When does F_m^L divide F_n? A combinatorial solution

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Arthur T. Benjamin
Dept. of Mathematics, Harvey Mudd College, Claremont, CA 91711
benjamin@hmc.edu

Jeremy A. Rouse
Dept. of Mathematics, Harvey Mudd College, Claremont, CA 91711
jrouse@hmc.edu

1 Introduction

One of the oldest theorems about Fibonacci numbers states that for $F_m > 1$,

$$F_m | F_n \text{ if and only if } m | n. \tag{1}$$

Indeed, Edouard Lucas proved in his classic paper [4] that this theorem remains true even when we replace the Fibonacci sequence with any Lucas sequence of the first kind defined recursively by $u_0 = 0$, $u_1 = 1$ and for any all $n \geq 2$ $u_n = au_{n-1} + bu_{n-2}$, where a and b are arbitrary integers. In 1970, Yuri Matijasevič [5] proved that for $F_m > 1$,

$$F_m^2 | F_{mr} \text{ if and only if } F_m | r. \tag{2}$$

which led to his solution to Hilbert's 10th Problem.

In this paper, we first present combinatorial proofs of (1) and (2), then extend our argument to characterize when F_m^3 divides F_n. Next, we give a combinatorial proof of

$$F_{mr} = \sum_{j=1}^{r} \binom{r}{j} F_j F_m^j F_{m-1}^{j-1},$$

which leads to a characterization of when F_m^L divides F_n for all $L \geq 1$. Finally, we generalize these results to any Lucas sequences of the first kind that are generated by non-negative integers.

2 Divisibility by F_m, F_m^2 and F_m^3

To introduce our main ideas, we begin with a combinatorial proof of statement (1). It is well known [1, 7] that for $n \geq 0$, F_n counts the number of ways to tile a board of length $n-1$ with squares and dominoes. Now suppose that $m|n$. Thus $F_n = F_{mr}$ counts the number of ways to tile a board of length $mr-1$ with squares and dominoes. Such a tiled board can be broken into r segments (called *supertiles*) $S_1, S_2, \ldots, S_r$ by chopping the board immediately to the right of cells $m, 2m, 3m, \ldots, (r-1)m$. Notice that this chopping can result in a domino being split into two "half-dominoes" (not the same as two squares) whenever a domino covers cells jm and $jm+1$ for some $1 \leq j \leq r-1$. When this happens, we say that supertile S_j is open on the right and that S_{j+1} is open on the left. Otherwise we say that S_j is closed on the right and S_{j+1} is closed on the left. See Figure 1.

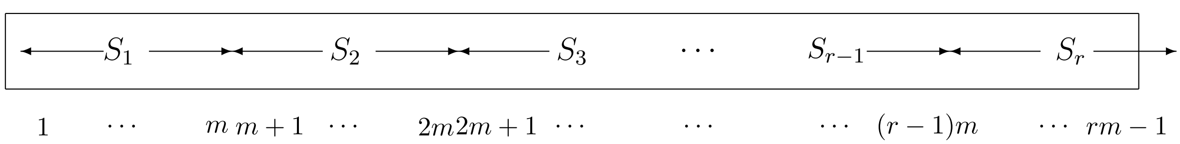

Figure 1: A board of length $rm-1$ (with a half-domino attached) can be split into r supertiles of length m.

For convenience, we append a half-domino to the end of our board so that S_r is open on the right. By doing so, all supertiles have length m, and this guarantees the existence of at least one supertile S_j that is closed on the left and open on the right. For $1 \leq j \leq r$, the number of tilings where supertile S_j is the first supertile of this form is $F_{m+1}^{j-1} F_m F_{(r-j)m-1}$ since the first $j-1$ supertiles can each be tiled F_{m+1} ways, S_j can be tiled F_m ways, a domino must cover cells jm and $jm+1$, and the remaining $(r-j)m-1$ cells can be covered $F_{(r-j)m-1}$ ways and end with a half-domino. (A slightly modified argument is needed when $j = r$ but since $F_{-1} = 1$, the formula remains valid.) Altogether, we have

$$F_n = F_{mr} = F_m \sum_{j=1}^{r} F_{(m+1)}^{j-1} F_{(r-j)m-1}.$$

Thus we have combinatorially demonstrated that if $m|n$, then $F_m|F_n$. For a slightly different combinatorial proof, see [2].

