ASYMPTOTIC STABILITY FOR KDV SOLITONS IN WEIGHTED H^s SPACES

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Abstract. In this work, we consider the stability of solitons for the KdV equation below the energy space, using spatially-exponentially-weighted norms. Using a combination of the I-method and spectral analysis following Pego and Weinstein, we are able to show that, in the exponentially weighted space, the perturbation of a soliton decays exponentially for arbitrarily long times. The finite time restriction is due to a lack of global control of the unweighted perturbation.

1. Introduction

Consider the initial value problem for the Korteweg-de Vries equation (KdV)

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^2) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

This is a well-known nonlinear dispersive partial differential equation modelling the behavior of water waves in a long, narrow, shallow canal.

It is well known that the KdV equation (1) is completely integrable. This means that, among other things, the equation possesses infinitely many conserved quantities, the first two of which are the mass

$$M[u] = \int_{\mathbf{R}} |u(x)|^2 dx$$

and the Hamiltonian

$$H[u] = \int_{\mathbf{R}} \left(u_x^2(x) - \frac{2}{3} u^3(x) \right) dx.$$

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The equation is also known to support traveling wave solutions known as solitons. Indeed, if one assumes that the solution of (1) is given by $Q_c(x,t) = \psi_c(x-ct)$ for some profile ψ_c and some speed $c>0$, we find that the soliton is given by

$$Q_{c,x_0}(x,t) = \frac{3c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}\Big(x-ct-x_0\Big)\right). \tag{2}$$

The stability of these solitons has been an area of intense study for many years and is the main topic of this paper.

One might first be interested in the orbital stability of the soliton. In the Sobolev space $H^s = H^s(\mathbf{R})$, this means that for all $\epsilon > 0$ there is a $\delta > 0$ so that if $\|u_0 - \psi_c\|_{H^s} < \delta$, then there is a continuous function $x_0 : [0,\infty) \to \mathbf{R}$ such that $\|u(t) - \psi_c(\cdot - ct - x_0(t))\|_{H^s} < \epsilon$ for all $t \geq 0$. The study of orbital stability in the energy space H^1 began with Benjamin [1] and Bona [2]; see also [3]. This work was made systematic by Weinstein [21], who established the orbital stability of solitons for nonlinear Schrödinger equations and for generalized KdV equations. The orbital stability of solitons in H^s with $s<1$ is not as well developed. Merle and Vega [12] showed that the solitons are orbitally stable in $H^0 = L^2$ using the Miura transform together with the stability theory for kink solutions of the mKdV equation in H^1. One might expect that the orbital stability results in L^2 and H^1 imply orbital stability in H^s with $0<s<1$. However, the natural interpolation argument fails because H^s functions need not be in H^1. In the case of H^s with $0<s<1$ the I-method has been used to show that any possible orbital instability of the solitons can be at most polynomial in time; see [20] and [18].

A stronger notion of stability is asymptotic stability in which one aims to show that there exist $c_+ \in (0,\infty)$ and $x_+ \in \mathbf{R}$ so that

$$\|u(t) - \psi_{c_+}(\cdot - c_+t - x_+)\|_X \to 0 \qquad \text{as} \ \ t \to +\infty \tag{3}$$

in some Banach space X. By perturbing the main soliton ψ_{c_+} by a very small soliton located sufficiently far to the left of the main soliton, we see that this notion of asymptotic stability cannot hold in a translation invariant space X. In order to investigate asymptotic stability of solitons in the Sobolev spaces H^s, the translation invariance of the space must be broken in some way. Within the current literature there appear to be three approaches to this problem:

1. Insert a spatial weight into the Sobolev space so that movement to the left registers as decay.
2. Replace strong convergence in (3) with weak convergence.
3. Truncate the Sobolev space in an appropriate time-dependent way.

The first results on asymptotic stability for KdV solitons were established by Pego and Weinstein in [17]. In that paper, the authors considered solutions of KdV in the exponentially weighted Sobolev space $H^1_a = \{f \mid \|e^{ax}f\|_{H^1} < \infty\}$, for appropriate choice of a. They were able to prove that solitons are asymptotically stable and that the rate of decay in (3) is exponential. Mizumachi [13] has since proved that the solitons are asymptotically stable in a weighted version of H^1 with polynomial type weight; the decay rate in (3) is shown to be polynomial.

Martel and Merle [9] have shown that KdV solitons are asymptotically stable in H^1 if we replace the strong convergence in (3) with weak convergence. Martel and Merle have gone on to show that for any $\beta > 0$, KdV solitons are asymptotically stable in the truncated Sobolev space $H^1(x > \beta t)$; see [10, 11]. Merle and Vega [12] used this approach together with asymptotic stability of kink solutions to the modified KdV equation to show that KdV solitons are asymptotically stable in L^2_{loc}. Buckmaster and Koch [4] have shown that for any $\beta > 0$ KdV solitons are asymptotically stable in $H^k(x > \beta t)$ for $k \geq -1$ an integer. More recently, Mizumachi and Tzvetkov [14] have used the approach of Pego and Weinstein together with the Miura transform to show that solitons for KdV are asymptotically stable in $L^2(x > \beta t)$, thus offering an alternative proof of the result of Merle and Vega.

The principal goal of this paper is to investigate the asymptotic stability of KdV solitons in H^s with $0 < s < 1$. As in the case of orbital stability, the standard interpolation argument does not enable us to conclude that solitons are asymptotically stable in H^s for fractional values of $s < 1$. Instead we turn to the I-method and implement it in the setting of the exponentially weighted spaces used by Pego and Weinstein. To that end we define $I : H^s \to H^1$ by $\widehat{If}(\xi) = m(\xi)\widehat{f}(\xi)$ where $\widehat{f}(\xi)$ denotes the Fourier transform of f and the multiplier m is given by

$$m(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq N, \\ N^{s-1}|\xi|^{1-s}, & \text{if } |\xi| > 2N, \end{cases}$$

with smooth, even patching on the intervening intervals, and with N a parameter that we will choose during the course of our analysis. We also define the space $H^s_a = \{f \mid \|e^{ax} f\|_{H^s} < \infty\}$.

Theorem 1. *There exist $\epsilon_1 > 0$ and $0 < r < 1$ and for every $T > 0$ there exists $\epsilon_2 > 0$ so that if $\|e^{ay} I_1 v(0)\|_{H^1} < \epsilon_1$, $|c(0) - c_0| < \epsilon_1$ and $\|I_1 v(0)\|_{H^1} < \epsilon_2$, then there exist piecewise differentiable functions $c(t)$, $\gamma(t)$ and a constant $C > 0$ so that for all $t \in [0, T]$:*

1. $\|e^{ay} I_1 v(t)\|_{H^1} < C\epsilon_1 r^t$,
2. $|\dot{c}| + |\dot{\gamma}| < C\epsilon_1 r^t$, *and*
3. $|c(t) - c_0| < 2C\epsilon_1$.

Remark. Theorem 1 represents the first asymptotic stability result for the KdV equation in the Sobolev space H^s with noninteger s.

Remark. Due to the absence of good commutator estimates between the exponential weight and the I-operator, we have chosen to work in the space with norm $\|e^{ay} I_1 f\|_{H^1}$.

The key difficulty in the proof of Theorem 1 is to accommodate both the exponential weight which occurs as a weight on the spatial variable and the I-operator which occurs as a weight on the frequency variable. We proceed by first establishing local well-posendess for the exponentially weighted soliton perturbation in a space $X^{s,1/2,1}$ which embeds into the Bourgain space $X^{s,1/2+}$, partially following the local well-posedness work of Molinet and Ribaud [15, 16], and Guo and Wang [7] on dispersive-dissipative equations. In so doing we establish multilinear estimates that accommodate the presence of the exponential weight. For technical reasons, this requires that $s > 7/8$. We then use the I-method to map our solutions into an exponentially-weighted version of H^1. Finally, we run an iteration scheme inspired by an analogous argument in [19] to establish global control of the perturbation in H^s and the exponentially weighted space H^s_a, concluding that the soliton is exponentially asymptotically stable in H^s_a for $s > 7/8$.

The paper is organized as follows: In section 2, we will set up our notation and establish basic results. In section 3, we will establish some necessary estimates to establish local well-posedness in section 4. In section 5, we will run the iteration scheme and establish the main result of the paper.

2. Notation and Basic Results

We will define the Fourier multiplier operator I_N by $\widehat{I_N f}(\xi) = m_N(\xi)\hat{f}(\xi)$, with m_N a smooth, even, decreasing function of $|\xi|$ which satisfies $m_N(\xi) = 1$ for $|\xi| < N$ and $m_N(\xi) = \frac{|\xi|^{s-1}}{N^{s-1}}$ for $|\xi| > 10N$. In this paper, N will be a function of our time-step n, and, in particular

$$N(n) = \kappa^{\left(-\frac{1}{\frac{7}{4}-s}+\eta_1\right)n}$$

for $\eta_1 > 0$ very small, where $1 > \kappa > \sqrt{1-\frac{b}{2}}$, and b are defined below.

We define $\tilde{v}_n(t) = I_{N(n)}v(y,t)$ and $\tilde{w}_n(t) = e^{ay}I_{N(n)}v(y,t)$, where $y = x - \int_0^t c(s)ds - \gamma(t)$, and $c(t)$, $\gamma(t)$ are chosen so that, at each time t, for appropriate value of n, $\|\tilde{w}_n(t)\|_{L^2}$ is minimized. In order to do so, we first need to consider the difference equations satisfied by $\tilde{v}$ and $\tilde{w}$, and consider their linearizations about the soliton.

Lemma 2.1. *The perturbation $\tilde{v}$ satisfies the difference equation*

$$\begin{aligned}(\tilde{v}_n)_t &= \partial_y(-\partial_y^2 + c_0 - 2\psi_c)\tilde{v}_n + I_{N(n)}\partial_y(v^2) + \partial_y(I_{N(n)}(\psi_c v) - \psi_c I_{N(n)}v) \\ &\quad + (\dot{\gamma}\partial_y + \dot{c}\partial_c)I_{N(n)}\psi_c + (\dot{\gamma} + c - c_0)\partial_y\tilde{v}\end{aligned} \tag{4}$$

Moreover, the perturbation $\tilde{w}_n(t)$ satisfies the difference equation

$$
\begin{aligned}
(\tilde{w}_n)_t &= e^{ay}\partial_y(-\partial_y^2 + c_0 - 2\psi_c)e^{-ay}\tilde{w}_n + (c - c_0 - \dot{\gamma})(\partial_y - a)\tilde{w} \\
&\quad -e^{ay}I_{N(n)}\partial_y(v^2) - e^{ay}(\dot{c}\partial_c + \dot{\gamma}\partial_y)I_{N(n)}\psi_c - e^{ay}\partial_y(I_{N(n)}(\psi_c v) - \psi_c I_{N(n)}v)
\end{aligned} \tag{5}
$$

Proof. From [17], we have that

$$v_t = p_y(-\partial_y^2 + c_0 - 2\psi_c)v + \partial_y(v^2) + (\dot{\gamma}\partial_y + \dot{c}\partial_c)\psi_c + (\dot{\gamma} + c - c_0)\partial_y v$$

and

$$
\begin{aligned}
w_t &= e^{ay}\partial_y(-\partial_y^2 + c_0 - 2u_c)e^{-ay}w + (c - c_0)(\partial_y - a)w + [e^{ay}(\dot{c}\partial_c + \dot{\gamma}\partial_y]u_c \\
&\quad + \dot{\gamma}(\partial_y - a)w + e^{ay}\partial_y(c - c_0 + v^2)e^{-ay}w].
\end{aligned}
$$

The result here comes from applying I to each equation. □

For fixed $c > 0$, define the operator $A_a = e^{ay}\partial_y(-\partial_y^2 + c - 2\psi_c)e^{-ay}$. We have the following from [17],[19]:

Proposition 1. *For $0 < a < \sqrt{\frac{c}{3}}$, the spectrum of A_a in H^1 consists of the following:*

1. *An eigenvalue of algebraic multiplicity 2 at $\lambda = 0$. A generator of the kernel of A_a is $\zeta_1 = e^{ay}\partial_y\psi_c$, and the second generator of the generalized kernel of A_a is $\zeta_2 = e^{ay}\partial_c\psi_c$.*
2. *A continuous spectrum S^a parametrized by $\tau \to i\tau^3 - 3a\tau^2 + (c - 3a^2)i\tau - a(c - a^2)$. For any element λ of this continuous spectrum, the real part of λ is at most $b := -a(c - a^2) < 0$.*

The spectrum contains no other elements.

