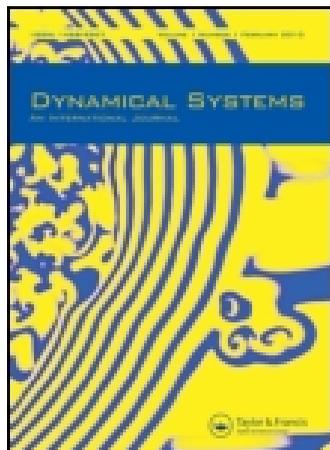


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### Dynamics near a minimal-mass soliton for a Korteweg-de Vries equation

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## Dynamics near a minimal-mass soliton for a Korteweg–de Vries equation

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We study soliton solutions to a generalized Korteweg–de Vries (KdV) equation with a saturated nonlinearity, following the line of inquiry of Marzuola, Raynor and Simpson for the nonlinear Schrödinger equation (NLS). KdV with such a nonlinearity is known to possess a minimal-mass soliton. We consider a small perturbation of a minimal-mass soliton and numerically shadow a system of ordinary differential equation (ODEs), which models the behaviour of the perturbation for short times. This connects nicely to analytic works of Comech, Cuccagna and Pelinovsky as well as of Grimshaw and Pelinovsky. These ODEs form a simple dynamical system with a single unstable hyperbolic fixed point with two possible dynamical outcomes. A particular feature of the dynamics is that they are non-oscillatory. This distinguishes the KdV problem from the analogous NLS one.

**Keywords:** soliton dynamics; Korteweg–de Vries; numerical approximation

### 1. Introduction

We consider a generalized Korteweg–de Vries (KdV) equation of the form

$$u_t + \partial_x(f(u)) + u_{xxx} = 0, \quad (1.1)$$

where  $f$  is a *saturated* nonlinearity, that is,  $f$  behaves subcritically at high intensities and supercritically at low intensities. An example of such a nonlinearity is

$$f(s) = \frac{s^p}{1 + \delta s^{p-q}}, \quad (1.2)$$

with  $p > 5$  and  $1 < q < 5$ , and with  $\delta > 0$  as an additional parameter. In the computations we present, we take  $p = 6$ ,  $q = 3$  and  $\delta = \frac{1}{4}$ . Analysis has previously been done on similar KdV problems in [1] and [2], which we will attempt to carefully verify and test numerically.

A travelling wave function  $u(x, t) = \phi_c(x - ct)$  is a *soliton* solution to (1.1) when the profile  $\phi_c$  satisfies the ordinary differential equation (ODE)

$$-c\phi_c + f(\phi_c) + \partial_{yy}\phi_c = 0. \quad (1.3)$$

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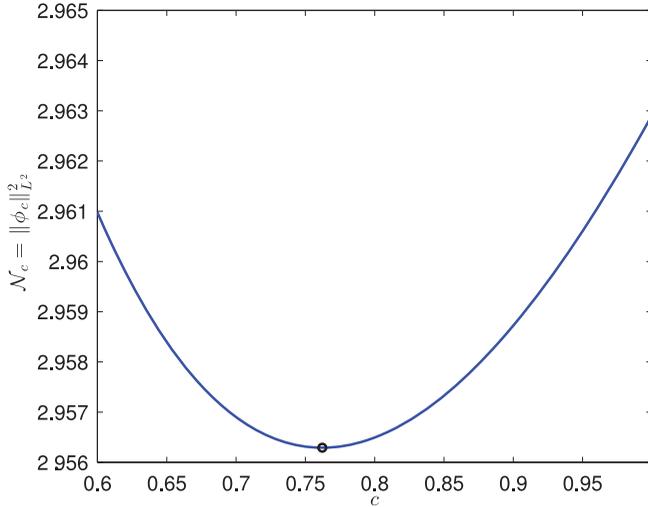


Figure 1.  $\mathcal{N}_c = \|\phi_c\|_{L^2}^2$  possesses a local minimum for the saturated nonlinearity given in (1.2) for  $p = 6, q = 3$  and  $\delta = \frac{1}{4}$ .

If we were considering a noncritical power nonlinearity,  $f(s) = s^p, 1 < p < 5$ , the equation would admit solitons of arbitrarily small  $L^2$ -norm. However, saturated nonlinearities are more restrictive. For the nonlinearities we consider, there will be a unique soliton of minimal  $L^2$ -norm. See Figure 1 for a plot of the  $L^2$ -norm as a function of  $c$  for one instance of (1.2). This critical soliton is denoted by  $\phi_{c^*}$ .

It is known that there exists an interval  $U \subset \mathbb{R}$  so that there is a soliton solution to (1.3) for each  $c \in U$  [3]; moreover,  $\phi_\infty(y)$  is a smooth function of  $c$  on  $U$ . Indeed, via elliptic theory we see that it holds generically that  $\phi \in C^{q+2}$ ; since  $\phi > 0$ , this then implies that  $\phi \in C^\infty$  for  $p, q \in \mathbb{Z}$ . Solitons can be interpreted as minimizers of the Hamiltonian energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 dx - \int_{\mathbb{R}} F(|u|) dx, \tag{1.4}$$

where  $F(s) = \int_0^s f(t) dt$ , subject to the fixed momentum condition

$$N(u) = \frac{1}{2} \int_{\mathbb{R}} u^2 dx, \tag{1.5}$$

with  $c$  acting as the Lagrange multiplier. Equation (1.1) also conserves the mass:

$$I(u) = \int_{\mathbb{R}} u dx. \tag{1.6}$$

By evaluating these conserved quantities at the soliton  $\phi_c$ , we get functions of  $c$ :

$$\mathcal{E}_c = E(\phi_c), \quad \mathcal{N}_c = N(\phi_c), \quad \mathcal{I}_c = I(\phi_c). \tag{1.7}$$

The minimal-momentum soliton is found at the value of  $c = c_*$  such that

$$\frac{d}{dc} \mathcal{N}_c = \langle \phi_c, \partial_c \phi_c \rangle = 0. \tag{1.8}$$

This first-order condition would also hold at a maximal momentum soliton.

