\Omega \phi_1 \, dx.
\end{aligned}$$

Hence

$$0 \geq \int_\Omega \phi_1 > 0,$$

which is a contradiction. □

In the strongly coupled case, the setup is identical except that we have that under condition $(C3b)$, with strong coupling, there exists some $R_0 > 0$ such that $\forall x \in \overline{\Omega}$

$$\begin{cases} g_1(x,u,v) > \lambda v + 1 & \forall u \geq 0 \text{ and } v > R_0 \\ g_2(x,u,v) > \lambda u + 1 & \forall v \geq 0 \text{ and } u > R_0. \end{cases}$$

As before, let

$$m_1(x,u,v) = \begin{cases} g_1(x,u,v) & v \geq R_0 \\ \max\{\lambda v + 1, g_1(x,u,v)\} & 0 \leq v < R_0, \end{cases}$$

and similarly

$$m_2(x,u,v) = \begin{cases} g_2(x,u,v) & u \geq R_0 \\ \max\{\lambda u + 1, g_2(x,u,v)\} & 0 \leq u < R_0, \end{cases}$$

and $\vec{m}(x,\vec{u}) = (m_1(x,u,v), m_2(x,u,v))$. Let

$$\vec{p_t}(x,\vec{u}) = (1-t)\vec{g}(x,\vec{u}) + t\vec{m}(x,\vec{u}),$$

where $\vec{g} = (g_1, g_2)$. Again we have the a priori bound (assuming the necessary Liouville result):

Lemma 4. *Suupose that* (1) *satisfies (H2) and is strongly coupled, and suppose that* (9) *has no nontrivial nonnegative solutions. Then there is an $T > 0$ such that any solution triple (t,u,v) of* (10) *satisfies $\|u\|_{L^\infty} + \|v\|_{L^\infty} < T$.*

Proof. The proof of this lemma is again a direct application of Proposition 1 exactly as in Lemma 2. □

We also have:

Lemma 5. *In the strongly coupled case, the BVP*

$$\begin{cases} -\Delta u & = m_1(x,u,v) \qquad \forall x \in \Omega \\ -\Delta v & = m_2(x,u,v) \qquad \forall x \in \Omega \\ u = v & = 0 \qquad\qquad\quad\;\; \forall x \in \partial\Omega \end{cases} \tag{12}$$

has no nonnegative solution.

Proof. Suppose (u, v) is a nonnegative solution of (12). Since $m_1(x, u, v) \geq \lambda_1 v + 1$, we have $-\Delta u = m_1(x, u, v) \geq \lambda_1 v + 1$. Similarly, since $m_2(x, u, v) \geq \lambda_1 u + 1$, we have $-\Delta v = m_2(x, u, v) \geq \lambda_1 u + 1$. Therefore, we have that

$$\Delta^2 u = -\Delta(-\Delta u) \geq -\Delta(\lambda v + 1) \geq \lambda^2 u + 1.$$

Multiplying both sides by the positive eigenfunction, ϕ_1, of the laplacian corresponding to λ_1 and integrating, we get

$$\begin{aligned} \lambda_1^2 \int_\Omega u\phi_1 \, dx &= \lambda_1 \int_\Omega u(-\Delta\phi_1) \, dx \\ &= -\lambda_1 \int_\Omega (\Delta u)\phi_1 \, dx \\ &= \int_\Omega (\Delta^2 u)\phi_1 \, dx \\ &\geq \int_\Omega (\lambda_1^2 u + 1)\phi_1 \, dx \\ &= \lambda_1^2 \int_\Omega u\phi_1 \, dx + \int_\Omega \phi_1 \, dx, \end{aligned}$$

after integrating by parts repeatedly. (Notice that our boundary conditions are exactly correct to prevent boundary terms in the integrations by parts.) Hence

$$0 \geq \int_\Omega \phi_1 > 0,$$

which is a contradiction. □

To conclude the proof of Theorem 1, it follows from Lemmata 3 and 5 that in either the weakly or strongly coupled case $deg(F', 0, (\underline{u}, T) \times (\underline{v}, T)) = 0$ and hence that

$deg(F, 0, (\underline{u}, T) \times (\underline{v}, T)) = 0$ also. By the excision property of Leray-Schauder degree we now see that
$deg(F, 0, (\underline{u}, T) \times (\underline{v}, T) \setminus (\underline{u}, \overline{u}) \times (\underline{v}, \overline{v})) = -1$, and thus (1) has a second solution, $(u_2, v_2) \in (\underline{u}, T) \times (\underline{v}, T)$, satisfying $(u_2(x_0), v_2(x_0)) > (\overline{u}(x_0), \overline{v}(x_0))$ at some point in $x_0 \in \Omega$. Hence we see that under very general abstract conditions two solutions to an elliptic system like (1) can be guaranteed.