We also need to consider the elements of the spectrum to A_a^*, which are $\eta_1 = e^{-ay}[\theta_1\partial_y^{-1}\partial_c\psi_c + \theta_2\psi_c]$ and $\eta_2 = e^{-ay}(\theta_3\psi_c)$, where $\partial_y^{-1}f$ is defined to be $\int_{-\infty}^{y} f(t)dt$ and θ_1, θ_2 and θ_3 are appropriate constants to obtain the biorthogonality relationship $\langle\zeta_j, \eta_k\rangle = \delta_{jk}$. We will define the L^2 spectral projections $Pw = \sum_{i=1}^{2}\langle w, \eta_i\rangle\zeta_i$ and $Qw = w - Pw$ onto the discrete and continuous spectrums of A_a respectively, with respect to the fixed initial value of c, c_0.

Returning to the difference equation (5), for each fixed t we select $\dot{c}_n(t)$ and $\dot{\gamma}_n(t)$ so that $P\tilde{w}_n = 0$, and $Q\tilde{w}_n = \tilde{w}_n$. Defining $\tilde{\mathcal{F}} = (c - c_0 - \dot{\gamma})(\partial_y - a)\tilde{w} - e^{ay}I_{N(n)}\partial_y(v^2) - e^{ay}(\dot{c}\partial_c + \dot{\gamma}\partial_y)I_{N(n)}\psi_c - e^{ay}\partial_y(I_{N(n)}(\psi_c v) - \psi_c I_{N(n)}v)$, and $\tilde{\mathcal{G}} = (c - c_0)(\partial_y - a)\tilde{w} - e^{ay}I_{N(n)}\partial_y(v^2) - e^{ay}\partial_y(I_{N(n)}(\psi_c v) - \psi_c I_{N(n)}v)$ we have that

$$w_t = A_a w + Q\tilde{\mathcal{F}},$$

and

$$\mathcal{A}\begin{bmatrix}\dot{\gamma} \\ \dot{c}\end{bmatrix} = \begin{bmatrix}\langle\tilde{\mathcal{G}}, \eta_1\rangle \\ \langle\tilde{\mathcal{G}}, \eta_2\rangle\end{bmatrix}, \tag{6}$$

where $\mathcal{A}$ is the matrix

$$\mathcal{A} = \begin{bmatrix} 1 + \langle e^{ay}(\partial_y \psi_c - \partial_y \psi_{c_0}), \eta_1\rangle - \langle \tilde{w}_n, \partial_y \eta_1\rangle & \langle e^{ay}(\partial_c \psi_c - \partial_c \psi_{c_0}), \eta_1\rangle \\ \langle e^{ay}(\partial_y \psi_c - \partial_y \psi_{c_0}), \eta_2\rangle - \langle \tilde{w}_N, \partial_y \eta_2\rangle & 1 + \langle e^{ay}(\partial_c \psi_c - \partial_c \psi_{c_0}), \eta_2\rangle \end{bmatrix}.$$

3. Linear and Multilinear Estimates

In this section we will review the construction of the space $X^{s,1/2,1}$ and mention the linear estimates which were developed in [19]. At the end of this section we prove a new bilinear estimate which is then used to establish a multilinear estimate that is necessary for the proof of Theorem 1.

First, we provide a version of the product rule that holds with the multiplier operator I in place of a derivative:

Lemma 3.1. *Suppose that $\|e^{ay} f_i\|_{L^2} < \infty$ and $\|I_N \partial_y f_i\|_{L^2} < \infty$ for $i = 1, 2$. Then*

$$\|e^{ay} I_N \partial_y (f_1 f_2)\|_{L^2} \leq 2\|I_N f_1\|_{H^1} \|e^{ay} I_N \partial_y f_2\|_{L^2} + 2\|I_N f_2\|_{H^1} \|e^{ay} I_N \partial_y f_1\|_{L^2}.$$

Proof. Define $\omega_R(y) = \chi_{\{y \leq R\}} e^{ay}$, and consider $\|\omega_R I_N \partial_y (f_1 f_2)\|_{L^2}$. Taking the Fourier transform and using duality, we find that this equals

$$\sup_{\|f\|_{L^2}=1} \int\int\int_{\Gamma_4} \widehat{\omega_R}(\xi_1) m(\xi_2 + \xi_3)(\xi_2 + \xi_3) \hat{f}_1(\xi_2) \hat{f}_2(\xi_3) f(\xi_4),$$

where $\Gamma_4 = \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbf{R}^4 \mid \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0\}$. Now, either $\xi_2 + \xi_3 \leq 2\xi_2$ or $\xi_2 + \xi_3 \leq 2\xi_3$. In the first case, note that $m(\xi_2 + \xi_3)(\xi_2 + \xi_3) \leq 2m(\xi_2)\xi_2$ by the properties of m, so we have, with $\xi_5 = \xi_2 + \xi_3$ and $\xi_6 = \xi_1 + \xi_5$,

$$\begin{aligned}
\|\omega_R I_N \partial_y (f_1 f_2)\|_{L^2} &\leq 2 \sup_{\|f\|_{L^2}=1} \int\int\int_{\Gamma_4} \widehat{\omega_R}(\xi_1) m(\xi_2)\xi_2 \hat{f}_1(\xi_2) \hat{f}_2(\xi_3) f(\xi_4) \\
&= 2 \sup_{\|f\|_{L^2}=1} \int\int\int_{\Gamma_4} \widehat{\omega_R}(\xi_1) (\widehat{I\partial_y f_1})(\xi_2) \hat{f}_2(\xi_3) f(\xi_4) \\
&= 2 \sup_{\|f\|_{L^2}=1} \int \int_{\xi_1+\xi_5+\xi_4=0} \hat{f}_2(\xi_5) (\omega_R I \partial_y f_1)\widehat{\ }(\xi_5) f(\xi_4) \\
&= 2 \sup_{\|f\|_{L^2}=1} \int_{\xi_6+\xi_4=0} (f_2 \omega_R I \partial_y f_1)\widehat{\ }(\xi_6) f(\xi_4) \\
&\leq 2 \sup_{\|f\|_{L^2}=1} \|f_2 \omega_R I \partial_y f_1\|_{L^2} \|f\|_{L^2} \\
&= 2\|f_2 \omega_R I p_y f_1\|_{L^2} \\
&\leq 2\|f_2\|_{L^\infty} \|\omega_R I p_y f_1\|_{L^2} \\
&\leq 2\|f_2\|_{H^s} \|e^{ay} I p_y f_1\|_{L^2} \\
&\leq 2\|I_N f_2\|_{H^1} \|e^{ay} I p_y f_1\|_{L^2}.
\end{aligned}$$

By the symmetry between the two cases, we obtain in total that

$$\|\omega_R I_N \partial_y (f_1 f_2)\|_{L^2} \leq 2\|I_N f_2\|_{H^1} \|e^{ay} I p_y f_1\|_{L^2} + 2\|I_N f_1\|_{H^1} \|e^{ay} I p_y f_2\|_{L^2}.$$

Now, letting $R \to \infty$, since $\chi_{\{y<R\}} |e^{ay} I_N p_y (f_1 f_2)(y)|^2$ is a pointwise-increasing function in R, by the Lebesgue monotone convergence theorem we see that

$$\begin{aligned}
\|e^{ay} I_N \partial_y (f_1 f_2)\|_{L^2}^2 &= \int |e^{ay} I_N p_y (f_1 f_2)(y)|^2 dy \\
&= \lim_{R\to\infty} \int \chi_{\{y<R\}} |e^{ay} I_N p_y (f_1 f_2)(y)|^2 \\
&= \lim_{R\to\infty} \|\omega_R I_N \partial_y (f_1 f_2)\|_{L^2}^2 \\
&\leq \lim_{R\to\infty} (2\|I_N f_2\|_{H^1} \|e^{ay} I p_y f_1\|_{L^2} + 2\|I_N f_1\|_{H^1} \|e^{ay} I p_y f_2\|_{L^2})^2 \\
&= (2\|I_N f_2\|_{H^1} \|e^{ay} I p_y f_1\|_{L^2} + 2\|I_N f_1\|_{H^1} \|e^{ay} I p_y f_2\|_{L^2})^2
\end{aligned}$$

as claimed. □

We next recall the definition of the space $X^{s,1/2,1}$. We define the sets A_j and B_k by

$$\begin{aligned}
A_j &:= \{(\tau, \xi) \in \mathbf{R}^2 \mid 2^j \leq \langle \xi \rangle \leq 2^{j+1}\}, \qquad j \geq 0 \\
B_k &:= \{(\tau, \xi) \in \mathbf{R}^2 \mid 2^k \leq \langle \tau - \xi^3 \rangle \leq 2^{k+1}\}, \qquad k \geq 0.
\end{aligned}$$

For $s, b \in \mathbf{R}$, the space $X^{s,b,1}$ is defined to be the completion of the Schwartz class functions in the norm

$$\|f\|_{X^{s,b,1}} := \left(\sum_{j\geq 0} 2^{2sj} \left(\sum_{k\geq 0} 2^{bk} \|\widetilde{f}\|_{L^2(A_j \cap B_k)} \right)^2 \right)^{1/2}.$$

In taking $b = 1/2$ we have the following embeddings:

$$X^{s,1/2+} \hookrightarrow X^{s,1/2,1} \hookrightarrow C_t^0 H_x^s.$$

We will work primarily in the spaces $X^{s,1/2,1}$ and $X^{s,-1/2,1}$, so we adopt the notation $X^s := X^{s,1/2,1}$ and $Y^s := X^{s,-1/2,1}$.

The spaces X^s, Y^s were used in the case when $s = 1$ to prove local well-posedness for the perturbations v and $w = e^{ay} v$ in $H^1(\mathbf{R})$, see [19]. We review some of the features of these spaces that were used in the aforementioned local well-posedness arguments. Let $W_1(t)$ denote the standard Airy evolution,

$$(W_1(t) f)\widehat{\ }(\xi) = e^{-it\xi^3} \widehat{f}(\xi).$$

Let $W_2(t)$ be the linear evolution defined for $t \geq 0$ by

$$(W_2(t) f)\widehat{\ }(\xi) = e^{-it\xi^3 - p_a(\xi)t} \widehat{f}(\xi),$$

where $p_a(\xi) = 3a\xi^2 + a(c_0^2 - a)$. We extend this to all of $t \in \mathbf{R}$ in defining

$$(W_2(t) f)\widehat{\ }(\xi) = e^{-i\xi^3 t - p_a(\xi)|t|} \widehat{f}(\xi).$$

While the Airy evolution $W_1(t)$ is the linear evolution associated with the unweighted perturbation v, the evolution $W_2(t)$ is the linear evolution associated with the weighted perturbation w. A key feature of the space X^s is that it accommodates both of the semigroups $W_1(t)$ and $W_2(t)$, as illustrated in the following linear estimates which are valid for all $s \in \mathbf{R}$:

$$\|\rho(t)W_1(t)f\|_{X^{s,1/2,1}} \lesssim \|f\|_{H^s}, , \tag{7}$$

$$\left\|\rho(t)\int_0^t W_1(t-s)F(s)ds\right\|_{X^{s,1/2,1}} \lesssim \|F\|_{X^{s,-1/2,1}}, , \tag{8}$$

and if $0 < a \leq \min(1, c_0)$, then

$$\|\rho(t)W_2(t)f\|_{X^{s,1/2,1}} \lesssim \|f\|_{H^s}, \tag{9}$$

$$\left\|\chi_{\mathbf{R}_+}(t)\rho(t)\int_0^t W_2(t-s)F(s)ds\right\|_{X^{s,1/2,1}} \lesssim \|F\|_{X^{s,-1/2,1}}. \tag{10}$$

Here $\rho : \mathbf{R} \to \mathbf{R}$ is a cutoff function such that

$$\rho \in C_0^\infty(\mathbf{R}), \qquad \operatorname{supp} \rho \subset [-2,2], \qquad \rho \equiv 1 \text{ on } [-1,1], \tag{11}$$

and $\chi_{\mathbf{R}_+}$ is the indicator function for the set $\mathbf{R}_+ := \{t \in \mathbf{R} \mid t \geq 0\}$. The estimates (7), (8) are proved in [8] while the proofs of (9), (10) are given in [19]. Also crucial for the result proved in [19] was the following bilinear estimate, valid for all $s \geq 0$ (see Proposition 3 in [19]):

$$\|uv_y\|_{Y^s} \lesssim \|u\|_{X^s}\|v\|_{X^s}. \tag{12}$$

In the case when $s = 1$ we have the following generalization of this result.