While the stability of soliton solutions of (1.1) is well understood away from such critical points, our understanding of the dynamics near this critical soliton remains incomplete. Due to the saturated nature of the nonlinearity, the equation is globally well posed in  $H^2$ , so there is no finite-time singularity. But does the perturbed soliton converge to some nearby stable state, disperse, or engage in some other dynamic? It is known (see the analytic proofs in [1,2]) that the minimal-mass soliton itself enjoys a purely nonlinear instability. The purpose of this work is to give a concise formulation of the model dynamics near a critical soliton, further examine the dynamics of this type of solutions, and observe the predicted dynamical system properties within a numerical integration of the full partial differential equation (PDE) model.

To better understand these dynamics, we consider perturbations of  $\phi_{c_*}$ . Beginning with the ansatz

$$u(x, t) = \phi_c(x - ct) + p(x - ct, t) \tag{1.9}$$

for a perturbed soliton, we obtain the following evolution equation for  $p$ :

$$p_t = \partial_y [-\partial_y^2 p + cp - f'(\phi_c)p] + O(p^2). \tag{1.10}$$

In order to analyse this equation, we first consider, in Section 2, the spectrum of the linearized operator,  $A_c$ , where

$$A_c \equiv \partial_y L_c, \quad L_c \equiv -\partial_{yy} + c - f'(\phi_c). \tag{1.11}$$

An examination of  $A_c$  reveals that its generalized kernel has a dimension of at least 2. At  $c_I$ , this dimension increases to least 3, and could be 4 under special circumstances. Thus, there will be secular growth of the perturbation at a critical point generated by the components of the solution parallel to elements of the generalized kernel. Even if the perturbation is initially orthogonal to these unstable directions, the higher order terms will likely generate unstable contributions.

Such secular growth is eliminated by making a more general ansatz that allows  $c$  to modulate in time. This is given in Section 3, where we introduce a three-dimensional set of scalar parameters, including the soliton speed and wave centre, and allow them to modulate. This permits for projection away from the linearly unstable modes. We separate the projection of  $p$  onto the discrete spectrum of  $A_c$  from its projection onto the continuous spectrum. We then discard the continuous projection component while modulating the remaining parameters to obtain a two-dimensional system of first-order ODEs.

Finally, in Section 4, we describe the numerical methods we use to compute key parameters and simulate (1.1). Section 5 presents the results of these computations and discusses their implications.

## 2. The linear operator

In this section, we survey the spectral properties of  $A_c$  as defined in (1.11), about  $c = c_I$ . For all values of  $c$  that admit a soliton, one can directly compute

$$A_c(-\partial_y \phi_c) = 0, \quad (2.1)$$

$$A_c \partial_c \phi_c = -\partial_y \phi_c. \quad (2.2)$$

For the remainder of this article, we will denote differentiation with respect to  $c$  by  $'$ . The first two elements of the generalized kernel of  $A$ , then, are, then, are

$$e_{1,c} = -\partial_y \phi_c, \quad e_{2,c} = \phi'_c. \quad (2.3)$$

At a minimal-momentum soliton (or indeed, any soliton satisfying the first-order condition (1.8)), there is a third independent function  $e_{3,c^*}$  in the generalized kernel of  $A_c$ , which satisfies

$$A_{c^*} e_{3,c^*} = e_{2,c^*}. \quad (2.4)$$

To see why such a state exists, consider the adjoint operator  $A_c^* = -L_c \partial_y$ . We immediately compute

$$A_c^* \phi_c = 0, \quad A_c^* D^{-1} \phi'_c = -\phi_c, \quad (2.5)$$

where

$$D^{-1} f \equiv \int_{-\infty}^y f. \quad (2.6)$$

Hence, we define

$$g_{1,c} = \phi_c, \quad g_{2,c} = D^{-1} \phi'_c. \quad (2.7)$$

Then, by the Fredholm alternative, we see that  $A_c f = e_{2,c}$  has a solution, provided has a solution, provided

$$\langle e_{2,c}, g_{1,c} \rangle = \int \phi'_c \phi = \frac{d}{dc} \int \frac{1}{2} \phi_c^2 = \mathcal{N}'_c \quad (2.8)$$

vanishes, which is condition (1.8). Thus, at a minimal-mass soliton, there is indeed a third element of the generalized kernel of  $A_{c^*}$ , which solves

$$A_{c^*} e_{3,c^*} = e_{2,c^*}.$$

Consequently, there is also a third element in the generalized kernel of  $A_{c^*}^*$ ,

$$A_{c^*}^* g_{3,c^*} = g_{2,c^*}. \quad (2.9)$$

To see if there is a fourth element in the generalized kernel, we again consider  $A_{c_*} f = e_{3,c_*}$  and recognize that we would need

$$\begin{aligned} \langle e_{3,c_*}, g_{1,c_*} \rangle &= \langle e_{3,c_*}, A_{c_*}^* g_{2,c_*} \rangle = \langle A_{c_*} e_{3,c_*}, g_{2,c_*} \rangle \\ &= \langle e_{2,c_*}, g_{2,c_*} \rangle = \int \phi'_{c_*} \left( \int_{-\infty}^y \phi'_{c_*} \right) dy \\ &= \int \frac{d}{dy} \frac{1}{2} \left( \int_{-\infty}^y \phi'_{c_*} \right)^2 = \frac{1}{2} (\mathcal{I}'_{c_*})^2 \end{aligned} \tag{2.10}$$

to vanish. Even at a critical value of  $c_*$  for which  $\mathcal{N}'_{c_*} = 0$ , it is not generic to observe  $\mathcal{I}'_{c_*} = 0$ . Indeed, for the particular nonlinearity  $f$  that we consider, our minimal-momentum soliton will not have this fourth element. It would be of interest to find a nonlinearity that does satisfy this additional degeneracy condition, and to study the dynamics near the resulting doubly-critical soliton.

