This problem is kind of like the strong induction problem we finished in class today.

Given a positive integer n, let $f(n)$ be the largest possible value of the product $\prod_{i=1}^{k} a_i$ if a_1, ..., a_k are positive integers with $\sum_{i=1}^{k} a_i = n$? (We also define $f(0) = 1$.)

For example, $f(1) = 1$, because $1 = 1$, $f(2) = 2$ because $2 = 2$ gives a product of 2, and $2 = 1 + 1$ gives a product of 1. We also have $f(3) = 3$, $f(4) = 4$ and $f(5) = 6$ (because $5 = 2 + 3$ gives a product of 6, which is better than the alternatives).

(a) Explain why if $n \geq 2$, then $f(n)$ is equal to the maximum value of $(n - k)f(k)$ as k varies from 0 to $n - 1$.

(b) Use (a) to calculate $f(n)$ for $n \leq 10$ and guess a formula for $f(n)$ for all positive integers n.

(c) Show that $f(n) \geq 2(n - 2)$ for $n \geq 3$. (Hint: $n = 2 + (n - 2)$.) Conclude that $f(n) > n$ if $n \geq 5$.

(d) Suppose that $n \geq 5$ is a positive integer and $n = a_1 + a_2 + \cdots + a_k$ is a decomposition of n with $a_1 \leq a_2 \leq \cdots \leq a_k$ and the product $a_1 a_2 \cdots a_k$ is maximal. Use (c) to show that $a_1 \leq 4$. Conclude that $f(n) = \max\{f(n-1), 2f(n-2), 3f(n-3), 4f(n-4)\}$.

(e) Use strong induction and (d) to prove the guess you made in part (b).