BOUNDS FOR COEFFICIENTS OF CUSP FORMS AND EXTREMAL LATTICES

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ABSTRACT. A cusp form $f(z)$ of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is determined uniquely by its first $\ell := \dim S_k$ Fourier coefficients. We derive an explicit bound on the nth coefficient of f in terms of its first ℓ coefficients. We use this result to study the non-negativity of the coefficients of the unique modular form of weight k with Fourier expansion

$$F_{k,0}(z) = 1 + O(q^{\ell+1}).$$

In particular, we show that $k = 81632$ is the largest weight for which all the coefficients of $F_{k,0}(z)$ are non-negative. This result has applications to the theory of extremal lattices.

1. Introduction and Statement of Results

An incredible number of interesting sequences appear as Fourier coefficients of modular forms. The analytic properties of these modular forms dictate the asymptotic behavior of the corresponding sequences.

The most famous example of such a sequence is the partition function $p(n)$, which counts the number of ways of representing an integer n as a sum of a non-increasing sequence of positive integers. Hardy and Ramanujan pioneered the use of the circle method to study the asymptotics for $p(n)$ and proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

by using the analytic properties of the generating function

$$f(z) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

where $q = e^{2\pi i z}$. (See Chapter 5 of [1] for a proof as well as for an exact formula for $p(n)$).

Another important example is given by the arithmetic of quadratic forms. Let Q be a positive-definite, integral, quadratic form in r variables, where r is even, and let

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$r_Q(n)$ denote the number of representations of the integer n by Q. It is well-known that the generating function

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n$$

is a holomorphic modular form of weight $\frac{r}{2}$ for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (see Chapter 10 of [9] for details).

To determine which integers are represented by Q, it is necessary to study the decomposition

$$\theta_Q(z) = E(z) + G(z)$$

where $E(z)$ is an Eisenstein series and $G(z)$ is a cusp form, and to determine explicit bounds on the coefficients of $E(z)$ and $G(z)$. If $r \geq 6$, formulas for the coefficients of Eisenstein series show that the coefficients of $E(z)$ are of size $n^{\frac{r}{2}-1}$, and if we write

$$G(z) = \sum_{i=1}^{\ell} c_i g_i(d_i z)$$

where the $g_i(z)$ are newforms, then Deligne's proof of the Weil conjectures implies that the nth coefficient of $G(z)$ is bounded by

$$\left(\sum_{i=1}^{\ell} |c_i| \right) d(n) n^{\frac{r-2}{4}}.$$

In [2], Bhargava and Hanke prove that a positive-definite quadratic form with integer coefficients represents every positive integer if and only if it represents the integers from 1 up to 290; in fact, it is only necessary for the form to represent 29 of these numbers. To prove this, they study about 6000 quadratic forms in four variables, and the most time-consuming part of their calculation comes from computing the constant

$$C(G) = \sum_{i=1}^{\ell} |c_i|.$$

In this paper, we find bounds for this constant $C(G)$ for general cusp forms G of weight k and full level.

If

$$\ell := \dim S_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \not\equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor - 1 & \text{if } k \equiv 2 \pmod{12}, \end{cases}$$

then any cusp form $G(z) = \sum_{n=1}^{\infty} a(n) q^n$ is determined uniquely by the coefficients $a(1)$, $a(2)$, …, $a(\ell)$. In fact, in [5, Theorem 3], Bruinier, Kohnen and Ono showed that the coefficients $a(n)$ of $G(z)$ may be explicitly computed recursively from the first ℓ coefficients of G. Specifically, $a(n)$ may be written as a polynomial with rational coefficients in the coefficients $a(n-i)$, the weight k, and the values of the j-function at points in the divisor of G.

Our first result is a bound on $\sum_{i=1}^{\ell} |c_i|$ (giving a bound on $|a(n)|$) in terms of the coefficients $a(1), a(2), \ldots, a(\ell)$.