A particular challenge, discussed below in Section 4, is that some of these generalized kernel elements, notably  $e_{3,c_*}$ ,  $g_{2,c_*}$  and  $g_{3,c_*}$  are not in  $L^2$ . While  $e_{3,c_*}$  vanishes exponentially fast at  $+\infty$ , it is only bounded at  $-\infty$ .  $g_{2,c_*}$  and  $g_{3,c_*}$  both vanish at  $-\infty$ , but at  $+\infty$  the former is only bounded and the latter grows linearly. The reader can find a discussion of the function spaces in which these kernel functions lie in [1].

For later use, we remark that away from  $c_*$ , using the implicit function theorem, there is a scalar  $\lambda_c$  and function  $e_{3,c}$ , both smooth in  $c$ , such that

$$(A_c - \lambda_c I) e_{3,c} = e_{2,c}, \quad \lambda_c \equiv - \frac{\mathcal{N}'_c}{\langle \phi_c, e_{3,c} \rangle}. \tag{2.11}$$

### 3. Modulation equations

To overcome the secular growth due to the generalized kernel, we now permit the equation parameters to modulate about the extremal soliton. First, we define the moving frame

$$y(x, t) \equiv x - \int_0^t c(\sigma) d\sigma - \xi(t). \tag{3.1}$$

Next, we consider a solution  $u(x, t)$  which is a perturbed, modulating soliton,

$$\begin{aligned} u(x, t) &= \phi_{c(t)} \left( x - \int_0^t c(\sigma) d\sigma - \xi(t) \right) \\ &\quad + p \left( x - \int_0^t c(\sigma) d\sigma - \xi(t), t \right) \\ &= \phi_c(y) + p(y, t). \end{aligned} \tag{3.2}$$

Substituting this into (1.1), we get

$$\begin{aligned} p_t + \phi'_c \dot{c} - \partial_y \phi_c \dot{\xi} - p_y \dot{\xi} \\ = \partial_y L_{c(t)} p - \frac{1}{2} \partial_y (f''(\phi_c) p^2) + \text{higher order terms.} \end{aligned} \tag{3.3}$$

We let

$$\eta(t) = c(t) - c_*, \tag{3.4}$$

and decompose

$$p(x, t) = \zeta(t)e_{3,c} + v = \zeta(t)e_{3,c_*} + \zeta(t)\eta(t)e'_{3,c_*} + v + O(\zeta\eta^2) \tag{3.5}$$

as in [1], Equations (3.19)– (3.24), which results in

$$v_t + A_c v = -\dot{\xi}e_{1,c} - (\dot{\eta} - \dot{\zeta})e_{2,c} - (\dot{\zeta} - \lambda_c \zeta)e_{3,c} - \dot{\eta}\partial_x p + \partial_x N.$$

To close the system, we introduce the constraints

$$\langle g_{1,c}, v \rangle = \langle g_{2,c}, v \rangle = \langle g_{3,c}, v \rangle = 0.$$

We make several observations about the resulting dynamical system. The term  $\partial_x N$  has quadratic and higher order terms. We will preserve only quadratic terms in our computations. We will also disregard all coupling to the continuous spectrum, though some of this may also be of quadratic order. The continuous spectrum for  $L_c$  is the interval  $[c, \infty)$ . However, for  $A_c$ , the continuous spectrum becomes the entire interval and in particular subsumes the kernel of  $L_c$  with which we have done the spectral decomposition. For that purpose, many works on the KdV equations use weighted spaces  $L^2(e^{ax} dx)$ , such that the continuous spectrum of  $A_c$  separates from the discrete spectrum (see, for instance, [4]). Of course ignoring the continuous spectrum terms will introduce errors over time in comparison with the full KdV dynamics, and in the long run damping effects due to coupling to these radiative modes accounts for the asymptotic stability that has been observed for generalized KdV equations.

Continuing, the quadratic-order terms we are considering in our approximation can be taken to be  $\frac{1}{2}\zeta^2\partial_x(f''(\phi_c)e_{3,c}^2)$ . Projecting onto the canonical spectral functions, the finite-dimensional system then takes the form

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} - \dot{\zeta} \\ \dot{\zeta} - \lambda_c \zeta \end{pmatrix} = -\zeta^2 \mathcal{S}_c(\zeta)^{-1} \begin{pmatrix} \langle g_{1,c}, e'_{3,c} - \frac{1}{2}\partial_x(f''(\phi_c)e_{3,c}^2) \rangle \\ \langle g_{2,c}, e'_{3,c} - \frac{1}{2}\partial_x(f''(\phi_c)e_{3,c}^2) \rangle \\ \langle g_{3,c}, e'_{3,c} - \frac{1}{2}\partial_x(f''(\phi_c)e_{3,c}^2) \rangle \end{pmatrix}. \tag{3.6}$$

Under this approximation,  $\xi$  is slaved to  $\eta$  and  $\zeta$ . There is only weak coupling between  $\xi$  and the other parameters through  $v$ .