Proposition 2. *Let $\alpha_1 \in (3/4, 1], \alpha_2 \in (0, 1]$ and suppose that $u \in X^{\alpha_1}, v \in X^{\alpha_2}$. Then*

$$\|u_y v\|_{Y^1} \lesssim \|u\|_{X^{\alpha_1}}\|v\|_{X^{\alpha_2}}. \tag{13}$$

Proof. Since we work primarily in frequency space, we define $\widetilde{X}^{s,b,1}$ to be the completion of the Schwartz class functions in the norm

$$\|f\|_{\widetilde{X}^{s,b,1}} := \left(\sum_{j\geq 0} 2^{2sj}\left(\sum_{k\geq 0} 2^{bk}\|f\|_{L^2(A_j\cap B_k)}\right)^2\right)^{1/2}.$$

Here $f = f(\tau, \xi)$ is a function of the frequency variables τ and ξ. Adopting the notation $\widetilde{X}^1 = \widetilde{X}^{1,1/2,1}$ and $\widetilde{Y}^1 = \widetilde{X}^{1,-1/2,1}$, the estimate (13) reads

$$\|(|\xi_1|f) * g\|_{\widetilde{Y}^1} \lesssim \|f\|_{\widetilde{X}^{\alpha_1}}\|g\|_{\widetilde{X}^{\alpha_2}}.$$

Following the proof of the standard bilinear estimate (12) we decompose f and g on dyadic blocks as follows: Define $f_{j_1,k_1} := \chi_{A_{j_1}}\chi_{B_{k_1}} f$ and $g_{j_2,k_2} :=$

$\chi_{A_{j_2}}\chi_{B_{k_2}}g$. We thus have

$$f=\sum_{j_1\geq 0}\sum_{k_1\geq 0} f_{j_1,k_1} \qquad \text{and} \qquad g=\sum_{j_2\geq 0}\sum_{k_2\geq 0} g_{j_2,k_2}.$$

Our goal is to estimate

$$\sum_{j\geq 0} 2^{2j}\left(\sum_{k\geq 0}\sum_{j_1\geq 0}\sum_{k_1\geq 0}\sum_{j_2\geq 0}\sum_{k_2\geq 0} 2^{-k/2}2^{j_1}\|f_{j_1,k_1}*g_{j_2,k_2}\|_{L^2(A_j\cap B_k)}\right)^2. \quad (14)$$

Indeed, we wish to establish an estimate of the form

$$(14)\lesssim \|f\|^2_{\widetilde{X}^{\alpha_1}}\|g\|^2_{\widetilde{X}^{\alpha_2}}.$$

To simplify the exposition we adopt the following notation:

$$F_{j_1,k_1} := 2^{\alpha_1 j_1}2^{k_1/2}\|f_{j_1,k_1}\|_{L^2}, \qquad \text{and}$$
$$G_{j_2,k_2} := 2^{\alpha_2 j_2}2^{k_2/2}\|g_{j_2,k_2}\|_{L^2}.$$

The proof is divided into the following cases:

(1) At least two of j, j_1, j_2 are less than 20.
(2) $j_1, j_2 \geq 20$ and $j < j_1 - 10$.
(3) $j, j_1 \geq 20, |j - j_1| \leq 10$.

Case (1). Here we may assume that $j, j_1, j_2 \leq 30$. Applying Young's inequality followed by Hölder's inequality yields

$$\|f_{j_1,k_1}*g_{j_2,k_2}\|_{L^2}\lesssim 2^{j_2/2}2^{15k_1/32}2^{15k_2/32}\|f_{j_1,k_1}\|_{L^2}\|g_{j_2,k_2}\|_{L^2}.$$

After summing in k and summing over j (a finite sum), we find that

$$(14)\lesssim\left(\sum_{j_1=0}^{30}\sum_{k_1\geq 0}2^{j_1}2^{15k_1/32}\|f_{j_1,k_1}\|_{L^2}\right)^2\left(\sum_{j_2=0}^{30}\sum_{k_2\geq 0}2^{j_2/2}2^{15k_2/32}\|g_{j_2,k_2}\|_{L^2}\right)^2.$$

Note that the sum in j_2 is finite, so

$$\begin{aligned}
&\sum_{j_2=0}^{30}\sum_{k_2\geq 0}2^{j_2/2}2^{15k_2/32}\|g_{j_2,k_2}\|_{L^2}\\
\leq&\left(\sum_{j_2=0}^{30}2^{(1-2\alpha_2)j_2}\right)^{1/2}\left(\sum_{j_2=0}^{30}2^{2\alpha_2 j_2}\left(\sum_{k_2\geq 0}G_{j_2,k_2}\right)^2\right)^{1/2}\\
\lesssim&\|g\|_{\widetilde{X}^{\alpha_2}}.
\end{aligned}$$

A similar argument shows that

$$\sum_{j_1=0}^{30}\sum_{k_1\geq 0}2^{j_1}2^{15k_1/32}\|f_{j_1,k_1}\|_{L^2}\lesssim\|f\|_{\widetilde{X}^{\alpha_1}},$$

which completes the argument.

Case (2). We may assume that $|j_1 - j_2| \leq 1$, since otherwise $f_{j_1} * g_{j_2} = 0$ on A_j. For $(\tau_1, \xi_1) \in A_{j_1} \cap B_{k_1}$ and $(\tau_2, \xi_2) \in A_{j_2} \cap B_{k_2}$ we have

$$(\tau_1 + \tau_2) - (\xi_1 + \xi_2)^3 - (\tau_1 - \xi_1^3) - (\tau_2 - \xi_2^3) = -3\xi\xi_1\xi_2. \tag{15}$$

It follows that $f_{j_1,k_1} * g_{j_2,k_2} = 0$ on $A_j \cap B_k$ unless $2^{k_{max}} \gtrsim 2^j 2^{j_1} 2^{j_2} \sim 2^{j+2j_1}$ where $k_{max} = \max\{k, k_1, k_2\}$.

Suppose that $k = k_{max}$. In order for $f_{j_1,k_1} * g_{j_2,k_2}$ to have low frequency support we require that whenever $(\tau_1, \xi_1) \in \text{supp } f_{j_1,k_1}, (\tau_2, \xi_2) \in \text{supp } g_{j_2,k_2}$, ξ_1 and ξ_2 must have opposite signs. It follows that supp f_{j_1} and supp g_{j_2} are separated by $K \sim 2^{j_1}$. In light of Lemma 3.3 in [19], we thus have

$$\|f_{j_1,k_1} * g_{j_2,k_2}\|_{L^2(A_j \cap B_k)} \lesssim 2^{-j/2} 2^{-j_1/2} 2^{-\alpha_1 j_1} 2^{-\alpha_2 j_2} F_{j_1,k_1} G_{j_2,k_2}.$$

Therefore, using $2^{-k/2} \lesssim 2^{-j/2-j_1}$, we have

$$\begin{aligned}
(14) &\lesssim \sum_{j\geq 0} \left(\sum_{j_1 \geq j+11} \sum_{k_1 \geq 0} \sum_{j_2 = j_1 - 1}^{j_1+1} \sum_{k_2 \geq 0} 2^{-j_1/2} 2^{-\alpha_1 j_1} 2^{-\alpha_2 j_2} F_{j_1,k_1} G_{j_2,k_2} \right)^2 \\
&\lesssim \sum_{j \geq 0} 2^{-j/2} \left(\sum_{j_1 \geq 0} \sum_{k_1 \geq 0} \sum_{j_2 \geq 0} \sum_{k_2 \geq 0} 2^{-j_1/8 - \alpha_1 j_1} 2^{-j_2/8 - \alpha_2 j_2} F_{j_1,k_1} G_{j_2,k_2} \right)^2 \\
&\lesssim \|f\|^2_{\widetilde{X}^{\alpha_1}} \|g\|^2_{\widetilde{X}^{\alpha_2}}.
\end{aligned}$$

Next we suppose that $k_1 = k_{max}$. In this case we require $2^{k_1} \gtrsim 2^{j+2j_1}$. We apply Lemma 3.4 from [19] with $K \sim 2^{j_1}$ to see that

$$\|f_{j_1,k_1} * g_{j_2,k_2}\|_{L^2(A_j \cap B_k)} \lesssim 2^{k/2} 2^{-j_1} 2^{-k_1/2} 2^{-\alpha_1 j_1} 2^{-\alpha_2 j_2} F_{j_1,k_1} G_{j_2,k_2}.$$

Observe that

$$2^{-k_1/2} \lesssim 2^{-k/16} 2^{-7k_1/16} \lesssim 2^{-k/16} 2^{-7j/16} 2^{-7j_1/8}.$$

It follows that

$$\begin{aligned}
(14) &\lesssim \sum_{j \geq 0} 2^{-j/16} \left(\sum_{\substack{j_1 \geq 0 \\ k_1 \geq 0}} \sum_{\substack{j_2 \geq 0 \\ k_2 \geq 0}} 2^{-j_1/8} 2^{-j_2/8} 2^{-\alpha_1 j_1} 2^{-\alpha_2 j_2} F_{j_1,k_1} G_{j_2,k_2} \right)^2 \\
&\lesssim \|f\|^2_{\widetilde{X}^{\alpha_1}} \|g\|^2_{\widetilde{X}^{\alpha_2}}.
\end{aligned}$$

Finally we consider the case when $k_2 = k_{max}$. Since the expression to be estimated is symmetric in (j_1, k_1) and (j_2, k_2), we can argue as in the case where $k_1 = k_{max}$ to obtain the desired estimate.

Case (3). In this case we may assume that $j_2 \leq j + 11$. In light of (15) we require $2^{k_{max}} \gtrsim 2^{2j+j_2}$. We begin by assuming that $k = k_{max}$. Lemma 3.3

from [19] gives

$$\|f_{j_1,k_1} * g_{j_2,k_2}\|_{L^2(A_j\cap B_k)} \lesssim 2^{-j/4}2^{-\alpha_1 j_1}2^{-\alpha_2 j_2}F_{j_1,k_1}G_{j_2,k_2}.$$

Therefore, since $2^{-k/2} \lesssim 2^{-j-j_2/2}$, we find

$$\begin{aligned}(14) &\lesssim \sum_{j\geq 0} 2^{-j\epsilon}\left(\sum_{j_1=j-10}^{j+10}\sum_{k_1\geq 0}\sum_{j_2=0}^{j+11}\sum_{k_2\geq 0} 2^{j_1(\frac{3}{4}-\alpha_1+)}2^{(-\alpha_2-1/2)j_2}F_{j_1,k_1}G_{j_2,k_2}\right)^2\\ &\lesssim \|f\|^2_{\widetilde{X}^{\alpha_1}}\|g\|^2_{\widetilde{X}^{\alpha_2}},\end{aligned}$$

provided $\alpha_1 > 3/4$ and $\epsilon > 0$ is chosen appropriately small.

Suppose that $k_1 = k_{max}$, meaning that $2^{k_1} \gtrsim 2^{2j+j_2}$. We apply Lemma 3.4 from [19] to estimate

$$\|f_{j_1,k_1} * g_{j_2,k_2}\|_{L^2(A_j\cap B_k)} \lesssim 2^{k/4}2^{-j_1/4}2^{-\alpha_1 j_1}2^{-\alpha_2 j_2}2^{-k_1/2}F_{j_1,k_1}G_{j_2,k_2}.$$

After using $2^{-k_1/2} \lesssim 2^{-j-j_2/2}$

$$\begin{aligned}(14) &\lesssim \sum_{j\geq 0} 2^{-j\epsilon}\left(\sum_{j_1\geq 0}\sum_{k_1\geq 0}\sum_{j_2\geq 0}\sum_{k_2\geq 0} 2^{j_1(-\alpha_1+\epsilon+\frac{3}{4})}2^{-j_2/2}2^{-\alpha_2 j_2}F_{j_1,k_1}G_{j_2,k_2}\right)^2\\ &\lesssim \|f\|^2_{\widetilde{X}^{\alpha_1}}\|g\|^2_{\widetilde{X}^{\alpha_2}},\end{aligned}$$

again provided $\alpha_1 > 3/4$ and $\epsilon > 0$ is chosen to be sufficiently small.