3. An Auxiliary Problem

Our next goal is to find some explicit situations in which we know that the conditions of Theorem 1 are satisfied. In this section we study an important auxiliary problem which has a simplified semipositone structure. We prove an existence result that generalizes the positone result in Drábek-Robinson [10] and the radially symmetric positone result in Robinson-Rudd [22]. In the next section we will use the solution of this problem to construct a positive lower solution for (1).

Throughout this paper we will denote by z the positive solution of (5) and let $M = \|z\|_\infty$. Additionally, ν will always represent the outward unit normal to the boundary of Ω.

Consider the auxiliary problem

$$\begin{cases} -\Delta\psi &= -k\chi_{\{\psi<1\}} + K\chi_{\{\psi\geq 1\}} \qquad \forall x \in \Omega \\ \quad \psi &= 0 \qquad\qquad\qquad\qquad\qquad \forall x \in \partial\Omega \end{cases} \tag{13}$$

where $\chi_{\{\psi<1\}}$ represents the standard characteristic function on the set $\{x \in \Omega : \psi(x) < 1\}$ and $\chi_{\{\psi\geq 1\}}$ is defined similarly.

Lemma 6. *For each fixed $k > 0$ there exists $K > 0$ such that (13) has a positive solution.*

Proof. Let $k > 0$ be fixed and let $B := B_r(x_0) \subset\subset \Omega$. Consider the sub-auxiliary problem

$$\begin{cases} -\Delta w &= -k\chi_{\overline{B}^c} + K\chi_{\overline{B}} \qquad \forall x \in \Omega \\ \quad w &= 0 \qquad\qquad\qquad\qquad \forall x \in \partial\Omega. \end{cases} \tag{14}$$

Let w_K represent the unique solution to this problem. Then $v_K := \frac{w_K}{K}$ satisfies

$$\begin{cases} -\Delta v_K &= -\frac{k}{K}\chi_{\overline{B}^c} + \chi_{\overline{B}} \qquad \forall x \in \Omega \\ \quad v_K &= 0 \qquad\qquad\qquad\qquad \forall x \in \partial\Omega \end{cases}$$

Since the right hand side of (14) satisfies a uniform $L^\infty(\Omega)$ bound we have, without loss of generality, that $v_K \longrightarrow v$ in $C^1(\overline{\Omega})$ as $K \to \infty$, where v solves

$$\begin{cases} -\Delta v &= \chi_{\overline{B}} \qquad \forall x \in \Omega \\ \quad v &= 0 \qquad \forall x \in \partial\Omega. \end{cases}$$

By the maximum principle, $v > 0$ in Ω and $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$. This implies that $w_K > 0$ in Ω for large K. Moreover, it is clear that for all K large enough we have $w_K > 1$ on $\overline{B}$ and $\frac{\partial w_K}{\partial \nu} < 0$ on $\partial\Omega$.

Since $\overline{B}^c \supset \{x \in \Omega : w_K < 1\}$, for large K, w_K satisfies

$$-\Delta w_K = -k\chi_{\overline{B}^c} + K\chi_{\overline{B}} \leq -k\chi_{\{w_K<1\}} + K\chi_{\{w_K \geq 1\}}$$

in Ω. Thus w_K is a lower solution of (13).

Let $K > 0$ be chosen so that w_K is a lower solution of (13). Then Kz is an upper solution of (13), because $-\Delta(Kz) = K \geq -k\chi_{Kz<1} + K\chi_{Kz\geq 1}$. By the same calculation, $-\Delta(Kz) \geq -\Delta w_K$. Hence, by the maximum principle, $Kz \geq w_K$, so Kz and w_K are well-ordered. Note that the function $h(t) := -k\chi_{\{t<1\}} + K\chi_{\{t\geq 1\}}$ is nondecreasing and continuous from the right. It follows that, given well-ordered lower and upper solutions, (13) has a solution obtained via monotone iteration from the upper solution. $\square$

In the arguments that follow we will refer to a solution, u, of (13) as *maximal*, if every other solution, v, of (13) for the same values of k and K satisfies $v \leq u$ in $\overline{\Omega}$. The solution found in the previous lemma will be maximal relative to other solutions lying between the given lower and upper solutions because it was found by monotone iteration from the given upper solution.