Theorem 1. *Assume the notation above. Then*

$$|a(n)| \leq \sqrt{\log(k)} \left(11 \cdot \sqrt{\sum_{m=1}^{\ell} \frac{|a(m)|^2}{m^{k-1}}} + \frac{e^{18.72}(41.41)^{k/2}}{k^{(k-1)/2}} \cdot \left| \sum_{m=1}^{\ell} a(m) e^{-7.288m} \right| \right) \cdot d(n) n^{\frac{k-1}{2}}.$$

We apply this result to the study of extremal lattices. An even, unimodular lattice is a free $\mathbb{Z}$-module Λ of rank r, together with a quadratic form $Q : \Lambda \to \mathbb{Z}$ with the property that the inner product

$$\langle \vec{x}, \vec{y} \rangle = Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y})$$

is positive definite on $\mathbb{R} \otimes \Lambda$ and is an integer for all pairs $\vec{x}, \vec{y} \in \Lambda$; additionally, we require that $\langle \vec{x}, \vec{x} \rangle$ is even for all $\vec{x} \in \Lambda$, and that the dual lattice

$$\Lambda^{\#} := \{ \vec{x} \in \mathbb{R} \otimes \Lambda : \langle \vec{x}, \vec{y} \rangle \in \mathbb{Z} \text{ for all } \vec{y} \in \Lambda \}$$

is equal to Λ. For such a lattice, we must have $r \equiv 0 \pmod 8$, so the theta function θ_Q is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $k \equiv 0 \pmod 4$.

For example, if

$$Q = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 - x_1x_3 - x_2x_4 - x_3x_4 - x_4x_5 - x_5x_6 - x_6x_7 - x_7x_8,$$

then Λ is the E_8 lattice and

$$\theta_Q(z) = E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n.$$

An even, self-dual lattice Λ is called extremal if $r_Q(n) = 0$ for $1 \leq n \leq \lfloor \frac{r}{24} \rfloor$. This means that if Q is the quadratic form corresponding to Λ, then

$$\theta_Q(z) = 1 + O(q^{\ell+1}) \in M_{\frac{r}{2}}.$$

An example is given by the famous Leech lattice Λ_{24}. It is the unique extremal lattice of dimension 24, and $\mathrm{Aut}(\Lambda_{24})$ is a perfect group whose quotient by -1 is Co_1, the first sporadic finite simple group discovered by John H. Conway.

Little is known about the set of dimensions in which extremal lattices exist, and examples are known only in dimensions ≤ 88. Cases where the rank is a multiple of 24 are particularly challenging, and Nebe [11] recently succeeded in constructing a 72-dimensional extremal lattice.

If Λ is an extremal lattice of dimension r, then the definition of $r_Q(n)$ implies that all the Fourier coefficients of the modular form

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n = 1 + O(q^{\ell+1}) \in M_{\frac{r}{2}}$$

are non-negative. In [10], Mallows, Odlyzko, and Sloane use this to show that extremal lattices fail to exist in large dimensions (larger than about 164,000) by showing that the unique modular form of weight k with Fourier expansion

$$F_{k,0}(z) = \sum_{n=0}^{\infty} a(n) q^n = 1 + O(q^{\ell+1}),$$

has $a(\ell+2) < 0$ if k is large enough. (In [13], Siegel proved that $a(\ell+1) > 0$ for all $k \equiv 0 \pmod 4$).

As an application of Theorem 1, we give an explicit estimate on the largest index negative coefficient of $F_{k,0}(z)$.

Theorem 2. *Suppose that $k \equiv 0 \pmod 4$, and $F_{k,0}(z) \in M_k$ is the unique modular form of weight k with*

$$F_{k,0}(z) = 1 + O(q^{\ell+1}) = \sum_{n=0}^{\infty} a(n) q^n.$$

We have $a(n) > 0$ if

$$n \geq e^{58.366/(k-2)} (\ell^3 \log(k))^{\frac{1}{k-2}} 1.0242382\ell.$$

Remark. *The result above is surprisingly strong. The factor preceding 1.0242382ℓ tends to 1 as $k \to \infty$, and since $a(n) = 0$ for $n \leq \ell$, the only region in which negative coefficients could occur is (asymptotically)*

$$\ell < n < 1.0242382\ell.$$

We now use this bound to determine the largest weights k in which all the coefficients of $F_{k,0}(z)$ are non-negative. This depends on $k \pmod{12}$, and so we have three cases.