To prove the converse statement, suppose $n = mr + s$, where $0 < s < m$, and $F_m > 1$. We apply the same argument as before, but now we end with a length s supertile S_{r+1}. Adjusting the previous argument for this, we obtain

$$F_n = F_{mr+s} = F_{m+1}^r F_s + F_m \sum_{j=1}^{r} F_{(m+1)}^{j-1} F_{(r-j)m+s-1},$$

where the $F_{m+1}^r F_s$ term accounts for those tilings where supertiles $S_1, S_2, \ldots, S_r$ are all closed on the left and right. Consequently, $F_n \equiv F_{m+1}^r F_s \not\equiv 0 \pmod{F_m}$, since F_{m+1} is relatively prime to F_m and $0 < F_s < F_m$.

A similar argument leads to divisibility criteria for F_m^2 and F_m^3. By statement (1), we need only consider situations where $n = mr$ for some non-negative integer r. Begin by observing that any tiling of length mr ending with a half-domino must contain an odd number of supertiles that are closed on one end and open on the other. The number of ways to create a tiling with 3 or more of these supertiles is a multiple of F_m^3 since the first 3 of these supertiles can each be independently tiled in F_m ways. Thus to determine $F_{mr} \pmod{F_m^2}$, we need only count those tilings that have exactly one supertile S_j that is closed on the left and open on the right and therefore have no supertiles that are closed on the right and open on the left. All of the supertiles that precede S_j are necessarily closed on both sides and all supertiles that come after S_j are necessarily open on both sides. If at least one of the supertiles S_i preceding S_j is to end with a square then S_i and S_j can each be independently tiled F_m ways; consequently the number of ways for this to occur must be a multiple of F_m^2. Thus F_{mr} must be congruent $\pmod{F_m^2}$ to the number of tilings with only one half-open supertile S_j and where all supertiles that precede it end with a domino. Since j can be chosen r ways, and all of the supertiles, besides S_j can each be tiled F_{m-1} ways, there are $rF_mF_{m-1}^{r-1}$ such tilings. Since F_m is relatively prime to F_{m-1}, we conclude that F_{mr} is divisible by F_m^2 if and only if F_m divides r.

To determine divisibility by F_m^3, we proceed exactly as above, but with one modification. This time we must also count those tilings with exactly one half-open supertile S_j that is preceded by exactly one closed supertile S_k which ends with a square. Thus for $1 \leq i \leq j-1$ and $i \neq k$, S_i is closed and ends with a domino. Since S_j and S_k can be chosen $\binom{r}{2}$ ways and can each be tiled F_m ways, and since all other supertiles can each be tiled F_{m-1} ways, it follows that

$$F_{mr} \equiv rF_mF_{m-1}^{r-1} + \binom{r}{2}F_m^2F_{m-1}^{r-2} \pmod{F_m^3}.$$

Factoring out F_{m-1}^{r-2} (which is relatively prime to F_m) and dividing everything (including the modulus) by F_m^2, gives us the following theorem.

Theorem 1. *F_m^3 divides F_{mr} if and only if F_m divides $\binom{r}{2} + r\frac{F_{m-1}}{F_m}$.*

We point out that the quantity $\binom{r}{2} + r\frac{F_{m-1}}{F_m}$ is an integer if and only if F_m divides r, i.e., when F_m^2 divides F_n. Thus, our theorem is equivalent to saying that F_m divides F_n if and only if $n = mr$ for some integer r and F_m divides r and F_m divides $\binom{r}{2} + r\frac{F_{m-1}}{F_m}$. Notice that when r is a multiple of F_m^2 then $\binom{r}{2} + r\frac{F_{m-1}}{F_m}$ is a multiple of F_m, resulting in the following corollary.

Corollary 2. *For $m, r \geq 0$, If F_m^2 divides r, then F_m^3 divides F_{mr}.*

3 Divisibility by F_m^L

Our general result will depend on the following theorem, [3] which we prove combinatorially.

Theorem 3. *For $m, r \geq 0$,*

$$F_{mr} = \sum_{j=1}^{r} \binom{r}{j} F_j F_m^j F_{m-1}^{r-j}.$$

Proof. As in the proof of Theorem 1, F_{mr} counts the number of ways to tile a board of length mr with squares and dominoes such that the last cell ends with a half-domino. Such a tiled board can be broken into r supertiles $S_1, S_2, \ldots, S_r$, each of length m. We observe that supertiles can be partitioned into 5 types:

A) Closed on the left and open on the right,
B) Open on the left and closed on the right,
C) Closed on both sides and ending with a square,
D) Closed on both sides and ending with a domino, or
E) Open on both sides.

The crucial observation here is that all supertiles of type A, B, or C have one restricted cell and $m-1$ cells that can be tiled freely in F_m ways. Supertiles of type D or E have two restricted cells and $m-2$ cells that can be tiled freely in F_{m-1} ways.