Continuing with the above assumptions,

$$\mathcal{S}_c(\zeta) = \mathcal{T}_c + \zeta \begin{pmatrix} -\langle g_{1,c}, \partial_x e_{3,c} \rangle & \langle g_{1,c}, e'_{3,c} \rangle & 0 \\ -\langle g_{2,c}, \partial_x e_{3,c} \rangle & \langle g_{2,c}, e'_{3,c} \rangle & 0 \\ -\langle g_{3,c}, \partial_x e_{3,c} \rangle & \langle g_{3,c}, e'_{3,c} \rangle & 0 \end{pmatrix} = \mathcal{T}_c + \zeta \hat{\mathcal{S}}_c, \tag{3.7}$$

where  $(\mathcal{T}_c)_{jk} = \langle g_{j,c}, e_{k,c} \rangle$ . Assuming  $\mathcal{T}_c$  and  $\mathcal{S}_c$  are  $O(1)$  and  $\zeta$  is sufficiently small, we can make the approximation  $\mathcal{S}_c \approx \mathcal{T}_c$ , and thus obtain, for appropriate vectors  $R_c$  and  $Q_c$ ,

$$\begin{aligned} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} - \zeta \\ \dot{\zeta} - \lambda_c \zeta \end{pmatrix} &= -\zeta^2 \mathcal{S}_c(\zeta)^{-1} \begin{pmatrix} \langle g_{1,c}, e'_{3,c} - \frac{1}{2} \partial_x(f''(\phi_c)e_{3,c}^2) \rangle \\ \langle g_{2,c}, e'_{3,c} - \frac{1}{2} \partial_x(f''(\phi_c)e_{3,c}^2) \rangle \\ \langle g_{3,c}, e'_{3,c} - \frac{1}{2} \partial_x(f''(\phi_c)e_{3,c}^2) \rangle \end{pmatrix} \\ &= -\zeta^2 \mathcal{T}_c^{-1} \begin{pmatrix} \langle g_{1,c}, e'_{3,c} - \frac{1}{2} \partial_x(f''(\phi_c)e_{3,c}^2) \rangle \\ \langle g_{2,c}, e'_{3,c} - \frac{1}{2} \partial_x(f''(\phi_c)e_{3,c}^2) \rangle \\ \langle g_{3,c}, e'_{3,c} - \frac{1}{2} \partial_x(f''(\phi_c)e_{3,c}^2) \rangle \end{pmatrix} + O(\zeta^3) \\ &= -\zeta^2 \mathcal{T}_c^{-1} R_c + O(\zeta^3) \\ &= -\zeta^2 Q_c + O(\zeta^3). \end{aligned}$$

Therefore, the leading-order equations, subject to these approximations, are

$$\dot{\xi} = -Q_{1,c} \zeta^2, \tag{3.8a}$$

$$\dot{\eta} - \zeta = -Q_{2,c} \zeta^2, \tag{3.8b}$$

$$\dot{\zeta} - \lambda_c \zeta = -Q_{3,c} \zeta^2. \tag{3.8c}$$

Making the Taylor expansions of  $\lambda_c$  the  $Q_{j,c}$  about  $c = c_*$ , and omitting the  $\xi$  equation, we obtain the quadratically nonlinear ODE system

$$\dot{\eta} - \zeta = -Q_{2,c_*} \zeta^2, \tag{3.9a}$$

$$\dot{\zeta} - \lambda'_{c_*} \eta \zeta = -Q_{3,c_*} \zeta^2. \tag{3.9b}$$

Critical points of the system are found at  $\zeta = 0$ ,  $\eta = \eta_0$ , where  $\eta_0$  is arbitrary, and at  $\zeta = \frac{1}{Q_2}$ ,  $\eta = \frac{Q_3}{\lambda'_{c_*} Q_2}$ . The latter isolated critical point is a saddle point in the first quadrant. We note here that we have not attempted to preserve the underlying Hamiltonian structure of the KdV equation in our leading-order approximation here, but simply chosen terms of the lowest order in terms of our small parameters. As a result, it is certainly true that some of the structure of the equation has been lost at the gain of computational simplicity.

For the  $\zeta = 0$  critical points, the linearized problem is

$$\begin{pmatrix} 0 & 1 \\ 0 & \lambda'_{c_*} \eta_0 \end{pmatrix}.$$

The eigenvalues are  $\lambda'_{c_*} \eta_0$  and 0. Thus, depending on the sign of  $\lambda'_{c_*} \eta_0$ , the solution is either linearly stable or unstable. It is worth noting in this context that  $\lambda_c = \frac{-\mathcal{N}'_c}{\langle \phi_c, e_{3,c} \rangle}$ . Thus,  $\lambda'_{c_*} = \frac{-\mathcal{N}''_{c_*}}{\langle \phi_{c_*}, e_{3,c_*} \rangle}$  since  $\mathcal{N}'_{c_*} = 0$ . Also note that, according to [1], near  $c_I$  we have that  $\langle \phi_c, e_{3,c} \rangle > 0$ . Thus, we expect that the sign of  $\lambda'_{c_*}$  depends solely on whether we are at a minimal or maximal soliton. In each case, we expect to see that, depending on the sign of  $\eta_0$ , the critical point at  $(0, \eta_0)$  is either linearly stable or linearly unstable. There is semi-stability at  $(0, 0)$ , depending on the sign of the initial perturbation  $\eta_0$ . See [Figure 2](#) for a representative

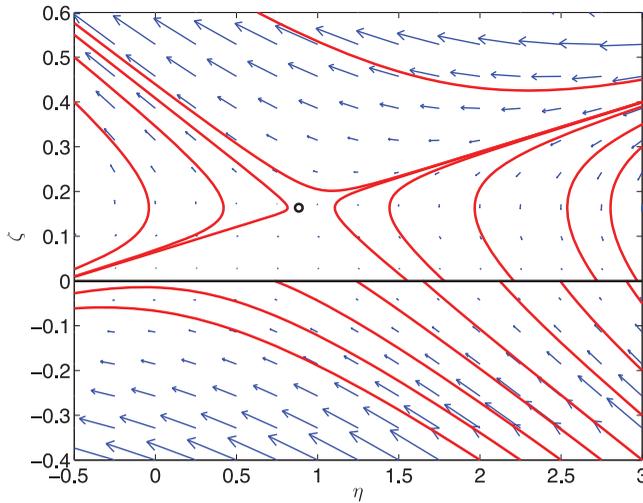


Figure 2. A plot of a representative phase-plane diagram. The hyperbolic fixed point is indicated by  $\circ$ , while the nonisolated critical points are indicated by the solid line  $\zeta = 0$ .

saturated KdV phase-plane diagram. Such a phase plane is rather simple as we are studying the dynamics of a single soliton near the minimal-mass threshold. Much more complex and rich dynamics can arise from the study of the many-body problem of multiple soliton pulses interacting. See [5] for a beginning approach to this. Note the computational methods applied below certainly be generalized to study many-soliton solutions and their effective dynamics.