Corollary 3. *The largest weight k for which all coefficients of $F_{k,0}(z)$ are non-negative is*

$$\begin{aligned} k &= 81288 \text{ if } k \equiv 0 \pmod{12}, \\ k &= 81460 \text{ if } k \equiv 4 \pmod{12}, \text{ and} \\ k &= 81632 \text{ if } k \equiv 8 \pmod{12}. \end{aligned}$$

Remark. *As a consequence, the largest possible dimension of an extremal lattice is* 163264.

Our approach to proving our results is to study the basis of cusp forms

$$F_{k,m}(z) = q^m + \sum_{n=\ell+1}^{\infty} A_k(m,n)q^n \in S_k.$$

Theorem 2 of [7] gives a generating function for the forms $F_{k,m}(z)$, and by integrating this generating function we are able to isolate individual coefficients of these forms. Using this method leads to a bound of the form

$$|A_k(m,n)| \leq c_1 \cdot c_2^{\ell} e^{c_3 m + c_4 n}$$

where $c_1, c_2 > 0$, $c_3 < 0$ and $0 < c_4 < \sqrt{3}/2$. Given that the coefficients of a cusp form of weight k are bounded by $O(d(n)n^{\frac{k-1}{2}})$, this bound is not useful by itself. Next, we estimate the Petersson norm $\langle F_{k,m}, F_{k,m}\rangle$ which is (essentially) the infinite sum

$$\sum_{n=1}^{\infty} \frac{|A_k(m,n)|^2}{n^{k-1}} \int_{2\pi\sqrt{3}n}^{\infty} y^{k-2}e^{-y}\, dy.$$

The exponential decay in the integral now cancels the exponential growth from the bound on $|A_k(m,n)|$. Finally, we translate the bound on $\langle F_{k,m}, F_{k,m}\rangle$ to a bound on the constant $\sum_{i=1}^{\ell} |c_i|$ using methods similar to those in [12].

An outline of the paper is as follows. In Section 2 we review necessary background material about modular forms. In Sections 3 and 4 we prove Theorems 1 and 2, respectively. In Section 5, we prove Corollary 3.

2. Preliminaries

Let M_k denote the $\mathbb{C}$-vector space of all holomorphic modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$, and let S_k denote the subspace of cusp forms. For even $k \geq 4$, we have the classical Eisenstein series

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \in M_k,$$

where B_k is the kth Bernoulli number and $\sigma_{k-1}(n)$ is the sum of the $k-1$st powers of the divisors of n. We will also use the standard Δ-function

$$\Delta(z) = \frac{E_4^3 - E_6^2}{1728} = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n \in S_{12}$$

and the classical modular j-function

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884q + \dots,$$

a weakly holomorphic modular form of weight 0. (Weakly holomorphic modular forms are holomorphic on the upper half plane and satisfy the modular equation, but may have poles at the cusps.)

For each prime p, there is a Hecke operator $T_p : M_k \to M_k$ given by

$$\sum_{n=1}^{\infty} a(n)q^n | T_p := \sum_{n=1}^{\infty} \left(a(pn) + p^{k-1} a\left(\frac{n}{p}\right)\right) q^n.$$

The subspace S_k is stable under the action of the Hecke operators.

If $f, g \in S_k$, we define the Petersson inner product of f and g by

$$\langle f, g \rangle = \frac{3}{\pi} \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} f(x+iy)\overline{g(x+iy)} y^k \, \frac{dx\,dy}{y^2}.$$

It is well-known (see Theorem 6.12 of [9] for a proof) that the Hecke operators are self-adjoint with respect to the Petersson inner product, and this fact, together with the commutativity of T_p and T_q, implies that there is a basis for S_k consisting of Hecke eigenforms, each normalized so that the coefficient of q is equal to 1.

