Day 16 homework - Assigned 2/21 and due on 2/28

Starred problems below are extra-credit for undergraduates and required for graduate students.

5. Let $\gamma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 1 & 9 & 7 & 8 & 6 & 5 & 2 \end{bmatrix}$. Compute the cycle decomposition of γ. What is the order of γ?

6. Let $\alpha = (1,8)(2,4)(3,5)(6,7)$ and $\beta = (1,8,2)(4,5,6)$ be elements of S_8.

Compute the cycle decompositions of $\alpha\beta$ and $\beta\alpha$. What are the orders of α, β, $\alpha\beta$ and $\beta\alpha$?

7. A card-shuffling machine shuffles 13 cards, and it always rearranges the cards in the same way relative to the order in which they were given to it. All of the hearts arranged in order from ace to king were put into the machine, and then the shuffled cards were put into the machine again to be shuffled. If the cards emerged in the order A, 7, 10, 2, 3, 4, J, Q, 8, K, 6, 5, 9, in what order were the cards after the first shuffle? (This is a variation of Problem 70 from the textbook. You only need to find the correct answer to the question. The answer is unique, but you do not need to prove that.)

8. Let $a = (a_1, a_2, \ldots, a_k)$ be a cycle of length k and let $b \in S_n$. Show that

$$bab^{-1} = (b(a_1), b(a_2), b(a_3), \ldots, b(a_k)),$$

which is also a cycle of length k.

9. $*$ Let $a = (a_1, a_2, \ldots, a_k)$ be a cycle and suppose that $b \in S_n$ is a permutation that commutes with a. Let $S = \{a_1, a_2, \ldots, a_k\}$.

(a) Show that $b(a_i) \in S$ for all i.

(b) Suppose that $b(a_1) = a_r$. Prove that $b = a^{r-1} \cdot c$ where c is a permutation with $c(a_i) = a_i$ for all i. Conclude that the centralizer of a has order $k \cdot (n-k)!$.