#### 4. Computational methods

In this section, we briefly outline the computations from [1] needed to make a comparison between (3.9) and the KdV equation (1.1). Motivated by (3.2), (3.4) and (3.5), we will take initial conditions of the form

$$u_0 = \phi_{c_*}(x) + \eta_0 e_{2,c_*} + \zeta_0 e_{3,c_*}. \quad (4.1)$$

##### 4.1. Spectral computations

To compute the coefficients appearing in (3.9),  $Q_j$  and  $\lambda'_{c_*}$ , we must compute

- the generalized kernels of  $A_{c_*}$  and  $A_{c_*}^*$  from which we can get the inner products  $\langle g_{j,c_*}, e_{k,c_*} \rangle$  which the matrix  $\mathcal{T}_{c_*}$  comprises;
- $\partial_x e_{3,c_*}$  and  $e'_{3,c_*}$  to obtain the  $R_{c_*}$  vector, which, with  $\mathcal{T}_{c_*}$ , allows us to obtain the  $Q_{c_*}$  vector;
- $\lambda'_{c_*}$ , which can be obtained by differentiating (2.11) and then computing  $\phi''_{c_*}$ .

The first few elements of the generalized kernel,  $e_{1,c_*} = \partial_x \phi_{c_*}$ ,  $e_{2,c_*} = \phi'_{c_*}$  and  $g_{1,c_*} = \phi_{c_*}$ , are readily obtained using the sinc discretization method previously used by the authors in [6]. Briefly, this approach solves equations like

$$L_c f = g \quad (4.2)$$

using a sinc discretization of  $f, g$  and  $L_c$ , provided  $g$  is localized. Derivatives of functions are easy to obtain using the discretized differentiation matrices, and  $L^2$  inner products are just finite-dimensional inner products, multiplied by the uniform grid spacing. To obtain the minimal-momentum soliton, we use a root-finding algorithm to solve  $\mathcal{N}'_c = 0$ . See [7–10] for additional details on the sinc discretization method.

A bit more care needs to be exercised while computing the other elements as they are not  $L^2$ -localized. First, let

$$\Theta(y) \equiv \int_{+\infty}^y e_{2,c_*}. \tag{4.3}$$

Then,  $e_{3,c_*}$  solves

$$L_{c_*} e_{3,c_*} = \Theta, \quad \lim_{y \rightarrow +\infty} e_{3,c_*}(y) = 0. \tag{4.4}$$

Given the sinc discretization of  $\phi'_{c_*}$ , we can readily integrate to obtain  $\Theta$  using the techniques given in [11]. The use of such a quadrature tool was not required in [6] as all of the nonlinear Schrödinger equation (NLS) kernel elements belonged to  $L^2$ . This also gives us  $g_{2,c_*}$ .

Assuming that  $e_{3,c_*}$  grows, at most, algebraically at  $-\infty$ , we can drop  $f'(\phi_{c_*})e_{3,c_*}$  at large negative values of the  $y$  term to estimate

$$-\partial_x^2 e_{3,c_*} + c_* e_{3,c_*} \approx -\mathcal{I}'_{c_*} \neq 0. \tag{4.5}$$

This approximation implies that  $e_{3,c_*}$  is actually bounded at  $-\infty$ :

$$\lim_{y \rightarrow -\infty} e_{3,c_*} = -\frac{1}{c_*} \mathcal{I}'_{c_*}. \tag{4.6}$$

One method of computing  $e_{3,c_*}$  is to split it into a piece which has the above asymptotics and a spatially localized piece,

$$e_3 \equiv e_3^{(1)} + e_3^{(2)}, \tag{4.7}$$

where  $e_3^{(1)} \in L^2$  and  $e_3^{(2)}$  vanishes at  $+\infty$ . Using the above estimate, we set

$$\begin{aligned} e_{3,c_*}^{(2)}(x) &\equiv e_{3,c_*}^- \cdot \chi_-(x) \\ &= -\frac{1}{c_*} \mathcal{I}'_{c_*} \cdot \frac{1}{2} (1 + \tanh(-x)). \end{aligned} \tag{4.8}$$

We have some flexibility in selecting  $\chi_-$ . The essential feature is that it should not contribute anything at  $+\infty$ , while capturing the known asymptotic behaviour at  $-\infty$ . We then solve

$$L_{c_*} e_{3,c_*}^{(1)} = \Theta - L_{c_*} e_{3,c_*}^{(2)}. \tag{4.9}$$

As the right-hand side is now localized at both  $\pm\infty$ , we obtain  $e_{3,c_*}^{(1)}$ .

The adjoint problem is similar, but slightly more care needs to be exercised. First, we solve

$$L_{c_*} h_{3,c_*} = g_{2,c_*} \tag{4.10}$$

and then integrate to obtain  $g_{3,c_*}$ . While  $e_{3,c_*}$  was asymptotically constant,  $g_{3,c_*}$  will have linear growth at  $+\infty$ . The other function which requires such an asymptotic splitting is  $e'_{3,c_*}$ ,

To compute  $\lambda'_{c_*}$ , we compute

$$\lambda'_{c_*} = -\frac{\mathcal{N}''_{c_*}}{\langle \phi_{c_*}, e_{3,c_*} \rangle} = -\frac{\langle \phi_{c_*}, \phi''_{c_*} \rangle + \langle \phi'_{c_*}, \phi'_{c_*} \rangle}{\langle \phi_{c_*}, e_{3,c_*} \rangle}, \quad (4.11)$$

$\phi''_{c_*}$  is  $L^2$ -localized and obtained by solving

$$L_{c_*} \phi''_{c_*} = -2\phi'_{c_*} + f''(\phi_{c_*})(\phi'_{c_*})^2.$$

Computing the various inner products, we obtain the matrix  $\mathcal{T}_{c_*}$  and the vector  $R_{c_*}$ , from which we can solve for  $Q_{c_*}$ . This provides us with all coefficients in the ODE system.