If

$$g(z) = \sum_{n=1}^{\infty} a(n) q^n$$

is such a Hecke eigenform, Deligne proves in [6] that if p is prime, then

$$|a(p)| \leq 2p^{\frac{k-1}{2}},$$

as a consequence of the Weil conjectures. It follows from this that $|a(n)| \leq d(n) n^{\frac{k-1}{2}}$ for all $n \geq 1$.

The self-adjoint property of the Petersson inner product implies that if g_i and g_j are two distinct Hecke eigenforms, then $\langle g_i, g_j \rangle = 0$. On the other hand, the second equation on p. 251 of [9] gives that

$$L(\mathrm{Sym}^2 g_i, 1) = \frac{\pi^2}{6} \cdot \frac{(4\pi)^k \langle g_i, g_i \rangle}{\Gamma(k)}.$$

Here, $L(\mathrm{Sym}^2 g_i, s)$ is the symmetric square L-function. In the appendix to [8], Goldfeld, Hoffstein and Lieman proved that $L(\mathrm{Sym}^2 g_i, s)$ has no Siegel zeroes, and in [12], the second author used this to derive the lower bound

$$L(\mathrm{Sym}^2 g_i, 1) \geq \frac{1}{64 \log(k)}.$$

3. Proof of Theorem 1

Let $\ell = \dim S_k$ and write $k = 12\ell + k'$, where $k' \in \{0, 4, 6, 8, 10, 14\}$. For each integer m with $1 \leq m \leq \ell$, we let $F_{k,m}(z)$ denote the unique weight k modular form with a Fourier expansion of the form

$$F_{k,m}(z) = q^m + \sum_{n=\ell+1}^{\infty} A_k(m,n)q^n.$$

In [7], Duke and the first author gave a generating function for the $F_{k,m}(z)$. Note that the notation in this paper differs slightly from theirs; $F_{k,m}$ is equal to the modular form $f_{k,-m}$ in [7].

Theorem (Lemma 2 of [7])**.** *We have*

$$F_{k,m}(z) = \frac{1}{2\pi i}\oint_C \frac{\Delta^\ell(z)E_{k'}(z)E_{14-k'}(\tau)}{\Delta^{1+\ell}(\tau)(j(\tau)-j(z))}p^{m-1}\,dp,$$

where $p = e^{2\pi i\tau}$ and C denotes a (counterclockwise) circle in the p-plane with sufficiently small radius.

Inspection of the integrand shows that the only poles of the integrand (as τ varies) occur when τ is equivalent to z under the action of $\mathrm{SL}_2(\mathbb{Z})$. We change variables by setting $\tau = u + iv$, $p = e^{2\pi i\tau}$, $dp = 2\pi i e^{2\pi i\tau}$, and let v and y be fixed constants. This gives

$$F_{k,m}(z) = \int_{-.5}^{.5} \frac{\Delta^\ell(z)E_{k'}(z)E_{14-k'}(\tau)}{\Delta^{1+\ell}(\tau)(j(\tau)-j(z))}e^{2\pi i m\tau}\,du,$$

which is valid provided no point with imaginary part at least v is equivalent to z under the action of $\mathrm{SL}_2(\mathbb{Z})$. It follows that

$$A_k(m,n) = \int_{-.5}^{.5}\int_{-.5}^{.5} \frac{\Delta^\ell(z)E_{k'}(z)E_{14-k'}(\tau)}{\Delta^{1+\ell}(\tau)(j(\tau)-j(z))}e^{2\pi i m\tau}e^{-2\pi i n z}\,du\,dx,$$

provided no point τ with $\mathrm{Im}\,\tau \geq v$ is equivalent to any point z with $\mathrm{Im}\,z = y$.

