Day 2 homework - Assigned 1/15 and due 1/24

Starred problems below are extra-credit.

1. You have two jugs, one of which holds exactly 8 liters, and another which holds exactly 5 liters. However, neither jug has any markings on it. You have a faucet that you can use to fill either jug, and you can dump jugs out, or dump the contents of one jug into another (stopping when the jug is empty, or when the other jug is full). How can you measure exactly one liter?

2. (a) Suppose that a and b are positive integers. Prove that $\gcd(a, b) = 1$ if and only if there is no prime number p that divides both a and b.

(b) Suppose that a, b and n are positive integers. If $a|n$, $b|n$ and $\gcd(a, b) = 1$, show that $ab|n$.

3. Suppose that a, b and n are integers with $\gcd(a, n) = \gcd(b, n) = 1$. Prove that $\gcd(ab, n) = 1$.

4. ∗ Suppose that a and b are odd positive integers. Prove by induction that that $a^{2^n} \equiv b^{2^n}$ (mod 2^{n+2}). (Note that a^{2^n} means to first let $c = 2^n$ and then compute a^c. On the other hand, $(a^2)^n = a^{2n} \neq a^{2^n}$.)