#### 4.2. A finite-difference method for KdV-type equations

Integrating (1.1) with initial conditions of the form

$$\phi_{c_*+\eta_0} + \zeta_0 e_{3,c_*} \quad (4.12)$$

requires some care to be exercised, as  $e_{3,c_*}$  is not spatially localized. However, since it is asymptotically constant, we can, to leading order, use approximate Neumann boundary conditions at  $\pm x_{\max}$ , the edges of our computational domain:

$$\partial_x u = \partial_x^2 u = 0, \quad \text{at } x = \pm x_{\max}. \quad (4.13)$$

Using this approximation, we can then solve (1.1) using a linearized implicit method formulated in [12]. This method has a second-order accuracy in time, with the spatial accuracy given by the quality of our finite-difference approximations of the derivatives. In this work, we use second-order symmetric estimates of the first and third derivatives.

#### 4.3. Extrapolating and matching discretizations

A challenge in using our sinc approximations of the kernel functions is that they are given on one discretized mesh which may not be sufficiently large to employ the approximate boundary conditions (4.13) for our time-dependent simulation. To overcome this, we use the far-field asymptotics of these elements to extrapolate onto a larger domain with a given mesh spacing.

Numerically, we discretize on a short interval,  $[-R_{\text{sol}}, R_{\text{sol}}]$  to compute the soliton using the iterative sinc method from [6]. We then asymptotically extend  $u_0$  to a much larger interval,  $[-R_{\text{as}}, R_{\text{as}}]$ , for  $R_{\text{as}}$  large relative to where we desire to have the boundary. In particular, we extend using the asymptotics

$$\begin{aligned} \phi_{c_*}(x) &\sim \alpha_1 e^{-\sqrt{c_*}|x|}, & \text{as } x \rightarrow \pm\infty, \\ e_{2,c_*}(x) &\sim \alpha_2 x e^{-\sqrt{c_*}|x|}, & \text{as } x \rightarrow \pm\infty, \\ e_{3,c_*}(x) &\sim \alpha_3 x^2 e^{-\sqrt{c_*}|x|}, & \text{as } x \rightarrow \infty, \end{aligned}$$

$$e_{3,c_*}(x) \sim \frac{1}{c} \partial_c \mathcal{I}' - \alpha_4 x^2 e^{-\sqrt{c_*}|x|}, \text{ as } x \rightarrow -\infty.$$

The asymptotics of  $\phi$  are standard, those of  $e_{2,c_*}$  arise from the commutator relation  $[\Delta, x]f = 2\partial_x f$ , and those for  $e_{3,c_*}$  arise from integration by parts, given the asymptotics of  $e_{2,c_*}$ . To observe that such a continuation is nicely continuous and avoid boundary effects from the iterative methods, we actually choose to extend  $\phi$ ,  $e_{2,c_*}$  and  $e_{3,c_*}$  from values determined of distance 1 from  $\pm R_{\text{sol}}$ . We include a log plot of an initial condition with  $\eta_0, \zeta_0 > 0$  in Figure 3. We then do a linear interpolation to have an evenly spaced grid on another still large interval,  $[-R_{\text{pde}}, R_{\text{pde}}]$ , with  $R_{\text{pde}} < R_{\text{as}}$ , on which the simulation is performed. Here,  $R_{\text{pde}}$  is chosen large enough to minimize boundary interactions during the numerical integration, which in KdV simulations appear quite quickly due to the dispersion relation (see Figure 4).

To summarize, we have selected three different domains,  $R_{\text{sol}}$ ,  $R_{\text{as}}$  and  $R_{\text{pde}}$ , on which we, respectively, compute the soliton, match asymptotics at  $\pm\infty$  and then linearly interpolate onto a uniform grid to solve the PDE, with  $R_{\text{sol}} < R_{\text{pde}} < R_{\text{as}}$ . The simulation then proceeds with a linearly implicit finite-difference scheme using a split step time discretization. In our results, our PDE simulation contains many oscillations that diminish as the boundary effects are minimized.

#### 4.4. Extracting parameters

As we aim to compare the ODE system (3.9) with the PDE, we will need to find a way to extract  $\xi$ ,  $\eta = c - c_*$ , and  $\zeta$  from  $u(x, t)$ ,

$$u(x, t) = \phi_c(x - f c - \xi) + \zeta e_{3,c}(x - f c - \xi) + v(x - f c - \xi). \tag{4.14}$$

We estimate the wave speed by computing the centre of mass of  $u$  at each time step, and estimating its speed by finite differences; this gives us  $c = c_* + \eta$ . Unfortunately, there is

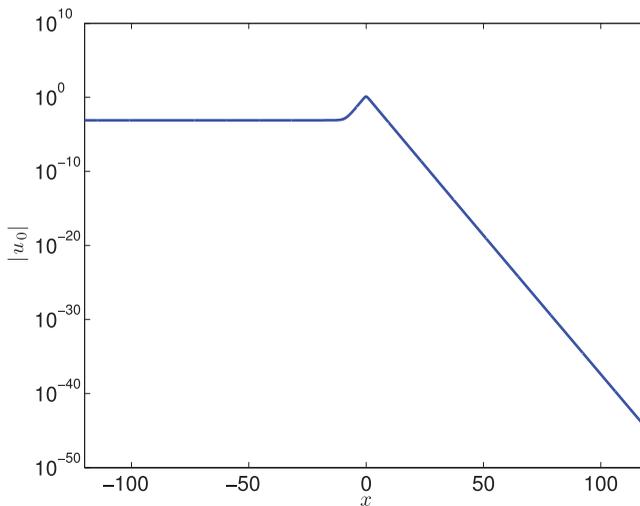


Figure 3. A log plot of the extended initial data.