Finally we consider the case for which $k_2 = k_{max}$, so that $2^{k_2} \gtrsim 2^{2j+j_2}$. We divide our analysis into the following two subcases:

(i) $|j_2 - j| \leq 5$.
(ii) $|j_2 - j| > 5$.

In case (i) we use Lemma 3.4 from [19] to estimate

$$\|f_{j_1,k_1} * g_{j_2,k_2}\|_{L^2(A_j\cap B_k)} \lesssim 2^{k/4}2^{-j_2/4}2^{-k_2/2}2^{-\alpha_1 j_1}2^{-\alpha_2 j_2}F_{j_1,k_1}G_{j_2,k_2}.$$

We thus obtain

$$\begin{aligned}(14) &\lesssim \sum_{j\geq 0}\left(\sum_{j_1=j-10}^{j+10}\sum_{k_1=0}^{k_2}\sum_{\substack{j_2\geq 0\\ |j-j_2|\leq 5}}\sum_{k_2\geq 0} 2^{j_1}2^{-3j_2/4}2^{-\alpha_1 j_1}2^{-\alpha_2 j_2}F_{j_1,k_1}G_{j_2,k_2}\right)^2\\ &\lesssim \sum_{j\geq 0} 2^{-j/2}\left(\sum_{j_1\geq 0}\sum_{k_1\geq 0}\sum_{j_2\geq 0}\sum_{k_2\geq 0} 2^{j_1(-1/4-\alpha_1)}2^{j_2(-1/4-\alpha_2)}F_{j_1,k_1}G_{j_2,k_2}\right)^2\\ &\lesssim \|f\|^2_{\widetilde{X}^{\alpha_1}}\|g\|^2_{\widetilde{X}^{\alpha_2}}.\end{aligned}$$

In case (ii) we again use Lemma 3.4 with $K \sim 2^j$ to estimate

$$\|f_{j_1,k_1} * g_{j_2,k_2}\|_{L^2(A_j\cap B_k)} \lesssim 2^{k/2}2^{-j/2}2^{-j_2/2}2^{-k_2/2}2^{-\alpha_1 j_1}2^{-\alpha_2 j_2}F_{j_1,k_1}G_{j_2,k_2}.$$

Next we estimate

$$2^{-k_2/2} \lesssim 2^{-k/16}2^{-7k_2/16} \lesssim 2^{-k/16}2^{-7j/8}2^{-7j_2/16}.$$

We thus find that

$$\begin{aligned}(14) &\lesssim \sum_{j\geq 0}\left(\sum_{j_1=j-10}^{j+10}\sum_{k_1\geq 0}\sum_{j_2=0}^{j-5}\sum_{k_2\geq 0} 2^{j_1}2^{-3j/8}2^{-15j_2/16}2^{-\alpha_1 j_1}2^{-\alpha_2 j_2}F_{j_1,k_1}G_{j_2,k_2}\right)^2\\ &\lesssim \sum_{j\geq 0}2^{-j/8}\left(\sum_{j_1\geq 0}\sum_{k_1\geq 0}\sum_{j_2\geq 0}\sum_{k_2\geq 0} 2^{j_1(-\alpha_1+9/16)}2^{j_2(-\alpha_2-15/16)}F_{j_1,k_1}G_{j_2,k_2}\right)^2\\ &\lesssim \|f\|^2_{\widetilde{X}^{\alpha_1}}\|g\|^2_{\widetilde{X}^{\alpha_2}},\end{aligned}$$

since $\alpha_1 > 3/4$. □

In the proof of the modified local well-posedness result we will require the following estimate.

Proposition 3. *Let $s > 7/8$. Suppose that u and v are spacetime functions such that $u, v \in X^s$ and $e^{ay}Iu, e^{ay}Iv \in X^1$. Then*

$$\begin{aligned}&\left\|e^{ay}\partial_y\Big(I(uv) - IuIv\Big)\right\|_{Y^1}\\ \lesssim& N^{\frac{3}{4}-s+}\left(\|e^{ay}Iu\|_{X^1}\|Iv\|_{X^1} + \|Iu\|_{X^1}\|e^{ay}Iv\|_{X^1}\right).\end{aligned} \tag{16}$$

Remark. Since $s > 7/8$ we see that (16) implies

$$\begin{aligned}&\left\|e^{ay}\partial_y\Big(I(uv) - IuIv\Big)\right\|_{Y^1}\\ \lesssim& N^{-1/8+}\left(\|e^{ay}Iu\|_{X^1}\|Iv\|_{X^1} + \|Iu\|_{X^1}\|e^{ay}Iv\|_{X^1}\right).\end{aligned}$$

Proof of Proposition 3. For a function $u(t,x)$ of spacetime we let u_{N_j} denote the function whose Fourier transform is given by $\widehat{u}_{N_j} = \eta_{A_j}(\xi)\widehat{u}(\xi)$, where η_{A_j} is a smooth cutoff function adapted to the set $A_j := \{\xi \in \mathbf{R} \mid |\xi| \sim N_j\}$ with N_j dyadic.

We truncate the exponential weight using a spatial cutoff function. Specifically, for $R > 1$ we let $\vartheta_R : \mathbf{R} \to \mathbf{R}$ by

$$\vartheta_R(y) = \begin{cases} 1, & y < R\\ 0, & y > R,\end{cases}$$

and define $\omega_{a,R}(y) := \vartheta_R(y)e^{ay}$. Observe that $\omega_{a,R} \in H^s(\mathbf{R})$ for all $s \in \mathbf{R}$; in particular, it makes sense to speak of the Fourier transform of $\omega_{a,R}$. Furthermore, we have the following approximation result.

Lemma 3.2. *If $f \in H^1_a(\mathbf{R})$, then*

$$\lim_{R\to\infty}\|\omega_{a,R}f\|_{H^1} = \|e^{ay}f\|_{H^1}.$$

Proof. Arguing as in the proof of Lemma 3.1, we find that

$$\lim_{R\to\infty} \|\omega_{a,R} f\|_{L^2} = \|e^{ay} f\|_{L^2}. \tag{17}$$

Observe that $\|e^{ay} f\|_{H^1}^2 = \|e^{ay} f\|_{L^2}^2 + \|e^{ay}(af + f_y)\|_{L^2}^2$. One also checks that

$$\|\omega_{a,R} f\|_{H^1}^2 = \|\omega_{a,R} f\|_{L^2}^2 + \|\omega_{a,R}(af + f_y)\|_{L^2}^2.$$

In light of this calculation and (17), we obtain the conclusion of the lemma. □

To prove (16) it suffices to show that

$$\begin{aligned} &\|\widehat{g}_{N_1} |\xi_2+\xi_3| \big(m(\xi_2+\xi_3) - m(\xi_2)m(\xi_3)\big) \widehat{u}_{N_2} \widehat{v}_{N_3}\|_{\widetilde{Y}^1} \\ \lesssim &N^{\frac{3}{4}-s+} \Big(N_{12}^{0-} N_3^{0-} \|g_{N_1} I u_{N_2}\|_{X^1} \|I v_{N_3}\|_{X^1} \\ &\qquad + N_2^{0-} N_{13}^{0-} \|I u_{N_2}\|_{X^1} \|g_{N_1} I v_{N_3}\|_{X^1} \Big), \end{aligned} \tag{18}$$

where $g := \omega_{a,R}$. Note that by symmetry we may assume that $N_2 \geq N_3$. We adopt the notation N_{12} for $|\xi_1+\xi_2| \sim N_{12}$ when $|\xi_1| \sim N_1$ and $|\xi_2| \sim N_2$. We adopt similar definitions for N_{13} and N_{23}.

Case (1). $N_2 \ll N$. In this case we see that $m(\xi_2+\xi_3) - m(\xi_2)m(\xi_3) = 0$, so the expression to be estimated vanishes.

Case (2). $N_2 \gtrsim N \gg N_3$. We use the mean value theorem to see that

$$|m(\xi_2+\xi_3) - m(\xi_2)m(\xi_3)| \lesssim \frac{N_3}{N_2} m(N_2) m(N_3).$$

It follows that

$$\begin{aligned} &\big\|g_{N_1} |\xi_2+\xi_3| \big(m(\xi_2+\xi_3) - m(\xi_2)m(\xi_3)\big)\big\|_{\widetilde{Y}^1} \\ \lesssim &\frac{N_3}{N_2} \Big\|\widehat{g}_{N_1} |\xi_2+\xi_3| \widehat{Iu}_{N_2} \widehat{Iv}_{N_3}\Big\|_{\widetilde{Y}^1} \\ \lesssim &\frac{N_3}{N_2} \Big(\|g_{N_1} I u_{N_2} \partial_y I v_{N_3}\|_{Y^1} + \|g_{N_1} I v_{N_3} \partial_y I u_{N_2}\|_{Y^1} \Big) \\ \lesssim &\frac{N_3}{N_2} \Big(\|g_{N_1} I u_{N_2}\|_{X^{3/4+}} \|I v_{N_3}\|_{X^{3/4+}} + \|g_{N_1} I v_{N_3}\|_{X^{3/4+}} \|I u_{N_2}\|_{X^{3/4+}} \Big) \\ \lesssim &\frac{N_3}{N_2 \langle N_{12}\rangle^{1/4-} \langle N_3\rangle^{1/4-}} \|g_{N_1} I u_{N_2}\|_{X^1} \|I v_{N_3}\|_{X^1} \\ &+ \frac{N_3}{N_2 \langle N_{13}\rangle^{1/4-} \langle N_2\rangle^{1/4-}} \|g_{N_1} I v_{N_3}\|_{X^1} \|I u_{N_2}\|_{X^1} \end{aligned}$$

Notice that

$$\frac{N_3}{N_2 \langle N_{12}\rangle^{1/4-} \langle N_3\rangle^{1/4-}} \lesssim \frac{N_3^{3/4+}}{N_2 \langle N_{12}\rangle^{1/4-}} \lesssim N^{-1/4+} N_{12}^{0-} N_3^{0-},$$

and

$$\frac{N_3}{N_2\langle N_{13}\rangle^{1/4-}\langle N_2\rangle^{1/4-}} \lesssim \frac{N_3^{3/4+}}{N_2\langle N_{13}\rangle^{1/4-}} \lesssim N^{-1/4+}N_2^{0-}N_{13}^{0-}.$$

Case (3). $N_2 \geq N_3 \gtrsim N$. Here we split the expression to be estimated into two terms which are then estimated separately:

$$\begin{aligned}
&\|\widehat{g}_{N_1}|\xi_2+\xi_3|(m(\xi_2+\xi_3)-m(\xi_2)m(\xi_3))\widehat{u}_{N_2}\widehat{v}_{N_3}\|_{\widetilde{Y}^1}\\
\lesssim&\|g_{N_1}|\xi_2+\xi_3|m(\xi_2+\xi_3)\widehat{u}_{N_2}\widehat{v}_{N_3}\|_{\widetilde{Y}^1}\\
+&\|g_{N_1}|\xi_2+\xi_3|\widehat{Iu}_{N_2}\widehat{Iv}_{N_3}\|_{\widetilde{Y}^1}\\
=:&\text{ Term I } + \text{ Term II.}
\end{aligned}$$