We claim that for $1 \leq j \leq r$, the summand $\binom{r}{j} F_j F_m^j F_{m-1}^{r-j}$ counts the number of tilings that have exactly j supertiles of type A, B, or C. To count such a tiling, we must first select which supertiles will be of type A, B, or C. This can be done $\binom{r}{j}$ ways. Call these supertiles $S_{i_1}, \ldots, S_{i_j}$, listed from left to right. Next we must designate which of these supertiles are of type A, which are of type B, and which are of type C. We claim this can be done exactly F_j ways by creating a bijection between all possible designations and the set $\mathcal{T}$ of length j tilings that end with a half-domino. Specifically, let T be a tiling in $\mathcal{T}$. For $1 \leq k \leq j$, if cell k is covered by the first half of a domino, then S_{i_k} is designated type A, if cell k is covered by the second half of a domino, then S_{i_k} is designated type B, and if cell k is covered by a square, then S_{i_k} is designated type C. (Notice that S_{i_1} is guaranteed to be of type A or C and that S_{i_j} is guaranteed to be of type A and that S_{i_k} is closed on the right if and only if $S_{i_{k+1}}$ is closed on the left.) Thus, since $|\mathcal{T}| = F_j$, supertiles $S_{i_1}, \ldots, S_{i_j}$ can be designated F_j ways. Once this is done, the other supertiles of type D and E can be designated in precisely one way. (Specifically, before S_{i_1}, all supertiles must be of type D, all supertiles after S_{i_j} are of type E, and if $i_k < i < i_{k+1}$, S_i is of type D if and only if S_{i_k} is closed on the right.) Finally, with the type of each supertile designated, the tiling can be constructed in $F_m^j F_{m-1}^{r-j}$ ways. $\square$

As a simple consequence of this theorem, we obtain the following sufficient condition.

Corollary 4. *For $L, m, r \geq 0$, If F_m^L divides r, then F_m^{L+1} divides F_{mr}.*

Proof. By the previous theorem, it suffices to show that for each $1 \leq j \leq r$, the summand $\frac{r(r-1)\cdots(r-j+1)F_j F_m^j F_{m-1}^{r-j}}{j!}$ is divisible by F_m^{L+1}. Since $F_m^L | r$, the numerator is divisible by F_m^{L+j}. For any prime factor p of F_m, the largest power of p to divide $j!$ is p^α where $\alpha = \sum_{k=1}^{\infty} \left\lfloor \frac{j}{p^k} \right\rfloor < \sum_{k=1}^{\infty} \frac{j}{p^k} \leq \sum_{k=1}^{\infty} \frac{j}{2^k} = j$. Thus, since $\alpha \leq j-1$, the fraction is divisible by F_m^{L+1} as desired. $\square$

For a necessary and sufficient condition, we utilize the following definition.

Definition For $m, r \geq 1$, let $r_0 = 0$ and for $k \geq 1$,

$$r_k = F_k \binom{r}{k} + \frac{F_{m-1}}{F_m} r_{k-1}. \tag{3}$$

For example, $r_0 = 0$, $r_1 = r$, $r_2 = \binom{r}{2} + r\frac{F_{m-1}}{F_m}$, $r_3 = 2\binom{r}{3} + \binom{r}{2}\frac{F_{m-1}}{F_m} + r\left(\frac{F_{m-1}}{F_m}\right)^2$. Continuing in this way, it is easy to see that for $0 \leq k \leq r$,

$$r_k = \sum_{j=1}^{k} F_j \binom{r}{j} \left(\frac{F_{m-1}}{F_m}\right)^{k-j} \quad \text{for } 0 \leq k \leq r.$$

When $k > r$, $\binom{r}{k} = 0$. Thus equation (3) implies

$$r_k = \left(\frac{F_{m-1}}{F_m}\right)^{k-r} r^* \quad \text{for } k \geq r,$$

where

$$r^* = r_r = \sum_{j=1}^{r} F_j \binom{r}{j} \left(\frac{F_{m-1}}{F_m}\right)^{r-j}.$$

Theorem 5. *For $L, m, r \geq 1$, $F_m^L | F_{mr}$ if and only if $F_m | r_{L-1}$.*

Proof. Using Theorem 3, and ignoring all factors of F_m^L and higher, we have, for $1 \leq L \leq r+1$,

$$\begin{aligned} F_{mr} &\equiv \sum_{j=1}^{L-1} F_j \binom{r}{j} F_m^j F_{m-1}^{r-j} \qquad \pmod{F_m^L} \\ &= F_m^{L-1} F_{m-1}^{r+1-L} \sum_{j=1}^{L-1} F_j \binom{r}{j} \left(\frac{F_{m-1}}{F_m}\right)^{L-1-j} \\ &= F_m^{L-1} F_{m-1}^{r+1-L} r_{L-1}. \end{aligned}$$

Thus, since F_m is relatively prime to F_{m-1} we have, for $1 \leq L \leq r+1$,

$$F_m^L | F_{mr} \text{ if and only if } F_m | r_{L-1},$$

as desired.