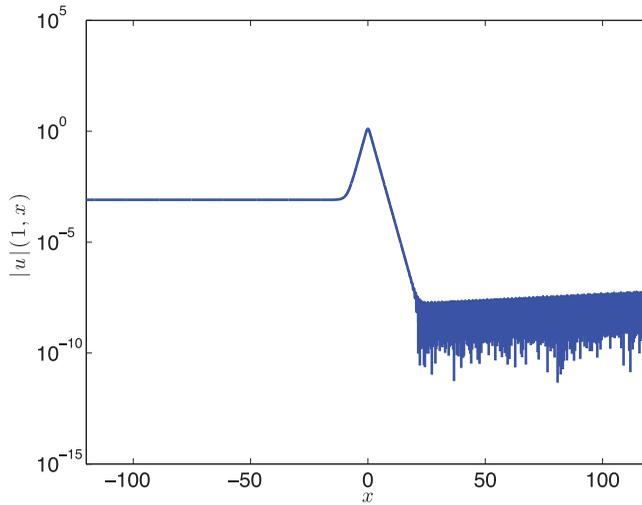


Figure 4. A semilog plot of the numerical solution to our PDE at  $t = 1$  showing the oscillatory back scattering from the boundary at  $x = R$ .

some ambiguity between the rate at which the wave moves due to the speed,  $c$ , and changes in the phase,  $\dot{\xi}$ ; we are only able to estimate, collectively,

$$\partial_t y(x, t) = -c - \dot{\xi}, \tag{4.15}$$

and assume that this is dominated by  $c$ , at least for the time scales we study. Indeed, on the time scales over which we numerically integrate, the  $\dot{\xi}$  term is quadratic in  $\zeta$ , which must remain small for our computations to remain accurate. We then integrate the wave speed by quadrature to estimate the shift.

Next, we estimate  $\zeta$  by projecting onto  $g_1$ :

$$\zeta(t) = \frac{\langle u(\cdot, t), g_{1,c} \rangle - \frac{1}{2}c(t)^2 \langle e'_{2,c}, g_{1,c} \rangle}{\langle e_{3,c}, g_{1,c} \rangle + c_* \langle e'_{3,c}, g_{1,c} \rangle}, \tag{4.16}$$

which is done by assuming that on the time scales we consider  $g_{1,c}$ ,  $e'_{2,c}$ ,  $e_{3,c}$  and  $e'_{3,c}$  are well approximated by their values at  $c = c_*$ .

### 5. Shadowing results

We now take as an initial condition (4.1) and study the evolution for different  $\eta_0$  and  $\zeta_0$ . In [2], it was shown by using an adiabatic approximation near the minimal-mass soliton that the nonlinear dynamics should either converge to a stable solitary wave or collapse to a spreading, dispersive-like, structure. We show below that the generalized KdV model we study behaves precisely so and hence demonstrate a range of parameters for which the adiabatic approximation captures the essential dynamics of the KdV model.

The results appear as Figures 5 and 6, in which we take initial data that begins in the first and third quadrants of the phase plane 2 and compare the projection of our integrated numerical PDE to the predicted ODE dynamics with domain size  $R_{\text{pde}} = 120.0$ ,  $N = 10^6$  spatial grid points, time of integration  $T = 30.0$  and time step  $h_t = 10^{-4}$ . The remaining

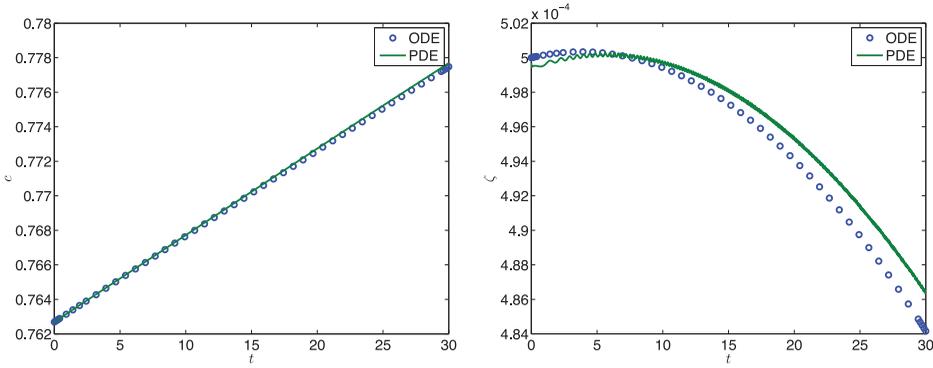


Figure 5. A comparison between the PDE and the ODE with  $\eta_0 = 5 \times 10^{-4}$  and  $\zeta_0 = 5 \times 10^{-4}$ .

cases display rather similar behaviours. In Figure 7, we observe that by taking a larger initial  $\eta_0 > 0$ ,  $\zeta_0 < 0$ , our solutions diverge from the predictive dynamics on a shorter time scale ( $T = 20.0$ ) with otherwise comparable parameters as above. Since the nature of KdV is to move to the right, in order to lessen boundary interaction, we solve the PDE in a moving

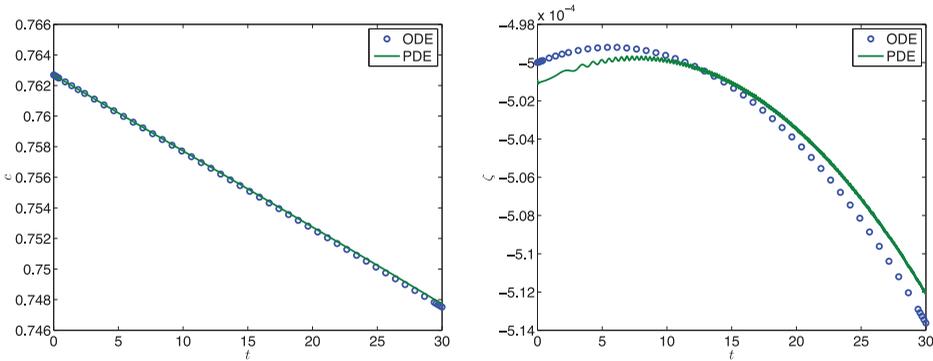


Figure 6. A comparison between the PDE and the ODE with  $\eta_0 = 5 \times 10^{-4}$  and  $\zeta_0 = -5 \times 10^{-4}$ .