From this, it is clear that we can take absolute values to obtain the bound

$$|A_k(m,n)| \leq \max_{|u|,|x|\leq .5} \left|\frac{\Delta(z)}{\Delta(\tau)}\right|^\ell \left|\frac{E_k'(z)E_{14-k'}(\tau)}{\Delta(\tau)(j(\tau)-j(z))}\right| e^{-2\pi m v}e^{2\pi n y}.$$

Since $\Delta(z) = q - 24q^2 + O(q^3)$, we have $|\Delta(z)| \leq e^{-2\pi y} + 24e^{-4\pi y} + B$, where B is a bound on the tail $\sum_{n=3}^{\infty}\tau(n)q^n$ of the series. We can bound the tail by $\sum_{n=3}^{\infty} d(n)n^{11/2}e^{-2\pi n y}$; using the bound $d(n) \leq 2\sqrt{n}$, we can exactly evaluate the sum that results in terms of y. This gives us an explicit upper bound for $|\Delta(z)|$ in terms of y. Similarly, we find an lower bound for $|\Delta(\tau)|$ in terms of v.

For each of the six choices of k', we bound $|E'_k(z)E_{14-k'}(\tau)|$ in terms of y and v by noting that $\sigma_{k-1}(n) \le 2\sqrt{n}n^{k-1} \le 2n^k$, so that

$$|E_k(z)| = \left|1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right| \le 1 + \frac{2k}{|B_k|}\sum_{n=1}^{\infty} 2n^k e^{-2\pi n y}.$$

This latter sum may be exactly evaluated in terms of y.

At this point, we set $y = .865$ and $v = 1.16$; these values satisfy the conditions above, since all points equivalent to $z = x + .865i$ under the action of $\mathrm{SL}_2(\mathbb{Z})$ have imaginary part less than 1.16, and give reasonable bounds for the quantities we are studying. With these choices, we find that

$$\left|\frac{\Delta(z)}{\Delta(\tau)}\right| \le 7.358,$$
$$\left|\frac{1}{\Delta(\tau)}\right| \le 1488.802,$$
$$|E'_k(z)E_{14-k'}(\tau)| \le 40.368.$$

It remains to bound the quantity $|j(\tau) - j(z)|$ on the appropriate intervals. We bound the tails of the two series, taking all terms with exponent 10 and above for $j(z)$ and all terms with exponent 5 and above for $j(\tau)$. Using the bounds given in [4], we find that the tail of $j(z)$ is bounded by

$$\sum_{n=10}^{\infty} e^{-2\pi n(.865)}\frac{1}{\sqrt{2}n^{3/4}}e^{4\pi\sqrt{n}}\left(1 - \frac{3}{32\pi\sqrt{n}} + \frac{.055}{n}\right) \le \frac{1.055}{\sqrt{2}}\sum_{n=10}^{\infty} e^{-2\pi\sqrt{n}(.865\sqrt{n}-2)}$$
$$\le \frac{1.055}{\sqrt{2}}\sum_{n=10}^{\infty} e^{-2\pi\sqrt{n}(.2\sqrt{n})} \le .000003636545.$$

Similarly, the tail of $j(\tau)$ is bounded by .000003636545.

We now bound the main terms of $|j(\tau) - j(z)|$. Writing $j(z) = q^{-1} + \sum c(n)q^n$, we must find a lower bound for

$$G(x,u) = \left|p^{-1} + \sum_{i=1}^{4} c(i)p^i - q^{-1} - \sum_{i=1}^{9} c(i)q^i\right|,$$

where $p = e^{2\pi i(u+1.16i)}$, $q = e^{2\pi i(x+.865i)}$, and $|u|, |x| \le .5$.

To bound $G(x,u)$, we examine the function $G(x,u)^2$, which can be written as an expression in $\cos(2\pi nx), \cos(2\pi nu), \sin(2\pi nx)$, and $\sin(2\pi nu)$. After finding bounds on the partial derivatives of G^2 with respect to x and u, we compute its values on a grid of points satisfying $|u|, |x| \le .5$ to see that $G^2 \ge 900$, implying that $G(x,u) \ge 30$ in this range. The computations were performed using Maple, and were shortened by noting that $G(x,u) = G(-x,-u)$; the bounds on derivatives were calculated by trivially bounding the second derivatives and, again, computing values on a grid of points.