Lemma 7. *If (k, K) is a pair such that (13) has a positive solution, then (13) has a maximal solution.*

Proof. by the maximum principle, $\overline{u} = Kz$ provides an upper bound on any solution, u, to (13), because $-\Delta\overline{u} = K \geq -k\chi_{\{u<1\}} + K\chi_{\{u\geq 1\}} = -\Delta u$. Hence, the solution obtained by monotone iteration from $\overline{u}$ is maximal relative to all solutions. $\square$

The following lemma characterizes the set

$$S_k := \{K > 0 : (13) \text{ has a positive solution}\}.$$

Lemma 8. *The set S_k is a closed ray. That is, for $k > 0$ fixed, let $K_k := \inf\{K > 0 : K \in S_k\}$. Then $S_k = [K_k, \infty)$.*

Proof. First, we already know that every sufficiently large $K \in \mathbb{R}$ is in S_k. We next need to show that S_k is a ray, i.e. if $K_0 \in S_k$ and $K > K_0$, then $K \in S_k$. But this follows immediately because ψ_{K_0} satisfies

$$-\Delta\psi_{K_0} = -k\chi_{\{\psi_{K_0}<1\}} + K_0\chi_{\{\psi_{K_0}\geq 1\}} < -k\chi_{\{\psi_{K_0}<1\}} + K\chi_{\{\psi_{K_0}\geq 1\}},$$

so ψ_{K_0} is a subsolution for (13) with respect to K. Moreover,Kz is a strictly larger supersolution to the problem, as above. Therefore, a solution to (13) for K must exist, and $K \in S_k$.

Finally, we must show that this is a closed ray, i.e. $K_k \in S_k$. Let $K_1, K_2 \in S_k$ with $K_1 > K_2$, and let ψ_1, ψ_2 represent the corresponding maximal solutions of (13). Then

$$-\Delta\psi_2 = -k\chi_{\{\psi_2<1\}} + K_2\chi_{\{\psi_2\geq 1\}} \leq -k\chi_{\{\psi_2<1\}} + K_1\chi_{\{\psi_2\geq 1\}},$$

so ψ_2 is a positive lower solution for (13) with $K = K_1$. Using the maximal property of solutions we get $\psi_1 \geq \psi_2$.

Now let $K_n \searrow K_k$ and let ψ_n be the corresponding maximal solutions. Then $\{\psi_n\}$ is monotonically decreasing, by the above argument, and thus the pointwise limit $\psi_k(x) := \lim_{n\to\infty} \psi_n(x)$ exists and

$$-k\chi_{\{\psi_n<1\}} + K_n\chi_{\{\psi_n\geq 1\}} \searrow -k\chi_{\{\psi_k<1\}} + K_k\chi_{\{\psi_k\geq 1\}}$$

pointwise, where we have used the fact that $h(t)$, as defined above, is continuous from the right. Since the right hand side of (13) is uniformly L^∞ bounded, we can apply standard regularity and imbedding theorems (see, for example, Gilbarg-Trudinger [Theorems 7.22, 9.11 and 9.15][12]), to derive a subsequence such that $\psi_n \longrightarrow \psi_k$ in $C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$ and ψ_k is a solution of (13) with $K = K_k$.

□

4. The Quasimonotone Nondecreasing Case: Weak Coupling

In this section we consider problem (1) with the assumption that Definition 1 is satisfied. We will give explicit conditions for the existence of the required ordered, strictly positive upper and lower solution pair. Our result is complementary to the existence and nonexistence results of Chhetri-Girg [5]. Our theorem here also complements the single equation positone results in Drábek-Robinson [10] and generalizes the single equation radially symmetric results in Robinson-Rudd [22].

Theorem 2. *Fix $k_1, k_2 > 0$ and choose $K_i > K_{k_i}$ for $i = 1, 2$. Let $m_{k_i} := \|\psi_{k_i}\|_\infty > 1$ where ψ_{k_i} is a solution of (13) with $K = K_{k_i}$. Suppose that $g_i(x, s, t)$ are such that (H1) and (H2) hold. Suppose there exist $C_1 > m_{k_1}/M$ and $C_2 > m_{k_2}/M$ such that the following hold uniformly for $x \in \overline{\Omega}$:*

(H3a): $\begin{cases} g_1(x,s,t) > -k_1 & \text{for} \quad 0 \le s \le 1 \quad \text{and} \quad t \le m_{k_2} \\ g_2(x,s,t) > -k_2 & \text{for} \quad 0 \le t \le 1 \quad \text{and} \quad s \le m_{k_1}, \end{cases}$

(H3b): $\begin{cases} g_1(x,s,t) > K_1 & \text{for} \quad 1 \le s \le m_{k_1} \quad \text{and} \quad t \le m_{k_2} \\ g_2(x,s,t) > K_2 & \text{for} \quad 1 \le t \le m_{k_2} \quad \text{and} \quad s \le m_{k_1}, \end{cases}$

(H3c): $g_i(x, s, t) < C_i$ *for* $0 \le s \le C_1 M$ *and* $0 \le t \le C_2 M$.