We estimate Term II as in Case (2) to see that

$$\begin{aligned}
\text{Term II} \lesssim& \frac{1}{\langle N_{12}\rangle^{1/4-}\langle N_3\rangle^{1/4-}}\|g_{N_1}Iu_{N_2}\|_{X^1}\|Iv_{N_3}\|_{X^1}\\
&+\frac{1}{\langle N_{13}\rangle^{1/4-}\langle N_2\rangle^{1/4-}}\|g_{N_1}Iv_{N_3}\|_{X^1}\|Iu_{N_2}\|_{X^1},
\end{aligned}$$

which is sufficient. Turning to Term I, we have

$$\begin{aligned}
\text{Term I} \lesssim& \, m(N_{23})\Big(\|g_{N_1}u_{N_2}\partial_y v_{N_3}\|_{Y^1}+\|g_{N_1}v_{N_3}\partial_y u_{N_2}\|_{Y^1}\Big)\\
\lesssim& \, m(N_{23})\Big(\|g_{N_1}u_{N_2}\|_{X^{3/4+}}\|v\|_{X^{3/4+}}+\|g_{N_1}v_{N_3}\|_{X^{3/4+}}\|u_{N_2}\|_{X^{3/4+}}\Big)\\
\lesssim& \frac{m(N_{23})}{\langle N_{12}\rangle^{1/4-}m(N_2)\langle N_3\rangle^{s-3/4-}}\|g_{N_1}Iu_{N_2}\|_{X^1}\|v_{N_3}\|_{X^s}\\
+& \frac{m(N_{23})}{\langle N_{13}\rangle^{1/4-}m(N_3)\langle N_2\rangle^{s-3/4-}}\|g_{N_1}Iv_{N_3}\|_{X^1}\|u_{N_2}\|_{X^s}\\
\lesssim& \frac{m(N_{23})}{\langle N_{12}\rangle^{1/4-}m(N_2)\langle N_3\rangle^{s-3/4-}}\|g_{N_1}Iu_{N_2}\|_{X^1}\|Iv_{N_3}\|_{X^1}\\
+& \frac{m(N_{23})}{\langle N_{13}\rangle^{1/4-}m(N_3)\langle N_2\rangle^{s-3/4-}}\|g_{N_1}Iv_{N_3}\|_{X^1}\|Iu_{N_2}\|_{X^1},
\end{aligned}$$

where in the final inequality we have used that $\|f\|_{X^s} \lesssim \|If\|_{X^1}$. Observe that since $N_2 \geq N_3$ and $s > 3/4$ we have

$$\langle N_2\rangle^{s-3/4-}m(N_3) \gtrsim N_3^{2s-7/4-}N^{1-s} \geq N^{s-3/4-},$$

since $s > 7/8$. It follows that

$$\frac{m(N_{23})}{\langle N_{13}\rangle^{1/4-}m(N_3)\langle N_2\rangle^{s-3/4-}} \lesssim N^{\frac{3}{4}-s+}N_{13}^{0-}N_2^{0-}. \tag{19}$$

To estimate the other multiplier expression we first note that if $N_{23} \gtrsim N_3$, then $m(N_{23}) \lesssim m(N_3)$ so that

$$\frac{m(N_{23})}{\langle N_{12}\rangle^{1/4-} m(N_2)\langle N_3\rangle^{s-3/4-}} \lesssim \frac{1}{\langle N_{12}\rangle^{1/4-}\langle N_3\rangle^{s-3/4-}},$$

which is acceptable. If $N_{23} \ll N_3$, then we must have $N_2 \sim N_3$ (with the relevant factors being supported at frequencies of opposite sign), in which case may estimate $\langle N_3\rangle m(N_2) \gtrsim N^{s-3/4-}$. The estimate is then completed as above in (19). $\square$

From Proposition 3 we have the following result.

Corollary 1. *Under the hypotheses of Proposition 3 we have*

$$\begin{aligned}&\left|\int_{t_0}^{t_0+\delta} \Big\langle e^{ay} Iv, e^{ay}\partial_y\Big(I(uv) - IuIv\Big)\Big\rangle_{H^1} dt\right| \\ \lesssim& N^{3/4-s+}\|e^{ay}Iv\|_{X^1}\left(\|e^{ay}Iu\|_{X^1}\|Iv\|_{X^1} + \|Iu\|_{X^1}\|e^{ay}Iv\|_{X^1}\right).\end{aligned}$$

Proof. We apply Cauchy-Schwartz together with the embedding $X^{1,1/2+} \hookrightarrow X^{1,1/2,1}$ to see that

$$\begin{aligned}&\left|\int_{t_0}^{t_0+\delta} \langle e^{ay} Iv, e^{ay}\partial_y\Big(I(uv) - IuIv\Big)\rangle_{H^1} dt\right| \\ \lesssim& \|e^{ay}Iv\|_{X^1}\|e^{ay}\partial_y(I(uv) - IuIv)\|_{Y^1} \\ \lesssim& N^{3/4-s+}\|e^{ay}Iv\|_{X^1}\Big(\|e^{ay}Iu\|_{X^1}\|Iv\|_{X^1} + \|Iu\|_{X^1}\|e^{ay}Iv\|_{X^1}\Big).\end{aligned}$$

$\square$

4. Modified Local Well-Posedness

This section is devoted to the proof of local well-posedness for the $\widetilde{v}$-equation and the $\widetilde{w}$-equation. We make the change of variables $y \mapsto y + \gamma(t) + \int_0^t c(s)ds$ and find that the initial value problem for $\widetilde{v} = I_N v$ is given by

$$\begin{cases} \partial_t\widetilde{v} + \partial_y^3\widetilde{v} + I_N\partial_y(v^2) + \partial_y(\psi_c\widetilde{v}) + I_N\partial_y(\psi_c v) + (\dot\gamma\partial_y + \dot c\partial_c)I_N\psi_c = 0, \\ \widetilde{v}(0,y) = \widetilde{v}_0(y). \end{cases} \tag{20}$$

The equation for $\widetilde{w} = e^{ay}I_N v$ is given by the modulation equation

$$\partial_t\widetilde{w} = A_a\widetilde{w} + Q\widetilde{\mathcal{F}},$$

where $A_a = e^{ay}\partial_y(-\partial_y^2 + c_0 - 2\psi_c)e^{-ay}$, Q is the spectral projection, and

$$\begin{aligned}\widetilde{\mathcal{F}} =& (c - c_0 + \dot\gamma)(\partial_y - a)\widetilde{w} - e^{ay}I_N\partial_y(v^2) - e^{ay}(\dot\gamma\partial_y + \dot c\partial_c)I_N\psi_c \\ &- e^{ay}\partial_y\left(I_N(\psi_c v) - \psi_c I_N v\right).\end{aligned}$$

Upon expanding the operator A_a, we find that the initial value problem for $\widetilde{w}$ is

$$\begin{cases} \partial_t \widetilde{w} + \partial_y^3 \widetilde{w} - 3a\partial_y^2 \widetilde{w} + (3a^2 - c_0)\partial_y \widetilde{w} + a(c_0 - a^2)\widetilde{w} \\ \quad +2(\partial_y - a)(\psi_c \widetilde{w}) - Q\widetilde{\mathcal{F}} = 0, \\ \widetilde{w}(0, y) = \widetilde{w}_0(y). \end{cases} \tag{21}$$

Before we proceed with our local well-posedness argument, we define the time-localized space X^s_δ to be the space with the norm

$$\|u\|_{X^s_\delta} := \inf\{\|w\|_{X^s} \mid w \equiv u \text{ on } [0, \delta]\}.$$

The main goal of this section is to prove the following modified local well-posedness result:

Proposition 4. *Let $0 < a < \sqrt{c_0/3}$, $s > 7/8$, and $N > 1$. There is an $r > 0$ such that the following statement holds: If $v_0 \in H^s(\mathbf{R})$ satisfies $\|\widetilde{v}_0\|_{H^1} < r$ and $\|\widetilde{w}\|_{H^1} < r$ where $\widetilde{v}_0 = I_N v_0$ and $\widetilde{w}_0 = e^{ay} I_N v$, then there is a $\delta > 0$ so that the initial value problems* (20) *and* (21) *admit solutions $\widetilde{v}(t, y), \widetilde{w}(t, y)$, respectively, on $[0, \delta]$. Moreover these solutions satisfy*

$$\|\widetilde{v}\|_{X^1_\delta} \lesssim \|\widetilde{v}_0\|_{H^1}, \qquad \textit{and} \qquad \|\widetilde{w}\|_{X^1} \lesssim \|\widetilde{w}_0\|_{H^1}.$$

Proof. Let $\rho : \mathbf{R} \to \mathbf{R}$ be a smooth cutoff function, as in (11), and let $\rho_\delta(\cdot) = \rho(\cdot/\delta)$. We begin by rewriting the equation for $\widetilde{v}(t, y)$, (20), using Duhamel's formula:

$$\widetilde{v} = W_1(t)\widetilde{v}_0 + \int_0^t W_1(t-s)\Big(I_N \partial_y(v^2) + 2\partial_y(\psi_c \widetilde{v}) + \partial_y(I_N(\psi_c v) - \psi_c I_N v)\Big)$$
$$+ \int_0^t W_1(t-s)(\dot{\gamma}\partial_y + \dot{c}\partial_c) I_N \psi_c ds.$$

We will show that the map Φ given by

$$\Phi\widetilde{v} := \rho_\delta(t) W_1(t)\widetilde{v}_0 + \rho_\delta(t) \int_0^t W_1(t-s)\Big(I_N \partial_y(v^2) + 2\partial_y(\psi_c \widetilde{v})\Big) ds$$
$$+ \rho_\delta(t) \int_0^t W_1(t-s)\Big(\partial_y(I_N(\psi_c v) - \psi_c I_N v) + (\dot{\gamma}\partial_y + \dot{c}\partial_c) I_N \psi_c\Big) ds$$

is a contraction on a small ball in X^1_δ. We estimate $\Phi\widetilde{v}$ in X^1_δ using (7) and (8):

$$\begin{aligned} \|\Phi\widetilde{v}\|_{X^1_\delta} &\lesssim \|\widetilde{v}_0\|_{H^1} + \|I_N \partial_y(v^2)\|_{Y^1_\delta} + \|\partial_y(\psi_c \widetilde{v})\|_{Y^1_\delta} \\ &\quad + \|\partial_y(I_N(\psi_c v) - \psi_c I_N v)\|_{Y^1_\delta} + \|(\dot{\gamma}\partial_y + \dot{c}\partial_c) I_N \psi_c\|_{Y^1_\delta} \\ &=: \|\widetilde{v}_0\|_{H^1} + \text{Term I} + \text{Term II} + \text{Term III} + \text{Term IV}. \end{aligned}$$

To estimate Term I we first note that

$$\|I_1 \partial_y(v^2)\|_{Y^1_\delta} \sim \|\partial_y(v^2)\|_{Y^s_\delta} \lesssim \|v\|^2_{X^s_\delta} \sim \|I_1 v\|^2_{X^1_\delta}.$$

In light of Lemma 12.1 from [**?**] we may conclude that

$$\|I_N\partial_y(v^2)\|_{Y^1_\delta} \lesssim \|I_N v\|^2_{X^1_\delta} = \|\widetilde{v}\|^2_{X^1}.$$

To estimate Term II we use the bilinear estimate (12) to see that

$$\text{Term II} \lesssim \|\psi_c\|_{X^1_\delta}\|\widetilde{v}\|_{X^1_\delta}.$$

Recall that for $\delta, \epsilon > 0$ sufficiently small we have

$$\|\psi_c\|_{X^1_\delta} \lesssim \delta^\epsilon.$$

Thus

$$\text{Term II} \lesssim \delta^\epsilon\|\widetilde{v}\|_{X^1_\delta}.$$

Turning to Term III we argue as for Terms I and II to find that

$$\text{Term III} \lesssim \|\partial_y I_N(\psi_c v)\|_{Y^1_\delta} + \|\partial_y(\psi_c I_N v)\|_{Y^1_\delta} \lesssim \delta^\epsilon\|\widetilde{v}\|_{X^1_\delta}.$$

Finally, for Term IV we recall that from the modulation equations we have

$$\|\dot{\gamma}\|_{L^\infty_t}, \|\dot{c}\|_{L^\infty_t} \lesssim \|\widetilde{w}\|_{X^1_\delta}$$

so that

$$\text{Term IV} \lesssim \delta^\epsilon\|\widetilde{w}\|_{X^1_\delta}.$$

Taken all together we have

$$\|\Phi\widetilde{v}\|_{X^1_\delta} \lesssim \|\widetilde{v}_0\|_{H^1} + \|\widetilde{v}\|^2_{X^1_\delta} + \delta^\epsilon\|\widetilde{v}\|_{X^1_\delta} + \delta^\epsilon\|\widetilde{w}\|_{X^1_\delta}. \tag{22}$$

For the $\widetilde{w}$ equation we expand the spectral projection $Qf = f - \sum_{j=1}^2\langle f, \eta_j\rangle\zeta_j$ and make the change of variables $y \mapsto y - ((3a^2 - c_0)t + \gamma(t) - \int_0^t c(s)ds)$, so that the equation for $\widetilde{w}$ reads

$$\begin{aligned}&\partial_t\widetilde{w} + \partial_y^3\widetilde{w} - 3a\partial_y^2\widetilde{w} + a(c_0 - a^2 - c + c_0)\widetilde{w} - a\dot{\gamma}\widetilde{w} - e^{ay}I_N\partial_y(v^2)\\ &- e^{ay}(\dot{\gamma}\partial_y + \dot{c}\partial_c)I_N\psi_c - e^{ay}\partial_y(I_N(\psi_c v) - \psi_c I_N v)\\ &+ \langle\widetilde{\mathcal{F}}, \eta_1\rangle\zeta_1 + \langle\widetilde{\mathcal{F}}, \eta_2\rangle\zeta_2 = 0.\end{aligned}$$

