Day 20 homework - Assigned 3/2 and due 3/27

Starred problems below are extra-credit for undergraduates and required for graduate students.

1. (a) Prove that A_5 is not isomorphic to D_{30}.

(b) Let $\mathbb{R}^\times$ be the multiplicative group of real numbers and $\mathbb{C}^\times$ be the multiplicative group of complex numbers. Prove that $\mathbb{R}^\times$ is not isomorphic to $\mathbb{C}^\times$. (Be careful, although it's not obvious, there is a bijection bewteen $\mathbb{R}^\times$ and $\mathbb{C}^\times$.)

2. Let G and $\overline{G}$ be isomorphic finite groups. Suppose that the subgroup of G generated by all elements in G of order 2 has order 48. Prove that the subgroup of $\overline{G}$ generated by all elements of order 2 in $\overline{G}$ also has order 48. [An element of the subgroup of a group G generated by an element of order 2 is a finite product of elements in G that have order 2. This question sounds random. The point is that "the subgroup generated by all elements of order 2 having order 48" is an intrinsic group theoretic property, and so it must be preserved by isomorphism.]

3. $*$ Let G be a group whose order is $2^k m$ where m is odd and $k \geq 1$. Suppose that G contains a cyclic subgroup of order 2^k. Prove that G contains a subgroup of order $2^{k-1}m$. Here are some hints.

(a) Let $\overline{G}$ be the subgroup of $S_{2^k m}$ that is isomorphic to G from Cayley's theorem. Show that $\overline{G}$ is not contained in $A_{2^k m}$.

(b) Let $K = \overline{G} \cap A_{2^k m}$. Show that K is a subgroup of $\overline{G}$ that contains half of the elements of $\overline{G}$. Make the desired conclusion.