Then (1) has at least two solutions.

The proof of this theorem follows from the series of lemmata established below.

Lemma 9. $(\underline{u}, \underline{v}) := (\psi_{k_1}, \psi_{k_2})$ *is a strict lower solution.*

Proof. This lemma follows from the strict inequalities in (H3a) and (H3b), i.e.

$$\begin{aligned} -\Delta \underline{u} &= -\Delta \psi_{k_1} \\ &= -k_1 \chi_{\{\psi_{k_1} < 1\}} + K_1 \chi_{\{\psi_{k_1} \ge 1\}} \\ &= -k_1 \chi_{\{0 \le \psi_{k_1} < 1\}} + K_1 \chi_{\{1 \le \psi_{k_1} \le m_{k_1}\}} \\ &< g_1(x, \psi_{k_1}, \psi_{k_2}) = g_1(x, \underline{u}, \underline{v}) \end{aligned}$$

because $0 < \psi_{k_2}(x) < m_{k_2}$ $\forall x \in \Omega$. The calculation for the $\underline{v}$ equation is similar. □

Lemma 10. $(\overline{u}, \overline{v}) := (C_1 z, C_2 z)$ *is a strict upper solution.*

Proof. The proof follows from the fact that $0 \le C_i z \le C_i M$ for all $x \in \overline{\Omega}$ and so from (H3c) we have the strict inequality

$$-\Delta(C_i z) = C_i > g_i(x, C_1 z, C_2 z)$$

for $i = 1, 2$. □

Observe that $C_i > g_i(x, \underline{u}, \underline{v})$ because $0 \le \underline{u} \le m_{k_1} < C_1 M$ and similarly for $\underline{v}$. It follows that $-\Delta \overline{u} = C_1 > g_1(x, \underline{u}, \underline{v}) > -\Delta \underline{u}$, and so an application of the maximum principle shows that $\underline{u} < \overline{u}$, i.e. the lower and upper solutions are well-ordered. An identical calculation can be done to show that $\underline{v} < \overline{v}$. By Theorem 1, Theorem 2 immediately follows.

Finally, we provide a few examples of reaction terms $g_i(x, s, t)$ that satisfy the conditions of Theorem 2.

Example 1. *For $n = 2$ or $n = 3$ let*

$$g_1(x, s, t) = \epsilon s^4 + Ae^{\frac{at}{1+t}} - \gamma$$
$$g_2(x, s, t) = \epsilon t^4 + Ae^{\frac{as}{1+s}} - \gamma,$$

where ϵ, γ, a and A are positive parameters. Due to the symmetry of this system, we may assume that $k_1 = k_2$ and $C_1 = C_2$ when checking the conditions.

- (H1) Since $g_i(x, 0, 0) = A - \gamma$, (H1) is satisfied if $A < \gamma$. (Recall that this condition is not actually required for the theorem, but the semipositone case was our original motivation.)
- (H2) Condition (H2) is satisfied with $q_{ii} = 4$, which is subcritical for $n = 2$ or $n = 3$, $q_{ij} = 0$ for $i \neq j$, $h_{ij} \equiv 0$ for $i \neq j$; $h_{ii} \equiv \epsilon$; $r_1(x, s, t) = Ae^{\frac{at}{1+t}} - \gamma$ and $r_2(x, s, t) = Ae^{\frac{as}{1+s}} - \gamma$.
- (H3) We show below that (H3a)-(H3b) are satisfied for $g_1(x, s, t)$. (The same arguments work for g_2.)
  - (H3a) Since $g_1(x, s, t) = \epsilon s^4 + Ae^{\frac{at}{1+t}} - \gamma$ is a nondecreasing function for $0 \leq s, t \leq 1$, to satisfy (H3a), it is enough to show $g_1(x, 0, 0) = A - \gamma > -k$. Since A is positive, (H3a) is satisfied for g_1 if $\gamma \geq k$ and $A \geq \gamma - k$.
  - (H3b) Since $g_1(x, s, t) = \epsilon s^4 + Ae^{\frac{at}{1+t}} - \gamma$ is a nondecreasing function for $1 \leq s, t \leq m_k$, to satisfy (H3b), it is enough to show $g_1(x, 1, 1) = \epsilon + Ae^{\frac{a}{2}} - \gamma > K$. This holds for g_1 if a is chosen large enough to satisfy $\epsilon + Ae^{\frac{a}{2}} > \gamma + K$.
  - (H3c) Using the fact that $g_1(x, s, t)$ is nondecreasing for $0 \leq s, t \leq CM$, it suffices to show $Ae^a + \epsilon(CM)^4 - \gamma < C$. This condition holds true if ϵ is sufficiently small and hence (H3c) holds.