Rewriting this equation using Duhamel's formula leads us to define the following operator

$$\begin{aligned}\Psi\widetilde{w} &= \rho_\delta(t)W_2(t)\widetilde{w}_0 + \rho_\delta(t)\int_0^t W_2(t-s)\Big(2(\partial_y - a)(\rho_\delta^2\psi_c\widetilde{w}) + a\rho_\delta\dot{\gamma}\widetilde{w}\Big)ds\\ &+\rho_\delta(t)\int_0^t W_2(t-s)\Big(a(c-c_0)\rho_\delta\widetilde{w} - e^{ay}I_N\partial_y(\rho_\delta^2 v^2)\Big)ds\\ &+\rho_\delta(t)\int_0^t W_2(t-s)\Big(-e^{ay}(\dot{\gamma}\partial_y + \dot{c}\partial_c)\rho_\delta I_N\psi_c + e^{ay}\partial_y(I_N(\psi_c v) - \psi_c I_N v)\Big)ds\\ &+\rho_\delta(t)\int_0^t W_2(t-s)\Big(\rho_\delta\langle\widetilde{\mathcal{F}}, \eta_1\rangle\zeta_1 + \rho_\delta\langle\widetilde{\mathcal{F}}, \eta_2\rangle\zeta_2\Big)ds,\end{aligned}$$

which we hope to show is a contraction on a ball in X^1_δ. We estimate $\Psi\widetilde{w}$ in X^1_δ using (9) and (10), which yields

$$\begin{aligned}\|\Psi\widetilde{w}\|_{X^1_\delta} &\lesssim \|\widetilde{w}_0\|_{H^1} + \|(\partial_y - a)\rho_\delta^2\psi_c\widetilde{w}\|_{Y^1_\delta} + \|\rho_\delta\dot{\gamma}\widetilde{w}\|_{Y^1_\delta} + \|(c-c_0)\rho_\delta\widetilde{w}\|_{Y^1_\delta}\\ &\quad + \|e^{ay}I_N\partial_y(\rho_\delta^2 v^2)\|_{Y^1_\delta} + \|e^{ay}(\dot{\gamma}\partial_y + \dot{c}\partial_c)\rho_\delta I_N\psi_c\|_{Y^1_\delta}\\ &\quad + \|e^{ay}\partial_y(I_N(\psi_c v) - \psi_c I_N v)\|_{Y^1_\delta} + \|\rho_\delta\langle\widetilde{\mathcal{F}},\eta_1\rangle\zeta_1\|_{Y^1_\delta} + \|\rho_\delta\langle\widetilde{\mathcal{F}},\eta_2\rangle\zeta_2\|_{Y^1_\delta}\\ &= \|\widetilde{w}_0\|_{H^1} + \text{Term I} + \text{Term II} + \text{Term III} + \text{Term IV}\\ &\qquad + \text{Term V} + \text{Term VI} + \text{Term VII} + \text{Term VIII}.\end{aligned}$$

To estimate Term I we use $e^{ay}\partial_y e^{-ay} = \partial_y - a$, $\widetilde{v} = e^{-ay}\widetilde{w}$, and the bilinear estimate (12) to see that

$$\begin{aligned}\text{Term I} &= \|e^{ay}\partial_y e^{-ay}\psi_c\widetilde{w}\|_{Y^1_\delta} = \|e^{ay}\partial_y\psi_c\widetilde{v}\|_{Y^1_\delta}\\ &\le \|e^{ay}\widetilde{v}\partial_y\psi_c\|_{Y^1_\delta} + \|e^{ay}\psi_c\partial_y\widetilde{v}\|_{Y^1_\delta}\\ &\lesssim \|\widetilde{w}\|_{X^1_\delta}\|\psi_c\|_{X^1_\delta} + \|e^{ay}\psi_c\|_{X^1_\delta}\|\widetilde{v}\|_{X^1_\delta}\\ &\lesssim \delta^\epsilon\|\widetilde{w}\|_{X^1_\delta} + \delta^\epsilon\|\widetilde{v}\|_{X^1_\delta}.\end{aligned}$$

In estimating Term II we use that $\|\dot{\gamma}\|_{L^\infty_t} \lesssim \|\widetilde{w}\|_{X^1_\delta}$, which gives

$$\text{Term II} \lesssim \|\widetilde{w}\|^2_{X^1_\delta}.$$

In order to estimate Term III we note that

$$|c(t) - c_0| \le \int_0^t |\dot{c}(s)|ds \lesssim \int_0^t \|\widetilde{w}(s)\|_{H^1_x} ds \lesssim \|\widetilde{w}\|_{L^1_t H^1_x}.$$

Since we are restricted to the interval $[0,\delta]$, Hölder's inequality gives

$$|c(t) - c_0| \le \delta^{1/2}\|\widetilde{w}\|_{L^2_t H^1_x} \lesssim \delta^{1/2}\|\widetilde{w}\|_{X^1_\delta}.$$

It follows that

$$\text{Term III} \lesssim \|c - c_0\|_{L^\infty_t}\|\widetilde{w}\|_{X^1_\delta} \lesssim \|\widetilde{w}\|^2_{X^1_\delta}.$$

To estimate Term IV we use (16) and (12) to see that

$$\begin{aligned}\text{Term IV} &\le \|e^{ay}\partial_y(I_N(\rho_\delta^2 v^2) - \rho_\delta^2(I_N v)^2)\|_{Y^1_\delta} + \|e^{ay}\partial_y(I_N v)^2\|_{Y^1_\delta}\\ &\lesssim \|e^{ay}I_N v\|_{X^1_\delta}\|I_N v\|_{X^1_\delta}\\ &= \|\widetilde{w}\|_{X^1_\delta}\|\widetilde{v}\|_{X^1_\delta}.\end{aligned}$$

The estimate for Term V is similar to the one we used for the analogous term in the $\widetilde{v}$ equation (term (IV)), yielding

$$\text{Term V} \lesssim \delta^\epsilon\|\widetilde{w}\|_{X^1_\delta}.$$

Term VI is estimated using (16), (12), and the fact that $\|I_N\psi_c - \psi_c\|_{X^1_\delta} \lesssim N^{-C}$ with C as large as need be:

$$\begin{aligned}\text{Term VI} &\leq \|e^{ay}\partial_y(I_N(\psi_c v) - I_N\psi_c I_N v)\|_{Y^1_\delta} + \|e^{ay}\partial_y(\psi_c - I_N\psi_c)I_N v\|_{Y^1_\delta} \\ &\lesssim N^{-1/8+}\delta^\epsilon\|\widetilde{v}\|_{X^1_\delta} + N^{-1/8+}\delta^\epsilon\|\widetilde{w}\|_{X^1_\delta} + N^{-C}\|\widetilde{v}\|_{X^1_\delta} + N^{-C}\|\widetilde{w}\|_{X^1_\delta},\end{aligned}$$

leaving us with

$$\text{Term VI} \lesssim \delta^\epsilon\|\widetilde{v}\|_{X^1_\delta} + \delta^\epsilon\|\widetilde{w}\|_{X^1_\delta}.$$

Turning to Terms VII and VIII we recall from Lemma 3.5 in [19] that

$$\|\langle f, \eta_j\rangle\zeta_j\|_{Y^1_\delta} \lesssim \|f\|_{Y^1_\delta}, \qquad j = 1, 2.$$

It follows that

$$\text{Term VII, Term VIII} \lesssim \|\widetilde{\mathcal{F}}\|_{Y^1_\delta} \lesssim \|\widetilde{w}\|^2_{X^1_\delta} + \|\widetilde{v}\|_{X^1_\delta}\|\widetilde{w}\|_{X^1_\delta} + \delta^\epsilon\|\widetilde{w}\|_{X^1_\delta} + \delta^\epsilon\|\widetilde{v}\|_{X^1_\delta}.$$

Altogether, then, we have

$$\|\Psi\widetilde{w}\|_{X^1_\delta} \lesssim \|\widetilde{w}_0\|_{H^1} + \delta^\epsilon\|\widetilde{w}\|_{X^1_\delta} + \delta^\epsilon\|\widetilde{v}\|_{X^1_\delta} + \|\widetilde{w}\|^2_{X^1_\delta} + \|\widetilde{w}\|_{X^1_\delta}\|\widetilde{v}\|_{X^1_\delta}.$$

Suppose that $\|\widetilde{v}_0\|_{H^1}, \|\widetilde{w}_0\|_{H^1} < r \ll 1$ and let

$$\mathcal{B} = \left\{\widetilde{v}, \widetilde{w} \in X^1_\delta \mid \|\widetilde{v}\|_{X^1_\delta} \leq 2cr,\ \|\widetilde{w}\|_{X^1_\delta} \leq 2cr\right\}.$$

Using the estimates that we have established, it transpires that $\Phi, \Psi : \mathcal{B} \to \mathcal{B}$ are contractions following the arguments from Proposition 4 of [19]. The desired result follows. $\square$

5. Iteration

In this section, we prove the main result of the paper, namely the exponential decay of the weighted perturbation given in Theorem 1. We will prove the result by induction. Define $\dot{c}_n$ and $\dot{\gamma}_n$ by (6), and let the variable y be defined accordingly as $y = x - \int_0^t c(s)ds - \gamma(t)$. Let $T > 0$ be given. Let $\kappa = (\max(1-b, \frac{3}{4}))^{\frac{1}{2+\frac{1-s}{s-\frac{3}{4}-}}-}$. Let $N(n) = \kappa^{(-\frac{1}{\frac{3}{4}-s+}+)n}$. Now, let ϵ_1 and c_2 be sufficiently small so that, whenever $\|e^{ay}I_{N(n)}w(t_n)\|_{H^1} < 2\epsilon_1$ and $\|I_{N(n)}v(t_n)\|_{H^1} < c_2$, it follows that $v(t)$ exists on $[t_0, t_0+\delta]$, and

$$\|w\|_{X^{1,b}_{[t_0,t_0+\delta]}} < C_0\epsilon_1 \qquad \text{and} \qquad \|v\|_{X^{1,b}_{[t_0,t_0+\delta]}} < C_0 c_2, \tag{23}$$

where C_0 is the implicit constant in the conclusion of Proposition 4. Additionally, assume that $c_2 < \frac{b}{10}$. Let $n_0 = \frac{T}{\delta}$. Finally, choose ϵ_2 sufficiently small that $Cr^{\frac{n_0}{2}}\epsilon_2 < c_2$, with r to be expressed later.

We must recall the known control on v. In [18] it is proven that, with $H(f) = \int |\partial_x f|^2 - \frac{2}{3} f^3$,

$$\|\tilde{v}_n(n)\|_{H^1}^2 \sim H(\psi + \tilde{v}_n(n)) - \left(\frac{\|\psi + \tilde{v}_n(n)\|_{L^2}}{\|\psi\|_{L^2}} \right)^{\frac{10}{3}} H(\psi)$$
$$= H(\psi + \tilde{v}_n(n)) - H(\psi) + (1 - \left(\frac{\|\psi + \tilde{v}_n(n)\|_{L^2}}{\|\psi\|_{L^2}} \right)^{\frac{10}{3}})H(\psi).$$