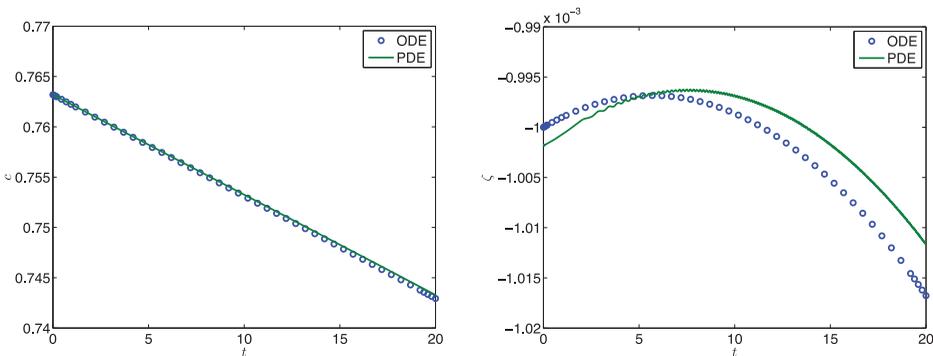


Figure 7. A comparison between the PDE and the ODE with  $\eta_0 = 10^{-3}$  and  $\zeta_0 = -10^{-3}$ .

reference frame around base velocity  $c_*$  and implement the schemes to project onto the ODE parameters  $c$  and  $\zeta$  as in Section 4.4.

Implementing these approximative schemes and comparing the evolution of the corresponding ODE in (3.9), the figures show that for long times the dynamics indeed fit the predicted dynamics. For small enough perturbations of the minimal-mass soliton, the dynamical system predicts that the orbits travel very slowly towards the stable or unstable manifolds. Hence, we only follow the orbits on time scales where the parameter  $\zeta$  has made a large motion in its orbit. The  $c$  component varies essentially linearly on this scale, however. As a result, we observe that (3.9) is a good model for perturbations close to the minimal mass. As in the NLS case, there is a perturbation of the minimal-mass solution leading to dynamics that move  $c$  to smaller values where the corresponding solitons are linearly unstable, which corresponds dynamically in the PDE with the mass radiation, as seen in [2]. However, continuing along this trajectory is forbidden by mass conservation, as seen in Figure 1. We postulate, as we did for the corresponding Schrödinger dynamics in [6], that this could be an energy transfer mechanism to the continuous spectrum in the infinite-dimensional system. This would eventually lead to dispersion. However, as we are here working with perturbations that are not in  $L^2$ , it is not possible to compare to known dispersive solutions as was done for Schrödinger.

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**Appendix A Details of numerical methods**

Using the sinc methods described in Section 4.1 and similarly applied in [6,8], we compute the parameters for system (3.9). The convergence of these parameters, as a function of the number of grid points, is given in Table A1.

Table A1. Convergence of the ODE system parameters as a function of the number of grid points,  $M$ .

$M$	$c_*$	$\lambda'_c$	$Q_1$	$Q_2$	$Q_3$
20	0.76419938	-0.18921573	4.90659351	4.14952911	-1.21924052
40	0.76214845	-0.19352933	3.33351916	5.02642785	-1.35439215
60	0.76218663	-0.19276719	2.51228473	5.22964908	-1.36903497
80	0.76218815	-0.19262966	2.21019564	5.28041633	-1.37169953
100	0.76218822	-0.19260110	2.10124978	5.29465657	-1.37230370
200	0.76218823	-0.19259143	2.03532641	5.30133115	-1.37253316
300	0.76218823	-0.19259139	2.03442654	5.30138737	-1.37253443
400	0.76218823	-0.19259139	2.03440251	5.30138854	-1.37253445
500	0.76218823	-0.19259139	2.03440202	5.30138858	-1.37253445

The kernel functions are given in Figure A1. As discussed,  $e_{3,c_*}$ ,  $g_{2,c_*}$  and  $g_{3,c_*}$  are clearly not in  $L^2$ .

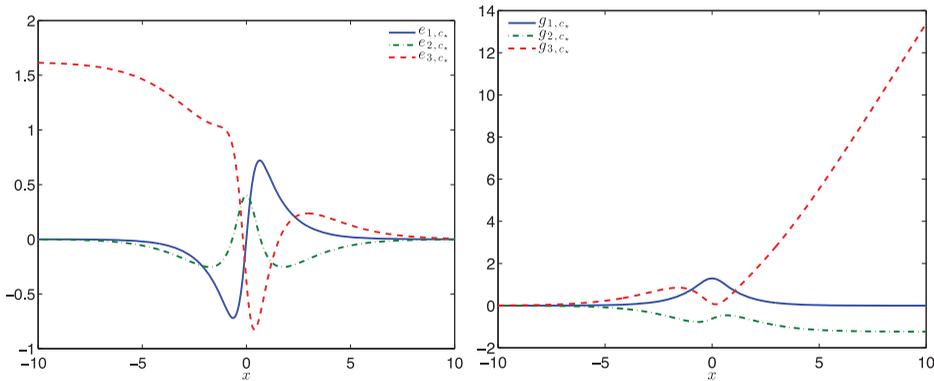


Figure A1. The elements of the generalized kernels of  $A_{c_*}$  and  $A_{c_*}$ , computed with  $M = 500$  grid points.