Remark 1. *For higher dimensions, an identical example can be constructed so long as the diagonal terms are subcritical and superlinear.*

Example 2. *For $n = 2, 3$, let*

$$g_1(x, s, t) = \epsilon s^4 + At^\theta - \gamma$$
$$g_2(x, s, t) = \epsilon t^4 + As^\theta - \gamma$$

where ϵ, γ and A are positive parameters and $0 < \theta < 1$. Due to the symmetry of this system, we may assume that $k_1 = k_2$ and $C_1 = C_2$ when checking the conditions.

(H1) Since $g_i(x, 0, 0) = -\gamma < 0$ (H1) is satisfied.

(H2) Condition (H2) is satisfied with $q_{ii} = 4$, which is subcritical in 2 or 3 space dimensions, $q_{ij} = \theta$ for $i \neq j$; $h_{ij} \equiv A$ for $i \neq j$; $h_{ii} \equiv \epsilon$ for $i = j$; $r_1(x, s, t) = -\gamma$ and $r_2(x, s, t) = -\gamma$.

(H3) We show below that (H3a)-(H3b) are satisfied for $g_1(x, s, t)$. (The same arguments work for g_2.)

(H3a) Since $g_1(x, s, t) = \epsilon s^4 + At^\theta - \gamma$ is a nondecreasing function for $0 \leq s, t \leq 1$, to satisfy (H3a), it is enough to satisfy $g_1(x, 0, 0) = -\gamma > -k$. Thus (H3a) is satisfied if $\gamma < k$.

(H3b) Since $g_1(x, s, t) = \epsilon s^4 + At^\theta - \gamma$ is a nondecreasing function for $1 \leq s, t \leq m_k$, to satisfy (H3b), it is enough to show $g_1(x, 1, 1) = \epsilon + A - \gamma > K$. This holds for g_1 if A is chosen large enough to satisfy $A > \gamma + K - \epsilon$.

(H3c) Using the fact that $g_1(x, s, t)$ is nondecreasing for $0 \leq s, t \leq CM$, it suffices to show $A(CM)^\theta + \epsilon(CM)^4 - \gamma < C$. This condition holds true if C is sufficiently large and ϵ is sufficiently small and hence (H3c) holds.

Remark 2. *As above, for higher dimensions an identical example can be constructed so long as the diagonal terms are subcritical and superlinear.*

5. The Quasimonotone Nondecreasing Case: Hamiltonian Systems

In this section, we consider the purely Hamiltonian case. Namely we prove the following theorem:

Theorem 3. *Consider a system of the form*

$$\begin{cases} -\Delta u & = g_1(v) \qquad \forall x \in \Omega \\ -\Delta v & = g_2(u) \qquad \forall x \in \Omega \\ u, v & > 0 \qquad\quad\;\; \forall x \in \Omega \\ u = v & = 0 \qquad\quad\;\; \forall x \in \partial\Omega \end{cases} \tag{15}$$

where $\Omega \subset \mathbb{R}^n$ is convex with C^3 boundary. Here $g_1, g_2 : [0, \infty) \to \mathbb{R}$ are C^1 monotone nondecreasing functions satisfying the following

(A1): *there exist* $a, b > 0$ *such that* $ab > \lambda_1^2$ *and*

$$\liminf_{s\to\infty} \frac{g_1(s)}{s} \geq a; \quad \liminf_{s\to\infty} \frac{g_2(s)}{s} \geq b$$

(A2): *there exist positive numbers* η_1, η_2 *such that*

$$\lim_{s\to\infty} \frac{g_1(s)}{s^p} = \eta_1; \quad \lim_{s\to\infty} \frac{g_2(s)}{s^q} = \eta_2$$

where $p, q \geq 1$ *and are subcritical in the following sense:*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n} \qquad n \geq 3.$$