To prove the theorem when $L > r+1$, observe that Theorem 3 says $F_{mr} = F_m^r r^*$. Thus for $L \geq r+1$, $F_m^L | F_{mr}$ if and only if $F_m | \frac{F_{mr}}{F_m^{L-1}} = \frac{r^*}{F_m^{L-1-r}}$. And since F_m is relatively prime to F_{m-1}, this is equivalent to the condition that $F_m | \left(\frac{F_{m-1}}{F_m}\right)^{L-1-r} r^* = r_{L-1}$ as desired. $\square$

4 Lucas sequences

The results and arguments of the last section can be generalized to Lucas sequences of the first kind, with non-negative integer coefficients a and b. Here, u_n is defined recursively by $u_0 = 0$, $u_1 = 1$, and for $n \geq 2$ $u_n = au_{n-1} + bu_{n-2}$. It is easy to see that u_n counts the number of *colored tilings* of a length n board ending with a half-domino, where all tiles (except for the terminating half-domino) are assigned a color. For each square, we have a choices of color, and for each domino, we have b choices of color. Using this interpretation, we present a combinatorial proof of the following generalization [6] of Theorem 3.

Theorem 6. *For $m, r \geq 0$,*

$$u_{mr} = \sum_{j=1}^{r} \binom{r}{j} u_j u_m^j b^{r-j} u_{m-1}^{r-j}.$$

Proof. Here u_{mr} counts the number of colored tilings of length mr that end with an uncolored half-domino. As in the proof of Theorem 3, we claim that for $1 \leq j \leq r$, $\binom{r}{j} u_j u_m^j b^{r-j} u_{m-1}^{r-j}$ counts the number of tilings that have exactly j supertiles of type A, B, or C. There are $\binom{r}{j}$ ways to choose which supertiles $S_{i_1}, \ldots, S_{i_j}$ will be of type A, B, or C. Next for any colored tiling T of length j that ends with an uncolored half-domino, we proceed as follows. If the k-th cell of T is covered by a colored square, then S_{i_k} is designated type C and its terminal square is given the same color as the square on the k-th cell of T. The rest of S_{i_k} can be tiled in u_m ways. If the k-th cell of T is covered by the beginning of a half-domino, then S_{i_k} is designated type A, temporarily ending with an uncolored half-domino. (Unless $i_k = r$, the color of this last half-domino will be assigned the same color as its second half. If $i_k = r$, then this half-domino will remain uncolored.) The rest of S_{i_k} can be tiled in u_m ways. If the k-th cell of T is covered by the second half of a domino, then S_{i_k} is of type B, with its initial half-domino given the same color as the domino ending on the k-th cell of T. The first half of this half-domino, at the end of supertile $S_{i_k - 1}$, will also be given the same color. The rest of S_{i_k} can be tiled in u_m ways. Each of the remaining $r - j$ supertiles is designated to be either type D or type E (depending on the same criteria as in the proof of Theorem 3.) If the supertile is of type D, then the color of its terminating domino can be chosen in b ways, and the rest of it can be tiled in u_{m-1} ways. If the supertile is of type E, then the color of its initial half-domino can be chosen b ways, the color of its terminating half-domino is determined by its second half, and the rest of the supertile can be tiled in u_{m-1} ways. Summarizing, for $1 \leq j \leq r$, a colored tiling of length mr ending with an uncolored half-domino with exactly j supertiles of type A,B, or C can be created $\binom{r}{j} u_j u_m^j (bu_{m-1})^{r-j}$ ways, as desired. $\square$

Replacing F with u, Corollary 4 and its proof immediately generalize to

Corollary 7. *If u_m^L divides r, then u_m^{L+1} divides u_{mr}.*

Finally, suppose that our Lucas sequence has a and b relatively prime. Then, inductively, for all $m \geq 1$, u_m and bu_{m-1} are relatively prime. For such a Lucas sequence, define,

Definition For $m, r \geq 1$, let $r_0 = 0$ and for $k \geq 1$,

$$r_k = u_k \binom{r}{k} + \frac{bu_{m-1}}{u_m} r_{k-1}. \tag{4}$$

After we substitute F_{mr}, F_j, F_m, and F_{m-1} with u_{mr}, u_j, u_m, and bu_{m-1}, respectively, throughout the proof of Theorem 5, we immediately obtain the following generalization.

Theorem 8. *For $L, m, r \geq 1$, $u_m^L | u_{mr}$ if and only if $u_m | r_{L-1}$.*

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