Then, since $H(\psi)$ is constant and $(1 - \left(\frac{\|\psi+\tilde{v}_n(n)\|_{L^2}}{\|\psi\|_{L^2}} \right)^{\frac{10}{3}})$ is very small ($\mathcal{O}(N^{-100})$, e.g.), it suffices to increment $H(\psi + \tilde{v}_n(n))$. It is then found in [18], as in [20], that $H(\psi + \tilde{v}_n(n+1)) - H(\psi + \tilde{v}_n(n)) \sim N(n)^{-1+} \|\tilde{v}_n(n)\|_{H^1}^2$. Therefore, when we increment $\tilde{v}_n$, we obtain that

$$\begin{aligned}
&\|\tilde{v}_{n+1}(n+1)\|_{H^1}^2 - \|\tilde{v}_n(n)\|_{H^1}^2 \\
=&\|\tilde{v}_{n+1}(n+1)\|_{H^1}^2 - \|\tilde{v}_n(n+1)\|_{H^1}^2 + \|\tilde{v}_n(n+1)\|_{H^1}^2 - \|\tilde{v}_n(n)\|_{H^1}^2 \\
\lesssim& \left(\frac{N(n+1)}{N(n)} \right)^{1-s} - 1)\|\tilde{v}_n(n+1)\|_{H^1}^2 + \|\tilde{v}_n(n+1)\|_{H^1}^2 - \|\tilde{v}_n(n)\|_{H^1}^2 \\
\lesssim& \left(\frac{N(n+1)}{N(n)} \right)^{1-s} - 1)(\|\tilde{v}_n(n+1)\|_{H^1}^2 - \|\tilde{v}_n(n)\|_{H^1}^2) + \|\tilde{v}_n(n+1)\|_{H^1}^2 \\
&\quad - \|\tilde{v}_n(n)\|_{H^1}^2 + \left(\frac{N(n+1)}{N(n)} \right)^{1-s} - 1)\|\tilde{v}_n(n)\|_{H^1}^2 \\
=& \left(\frac{N(n+1)}{N(n)} \right)^{1-s} (\|\tilde{v}_n(n+1)\|_{H^1}^2 - \|\tilde{v}_n(n)\|_{H^1}^2) \\
&\quad + \left(\frac{N(n+1)}{N(n)} \right)^{1-s} - 1)\|\tilde{v}_n(n)\|_{H^1}^2 \\
\leq& \left(\frac{N(n+1)}{N(n)} \right)^{1-s} (N(n)^{-1+}\|\tilde{v}_n(n)\|_{H^1}^2 + \left(\frac{N(n+1)}{N(n)} \right)^{1-s} - 1)\|\tilde{v}_n(n)\|_{H^1}^2 \\
=&(\kappa^{(-\frac{1-s}{\alpha+1-s}+\eta_1)}(N(n)^{-1+} + 1) - 1)\|\tilde{v}_n(n)\|_{H^1}.
\end{aligned}$$

Therefore, for n large,

$$\|\tilde{v}_{n+1}(n+1) \lesssim \kappa^{\frac{1-s}{\frac{3}{4}-s+}+}(N(n)^{-1+} + 1)\|\tilde{v}_n(n)\|_{H^1} \leq r\|\tilde{v}_n(n)\|_{H^1}^2,$$

where $r = 1.01\kappa^{\frac{1-s}{\frac{3}{4}-s+}+}$ is slightly larger than 1. Hence it follows that

$$\|\tilde{v}_n(n)\|_{H^1}^2 \leq Cr^n \epsilon_2^2. \tag{24}$$

Hence it follows that $\|\tilde{v}_n(t)\|_{H^1} < c_2$ on J_n for $0 \leq n \leq n_0$.

With all these preliminaries complete, we can state the induction lemma:

Lemma 5.1. *Define* $\tilde{w}_n(t,y) = e^{ay} I_{N(n)} v(t,y)$ *and* $\tilde{v}_n(t,y) = I_{N(n)} v(t,y)$ *on the time interval* $J_n := [t_n, t_{n+1})$*,where* $t_n = n\delta$*. Suppose* $\|\tilde{w}(0)\|_{H^1} < \epsilon_1$*,* $\|\tilde{v}(0)\|_{H^1} < \epsilon_2$*, and* $|c(0) - c_0| < \epsilon_1$*. Then, for all* $n \in \mathbf{N}$*, the following hold:*

(1) *Define* $c(t)$ *inductively starting at* $c(0)$ *by* $c(t) = c(t_n) + \int_{t_n}^{t} \dot{c}_n(t)dt$ *for* $t \in [t_n, t_{n+1})$*, and similarly for* $\gamma(t)$*. Then* $\dot{c}_n$ *and* $\dot{\gamma}_n$ *are continuous on* J_n *for all* n*, and* c, γ *are continuous functions of* t*.*
(2) $|\dot{c}_n(t_n)| < C\epsilon_1 \kappa^n$*,*
(3) $|\dot{\gamma}_n(t_n)| < C\epsilon_1 \kappa^n$*,*
(4) $|c(t_n) - c_0| < C\frac{1-\kappa^n}{1-\kappa}\epsilon_1$*, and*
(5) $\|\tilde{w}_n(t_n)\|_{H^1} < \epsilon_1 \kappa^n$*,*

where $C = 2\max\{(2 + \|u\|_{L^\infty} + \|p_y u\|_{L^\infty})(\|\eta_1\|_{L^2} + \|\eta_2\|_{L^2}), C_0^{\frac{3}{2}}, 1\}$*.*

Proof. Note that, for $n = 0$, $t = 0$ and $N(0) = 1$, so (4)-(5) are verified by hypothesis. Also note that the smoothness of $\dot{c}_n$ and $\dot{\gamma}_n$ on each J_n is a standard application of the implicit function theorem. Then c and γ are continuous by construction, so (1) holds for all n. Finally, we need to verify (2)-(3) at $n = 0$ in order to begin the induction. Note that

$$\begin{bmatrix} \dot{\gamma} \\ \dot{c} \end{bmatrix} = \mathcal{A}\left(\begin{bmatrix} \langle \tilde{\mathcal{G}}, \eta_1 \rangle_{L^2} \\ \langle \tilde{\mathcal{G}}, \eta_2 \rangle_{L^2} \end{bmatrix} \right),$$

where

$$\mathcal{A} = \left(\begin{bmatrix} 1 + \langle e^{ay}(\partial_y \psi_c - \partial_y \psi_{c_0}), \eta_1 \rangle - \langle \tilde{w}, \partial_y \eta_1 \rangle & \langle e^{ay}(\partial_c \psi_c - \partial_c \psi_{c_0}), \eta_1 \rangle \\ \langle e^{ay}(\partial_y \psi_c - \partial_y \psi_{c_0}), \eta_2 \rangle - \langle \tilde{w}, \partial_y \eta_2 \rangle & 1 + \langle e^{ay}(\partial_c \psi_c - \partial_c \psi_{c_0}), \eta_2 \rangle \end{bmatrix} \right)^{-1}.$$

At any time when $|c - c_0|$ and $\|\tilde{w}_n\|_{H^1}$ are sufficiently small, it follows that $\|\mathcal{A}\| \leq 2$, so that

$$\left| \begin{bmatrix} \dot{\gamma} \\ \dot{c} \end{bmatrix} \right| \leq 2 \left| \begin{bmatrix} \langle \tilde{\mathcal{G}}, \eta_1 \rangle_{L^2} \\ \langle \tilde{\mathcal{G}}, \eta_2 \rangle_{L^2} \end{bmatrix} \right| \leq 2(\max_{i=1,2} \|\eta_i\|_{H^1}) \|\tilde{\mathcal{G}}\|_{L^2}.$$

Finally, by Lemma 3.1

$$\begin{aligned} \|\tilde{\mathcal{G}}\|_{L^2} &= \|(c - c_0)(\partial_y - a)\tilde{w} - e^{ay} I(v^2)_y - e^{ay}\partial_y[I(uv) - uIv]\|_{L^2} \\ &\leq |c - c_0|\|\tilde{w}\|_{H^1} + \|e^{ay} I(v^2)_y\|_{L^2} + \|e^{ay}\partial_y[I(uv) - uIv]\|_{L^2} \\ &\leq |c - c_0|\|\tilde{w}\|_{H^1} + \|Iv\|_{H^1}\|e^{ay} I\partial_y v\|_{L^2} + 2\|u\|_{L^\infty}\|e^{ay} I\partial_y v\|_{L^2} \\ &\quad + \|\partial_y u\|_{L^\infty}\|e^{ay} p_y Iv\|_{L^2} \\ &\leq (|c - c_0| + \|Iv\|_{H^1} + 2\|u\|_{L^\infty} + \|\partial_y u\|_{L^\infty})\|\tilde{w}\|_{H^1} \\ &\leq (2 + 2\|u\|_{L^\infty} + \|\partial_y u\|_{L^\infty})\|\tilde{w}\|_{H^1} \end{aligned}$$

so long as $|c - c_0|$ and $\|Iv\|_{H^1}$ are at most unit size. Therefore (2)-(3) are satisfied at $t = 0$ because of our assumptions on the initial data, given our choice of C above.

It remains to make the inductive step. Assume that, at step n, (1)-(5) are valid. In order to step forward in time, we must first gain some a priori control of the various functions on the interval J_n. Without loss of generality, assume $\delta \leq 1$. Select η so that $24C\epsilon_1 < \eta^2$ and $\eta + c_2 < \frac{1}{20}$ (and assume ϵ_2 is sufficiently small to allow this). Define $L(t) = 8C\|\tilde{w}\|_{H^1} + |\dot{c}| + |\dot{\gamma}| + |c - c_0|$. Note that at $t = n$, $L(n) < 11C\epsilon_1 < \frac{\eta}{2}$. Hence, by continuity, there is a $\delta_0 > 0$ so that $L(t) < \eta$ on $[t_n, t_n + \delta_0)$. Let δ_1 be the largest such δ_0 which is at most δ. We want to show that $\delta_1 = \delta$. Suppose not; then $\delta_1 < \delta$. Then $L(t_n + \delta_1) = \eta$ by continuity. Define $J = [t_n, t_n + \delta_1]$. On J, as above, we have that $\dot{c} + \dot{\gamma} < C\|\tilde{w}\|_{H^1} < \frac{\eta}{6}$. Moreover, $|c - c_0(t)| \leq |c(n) - c_0| + \delta_1 \sup_J |\dot{c}| \leq 2C\epsilon_1 + \frac{\eta}{4} \leq \frac{\eta}{12} + \frac{\eta}{6} = \frac{\eta}{4}$. Finally, we must estimate $\|\tilde{w}(t_n + \delta_1)\|_{H^1}$.

We have:

$$
\begin{aligned}
\|\tilde{w}(t_n + \delta_1)\|_{H^1}^2 &= \|\tilde{w}(t_n)\|_{H^1}^2 + \int_J \frac{d}{dt}\|\tilde{w}\|_{H^1}^2 dt \\
&= \|\tilde{w}(t_n)\|_{H^1}^2 + 2\int_J \langle \tilde{w}, \tilde{w}_t \rangle_{H^1} dt \\
&= \|\tilde{w}(t_n)\|_{H^1}^2 + 2\int_J \langle \tilde{w}, A_a\tilde{w} + Q\mathcal{F} \rangle_{H^1} dt \\
&\leq \epsilon_1 + 2\int_J \langle \tilde{w}, A_a\tilde{w} \rangle_{H^1} dt + \int_J \langle \tilde{w}, Q\mathcal{F} \rangle_{H^1} dt \\
&\leq \epsilon_1 - \frac{2b\eta^2}{64C^2} + \int_J \langle \tilde{w}, Q\mathcal{F} \rangle_{H^1} dt \\
&\leq \frac{\eta^2}{20} - \frac{2b\eta^2}{64C^2} + \int_J \langle \tilde{w}, Q\mathcal{F} \rangle_{H^1} dt
\end{aligned}
$$

by Proposition 1, the inductive hypothesis, the a priori control on $\tilde{w}$ on J, and the fact that the length of J is at most 1. It remains to estimate

$$
\begin{aligned}
&\int_J \langle \tilde{w}, Q\mathcal{F} \rangle_{H^1} dt \\
=&\int_J \langle \tilde{w}, Q((c - c_0 - \dot{\gamma})(\partial_y - a)\tilde{w} - e^{ay} I_{N(n)}\partial_y(v^2) + e^{ay}(\dot{c}\partial_c + \dot{\gamma}\partial_y) I_{N(n)}\psi_c \\
&\qquad - e^{ay}\partial_y(I_{N(n)}(\psi_c v) - \psi_c I_{N(n)} v)) \rangle_{H^1} dt \\
=&\text{(I)+(II)+(III)+(IV)}.
\end{aligned}
$$