Further assume that for $i = 1, 2$ *there exist* $k_i > 0$ *and* $C_i > m_{k_i}/M$ *such that*

(A3a): $g_i(s) > -k_i$ *for* $0 \leq s \leq 1$
(A3b): $g_i(s) > K_i$ *for* $1 \leq s \leq m_{k_i}$
(A3c): $g_i(s) < C$ *for* $0 \leq s \leq C_i M$

where K_i *and* m_{k_i} *are as defined in Theorem 2. Then* (15) *has at least two solutions.*

Remark 3. *Note that, as in previous sections, condition (H1) is not actually required here; there is no condition on the sign of the nonlinearity at* 0*. However, being motivated by the semipositone case, we will provide a semipositone example satisfying the conditions of the theorem below.*

Remark 4. *Observe that we do not assume (H2) here which states that the nonlinearities are superlinear and subcritical. The conditions (A1) and (A2) represent superlinear and subcritical behavior respectively in the Hamiltonian setting, and are used here to apply the result of Clement-de Figueiredo-Mitidieri* [4]*.*

Remark 5. *Recall that we wish to apply Theorem 1 to establish the existence of two solutions to* (15)*. Clement, de Figueiredo and Mitidieri* [4]*(Theorem 2.1) proved that there exists an a priori bound for positive solutions to* (15) *when* g_1, g_2 *are monotone nondecreasing and satisfy (A1)-(A2). In fact, the result was proved in order to establish the existence of positive solution to a certain Hamiltonian system using degree theory. Therefore the conditions on the nonlinearities presented here can depend uniformly on a homotopy parameter* t*. To maintain the uniformity with the previous sections, we omit the explicit dependence* *on t in the statement of the theorem above; however, the proof here proceeds exactly as for Theorem 2 because the a priori bound holds uniformly for the homotopy described in Section 2, as we need it to.*

Proof. As in Section 4, $(\underline{u}, \underline{v}) := (\psi_{k_1}, \psi_{k_2})$ is a strict lower solution and $(\overline{u}, \overline{v}) := (Cz, Cz)$ is a strict upper solution, and this pair is ordered. Hypotheses (A1)-(A2) combined with the Hamiltonian structure and strong coupling ensures that every nonnegative solution of (15) is a priori bounded [Theorem 2.1][4]. This in turn implies that the condition (C3b) of Theorem 1 is satisfied. Therefore, by Theorem 1, there are two positive solutions to (15). $\square$

Finally, we provide an example satisfying the hypotheses of Theorem 3.

Example 3.

$$\begin{aligned} g_1(v) &= \eta v^p + Av^\theta - \gamma \\ g_2(u) &= \eta u^q + Au^\theta - \gamma \end{aligned}$$

where η, γ and A are positive parameters and $0 < \theta < 1$, and $p, q > 1$ satisfy the subcriticality condition. Due to the symmetry of this system, we may assume that $k_1 = k_2$ and $C_1 = C_2$ when checking the conditions.

Clearly g_i are monotone nondecreasing functions and $g_i(0) = -\gamma < 0$.

- (A1) Since $p, q > 1$ and g_i are monotone nondecreasing, (A1) is clearly satisfied.
- (A2) Obviously $\lim\limits_{v\to\infty} \frac{g_1(v)}{v^p} = \eta$ and $\lim\limits_{s\to\infty} \frac{g_2(u)}{u^q} = \eta$ since $0 < \theta < 1$.
- (A3) We will show below that the conditions (A3a)-(A3c) are satisfied for $g_1(v)$ (The same arguments work g_2.)
  - (A3a) Since $g_1(v) = \eta v^p + Av^\theta - \gamma$ is a nondecreasing function for $0 \le v \le 1$, to satisfy (A3a), it is enough to satisfy $g_1(0) = -\gamma > -k$. Thus (A3a) is satisfied if $\gamma < k$.
  - (A3b) Since g_1 is a nondecreasing function for $1 \le v \le m_k$, to satisfy (A3b), it is enough to show $g_1(1) = \eta + A - \gamma > K$. This holds for g_1 if A is chosen large enough to satisfy $A > \gamma + K - \eta$.
  - (A3b) Similarly, to satisfy (L3) it suffices to show $A(CM)^\theta + \eta(CM)^p - \gamma < C$. This condition holds true if η is sufficiently small and hence (A3c) holds.

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