For (I), note that $Q(\partial_y - a)\tilde{w} = (\partial_y - a)\tilde{w}$, and ∂_y is anti-symmetric, so (I)$=\int_J ((c - c_0) - \dot{\gamma})(-a)\|w\|_{H^1}^2 dt$, which is at most $\frac{2a\eta^3}{64C^2}$, which is certainly less than $\frac{\eta}{20}$. For (II), we have

$$
\begin{aligned}
&\int_J \langle \tilde{w}, -e^{ay} I_{N(n)} \partial_y (v^2) \rangle_{H^1} dt \\
=&\int_J \langle \tilde{w}, e^{ay} \partial_y [I_{N(n)} v^2 - (I_{N(n)} v)^2] \rangle_{H^1} dt + \int_J \langle \tilde{w}, e^{ay} \partial_y (I_{N(n)} v)^2 \rangle_{H^1} dt \\
\leq &\int_J \langle \tilde{w}, e^{-ay} \partial_y [I_{N(n)} (v^2) - (I_{N(n)} v)^2] \rangle_{H^1} dt + \|\tilde{w}\|_{X^{1,\frac{1}{2},1}} \|e^{ay} \partial_y (I_{N(n)} v)^2\|_{X^{1,-\frac{1}{2},1}} \\
\leq &2N(n)^{-\frac{1}{4}} \|\tilde{w}\|_{X^{1,\frac{1}{2},1}} \|\tilde{w}\|_{X^{1,\frac{1}{2},1}} \|\tilde{v}\|_{X^{1,\frac{1}{2},1}} + \|\tilde{w}\|_{X^{1,\frac{1}{2},1}} \|\tilde{w}\|_{X^{1,\frac{1}{2},1}} \|\tilde{v}\|_{X^{1,\frac{1}{2},1}} \\
\leq &(1 + 2N(n))^{-\frac{1}{4}}) \|\tilde{w}\|^2_{X^{1,\frac{1}{2},1}} \|\tilde{v}\|_{X^{1,\frac{1}{2},1}} \\
\leq &(1 + 2N(n))^{-\frac{1}{4}} C_0^3 \epsilon_1^2 c_2 \\
\leq &\frac{1}{5760} \frac{C_0^3}{C^2} \eta^2 \\
\leq &\frac{\eta^2}{20},
\end{aligned}
$$

by Corollary 1, Proposition 2, and the local well-posedness estimate (23). For (III), recall that $I_{N(n)} \psi_c - \psi_c = \mathcal{O}(N^{-C})$ for C arbitrarily large. So, since $Q(e^{ay} \partial_c \psi_{c_0}) = Q(e^{ay} \partial_y \psi_{c_0}) = 0$, we have

$$
\begin{aligned}
\text{(III)} &= \int_J \langle \tilde{w}, Q[e^{ay} (\dot{c} \partial_c + \dot{\gamma} \partial_y)((I_N(n) - 1)[\psi_c - \psi_{c_0}] + [\psi_c - \psi_{c_0}] + \psi_{c_0}))] \rangle_{H^1} dt \\
&= \int_J \langle \tilde{w}, Q[e^{ay} (\dot{c} \partial_c + \dot{\gamma} \partial_y)((I_N(n) - 1)[\psi_c - \psi_{c_0}] + [\psi_c - \psi_{c_0}])] dt \\
&\leq (1 + \tilde{C} N^{-\tilde{\tilde{C}}}) \int_J (|\dot{c}| + |\dot{\gamma}|) |c - c_0| \|\tilde{w}\|_{H^1} dt \\
&\leq \tilde{C} \frac{\eta}{4} \frac{\eta}{3} \frac{\eta}{8C} \\
&\leq \frac{\eta^2}{20}.
\end{aligned}
$$

Finally, for (IV), we have

$$\begin{aligned}
&\int_J \langle \tilde{w}, e^{ay}\partial_y(I_{N(n)}(\psi_c v) - \psi_c I_{N(n)} v))\rangle_{H^1} dt \\
&\quad = \int_J \langle \tilde{w}, e^{ay}\partial_y[I_{N(n)}(\psi_c v) - (I_{N(n)}\psi_c)(I_{N(n)} v] dt \\
&\qquad + \int_J \langle \tilde{w}, e^{ay}\partial_y[((I_{N(n)}\psi_c) - \psi_c)(I_{N(n)} v] dt \\
&\quad \leq \|\tilde{w}\|_{X^{1,\frac{1}{2},1}} N^{-\frac{1}{4}} (\|e^{ay} I_{N(n)}\psi_c\|_{X^{1,\frac{1}{2},1}} \|\tilde{v}\|_{X^{1,\frac{1}{2},1}} + \|I_{N(n)}\psi_c\|_{X^{1,\frac{1}{2},1}} \|\tilde{w}\|_{X^{1,\frac{1}{2},1}}) \\
&\qquad + \tilde{C} N^{-\tilde{\tilde{C}}} \|\tilde{w}\|_{X^{1,\frac{1}{2},1}} [\|e^{ay}\psi_c\|_{X^{1,\frac{1}{2},1}} \|\tilde{v}\|_{X^{1,\frac{1}{2},1}} + \|\psi_c\|_{X^{1,\frac{1}{2},1}} \|\tilde{w}\|_{X^{1,\frac{1}{2},1}}] \\
&\quad \leq 4\|\tilde{w}\|_{X^{1,\frac{1}{2},1}} (N^{-\frac{1}{4}} \|\tilde{v}\|_{X^{1,\frac{1}{2},1}} + \|\tilde{w}\|_{X^{1,\frac{1}{2},1}})) \\
&\quad \leq 4C_0\epsilon_1(c_2 + \eta) \\
&\quad \leq \frac{1}{120}\frac{C_0}{C}\eta^2 \\
&\quad \leq \frac{\eta^2}{20}
\end{aligned}$$

Adding it all together, we get that

$$\|\tilde{w}(t_n + \delta_1)\|_{H^1}^2 \leq \frac{\eta^2}{20} - \frac{2b\eta^2}{64C^2} + \frac{\eta^2}{20} + \frac{\eta^2}{20} + \frac{\eta^2}{20} + \frac{\eta^2}{20} < \frac{\eta^2}{4},$$

so, $L(t_n + \delta_1) < \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{2} = \eta$,and hence $\delta_1 = \delta$.

Now we are ready to make the inductive step. Consider (2)-(5) at time t_{n+1}. As above, we have that $|\dot{c}_n(t_{n+1})| + |\dot{\gamma}_n(t_{n+1})| \leq 2C\|\tilde{w}(t_{n+1})\|_{H^1}$, so (2) and (3) are validated whenever (5) is. Indeed, the estimates (2)-(3) hold on the entire interval J_n whenever $\|w\|_{H^1}$ is similarly controlled on the interval. Similarly, whenever (2) is valid on J_n, we have

$$\begin{aligned}
|c(t_{n+1}) - c_0| &\leq |c(t_n) - c_0| + \int_{J_n} |\dot{c}_n(t)| dt \\
&\leq C\frac{1-\kappa^n}{1-\kappa}\epsilon_1 + C\kappa^n\epsilon_1 \\
&\leq C\frac{1-\kappa^{n+1})}{1-\kappa}\epsilon_1,
\end{aligned}$$

so (4) is also validated. It therefore remains only to control $\|w_n(t)\|_{H^1}$ on J_n and estimate $\|w_{n+1}(n+1)\|_{H^1}^2 - \|w_n(n)\|_{H^1}^2$. We must therefore do two things: Estimate $\|w_{n+1}(n+1)\|_{H^1}^2 - \|w_n(n+1)\|_{H^1}^2$, and estimate $\|w_n(t)\|_{H^1}^2$ on J_n. In what follows, for notational simplicity, we will estimate $\|w_n(t_{n+1})\|_{H^1}^2$, but the same estimate is valid for any $t \in J_n$. Define $K_n(n) = \|w_n(t_n)\|_{H^1}^2$. Then, as computed above, we have the following increment:

$$\begin{aligned}
&K_n(n+1) - K_n(n) \\
=&2\int_{J_n} \langle \tilde{w}, A_a\tilde{w}\rangle_{H^1} dt + \int_{J_n} \langle \tilde{w}, Q\mathcal{F}\rangle_{H^1} dt \\
=&2\int_{J_n} \langle \tilde{w}, A_a\tilde{w}\rangle_{H^1} dt + 2\int_{J_n} \langle \tilde{w}, Q((c-c_0-\dot{\gamma})(\partial_y - a)\tilde{w} - e^{ay} I_{N(n)}\partial_y(v^2) \\
&\quad + e^{ay}(\dot{c}\partial_c + \dot{\gamma}\partial_y) I_{N(n)}\psi_c - e^{ay}\partial_y(I_{N(n)}(\psi_c v) - \psi_c I_{N(n)} v))\rangle_{H^1} dt \\
=&(0)+(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})
\end{aligned}$$

We estimate these terms as above. For (0), by Proposition 1, this is at most $-2b\int_{J_n} \|w\|_{H^1}^2 dt$. For (I), we get

$$\int_{J_n} (c-c_0) - \dot{\gamma})(-a)\|w\|_{H^1}^2 dt \le 4a\eta \int_{J_n} \|w(t)\|_{H^1}^2 dt.$$

For (II), we obtain, as above,

$$\int_{J_n} \langle \tilde{w}_n, e^{ay} I_{N(n)}\partial_y v^2\rangle_H^1 dt \le (1+2N(n))^{-\frac{1}{8}})\|\tilde{w}_n\|_{X^{1,\frac{1}{2},1}}^2 \|\tilde{v}_n\|_{X^{1,\frac{1}{2},1}} \le Cc_0 N(n).$$

Then, for (III), we get as above

$$(III) \le (1+\tilde{C}N^{-\tilde{\tilde{C}}}) \int_J (|\dot{c}| + |\dot{\gamma}|)|c-c_0|\|\tilde{w}\|_{H^1} dt \le 2\int_{J_n} \eta \|\tilde{w}_n(t)\|_{H^1}^2 dt.$$

Finally, for (IV), we have, as above, with τ a small positive number,

$$\begin{aligned}
(IV) &\le \|\tilde{w}\|_{X^{1,\frac{1}{2},1}} \left((N^{\frac{3}{4}-s+} + \tau)\|\tilde{w}_n\|_{X^{1,\frac{1}{2},1}} + N^{\frac{3}{4}-s+}\|v\|_{X^{1,\frac{1}{2},1}} \right) \\
&\le 2\tau N(n) + N^{\frac{3}{4}-s+} c_0 \sqrt{N(n)}.
\end{aligned}$$

Notice that $N(n)$ has been chosen so that $N(n)^{\frac{3}{4}-s+} \ll \kappa^n \le C\epsilon_1\kappa^n$. Therefore, putting everything together, we have that

$$\begin{aligned}
K_n(n+1) - K_n(n) \le (-2b + 4a\eta + 2\eta) &\int_{J_n} \|\tilde{w}_n(t)\|_{H^1}^2 dt \\
&+ (Cc_0 + 2\tau)N(n) + Cc_0\epsilon_1\kappa^n \sqrt{K_n(n)}.
\end{aligned}$$

Now, suppose that $K_n(n) \sim (\epsilon_1\kappa^n)^2$. Then by the same argument as in [19], it follows that $K_n(n+1) \le \max\{(1-b), \frac{3}{4}\} K_n(n) \le \kappa^{2+\frac{1-s}{s-\frac{3}{4}-}-} K_n(n)$. Finally, it remains to compare $K_{n+1}(n+1)$ to $K_n(n+1)$. By properties of the I_N multiplier, we have that

$$\begin{aligned} K_{n+1}(n+1) &\leq \left(\frac{N(n+1)}{N(n)}\right)^{1-s} K_n(n+1) \\ &\leq \kappa^{\frac{1-s}{\frac{3}{4}-s+}+} K_n(n+1) \\ &\leq \kappa^{\frac{1-s}{\frac{3}{4}-s+}+} \kappa^{2+\frac{1-s}{s-\frac{3}{4}-}-} K_n(n) \\ &\leq \kappa^2 K_n(n). \end{aligned}$$

On the other hand, if $K_n(n) \ll (\epsilon_1 \kappa^n)^2$, then the largest term on the right hand side is the last one, and we obtain that $K_n(n+1) \ll (\epsilon_1 \kappa^n)^2$. Then $K_{n+1}(n+1) \ll \kappa^{(\frac{1-s}{\frac{3}{4}-s+}+)}(\epsilon_1 \kappa^n)^2$, which can be taken to be at most $\epsilon_1^2 \kappa^{2(n+1)}$. In either case, after applying the inductive hypothesis, we obtain that $K_{n+1}(n+1) \leq (\epsilon_1 \kappa^{n+1})^2$, so $\|\tilde{w}_{n+1}(n+1)\|_{H^1} \leq \epsilon_1 \kappa^{n+1}$. Hence the inductive step holds and the proof of the lemma is complete.

$\square$

To conclude the proof of Theorem 1, let $r = \kappa^{\frac{1}{\delta}}$. Then (2) and (3) are immediate from the lemma. To conclude (1), note that $\|e^{ay} I_1 v(t)\|_{H^1} \leq \|e^{ay} I_N v(t)\|_{H^1} = \|\tilde{w}(t)\|_{H^1}$ for any N, by the properties of I_N and Lemma 3.2. Hence (1) follows from the last conclusion of the inductive lemma.

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