Putting together these computations, we see that

$$|A_k(m,n)| \leq 2003.34 \cdot 7.358^{\ell} e^{-2\pi m \cdot 1.16} e^{2\pi n \cdot 0.865}.$$

We now use this estimate on $|A_k(m,n)|$ to estimate $\langle G, G\rangle$, where $G = \sum_{m=1}^{\ell} a(m) F_{k,m}$. We have

$$\begin{aligned} \langle G, G\rangle &= \frac{3}{\pi} \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} |G(x+iy)|^2 y^{k-2}\, dy\, dx \\ &\leq \frac{3}{\pi} \int_{\sqrt{3}/2}^{\infty} \int_{-1/2}^{1/2} |G(x+iy)|^2 y^{k-2}\, dx\, dy. \end{aligned}$$

Plugging in the Fourier expansion $G(z) = \sum_{n=1}^{\infty} a(n) q^n$ and using the fact that we are integrating over a complete period gives

$$\langle G, G\rangle \leq \frac{3}{\pi} \sum_{n=1}^{\infty} |a(n)|^2 \int_{\sqrt{3}/2}^{\infty} y^{k-2} e^{-4\pi n y}\, dy.$$

Setting $u = 4\pi n y$, $du = 4\pi n\, dy$ gives

$$\langle G, G\rangle \leq \frac{12}{(4\pi)^k} \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^{k-1}} \int_{2\pi\sqrt{3}n}^{\infty} u^{k-2} e^{-u}\, du. \tag{1}$$

We have

$$a(n) = \sum_{m=1}^{\ell} a(m) A_k(m,n)$$

and so for $n \geq \ell + 1$, we have

$$|a(n)|^2 \leq (2003.34)^2 (7.358)^{2\ell} \left| \sum_{m=1}^{\ell} a(m) e^{-2\pi m \cdot 1.16} \right|^2 \cdot e^{4\pi n \cdot 0.865}.$$

For $1 \leq n \leq \ell$ we use the simple bound

$$\int_{2\pi\sqrt{3}n}^{\infty} u^{k-2} e^{-u}\, du \leq \int_0^{\infty} u^{k-2} e^{-u}\, du = (k-2)!.$$

Hence, the contribution to $\langle G, G\rangle$ from the terms with $1 \leq n \leq \ell$ is at most

$$\frac{12(k-2)!}{(4\pi)^k} \sum_{n=1}^{\ell} \frac{|a(n)|^2}{n^{k-1}}.$$

For $n \geq \ell + 1$ we use that

$$\int_{2\pi\sqrt{3}n}^{\infty} u^{k-2} e^{-u}\, du = e^{-2\pi\sqrt{3}n} \sum_{i=0}^{k-2} \frac{(k-2)!}{i!} (2\pi\sqrt{3}n)^i.$$

Since the highest power of n in this expression is $k-2$, the piece

$$\frac{1}{n^{k-1}} \sum_{i=0}^{k-2} \frac{(k-2)!}{i!} (2\pi\sqrt{3}n)^i$$

of the right side of equation (1) is a decreasing function of n and is therefore bounded by

$$\frac{1}{(\ell+1)^{k-1}} \sum_{i=0}^{\infty} \frac{(k-2)!}{i!} (2\pi\sqrt{3}(\ell+1))^i = \frac{(k-2)! e^{2\pi\sqrt{3}(\ell+1)}}{(\ell+1)^{k-1}}.$$

Hence, the contribution to $\langle G, G \rangle$ from the terms with $n \geq \ell + 1$ is at most

$$\frac{12}{(4\pi)^k} \cdot (2003.34)^2 (7.358)^{2\ell} \left| \sum_{m=1}^{\ell} a(m) e^{-2\pi m \cdot 1.16} \right|^2 \cdot \frac{(k-2)! e^{2\pi\sqrt{3}(\ell+1)}}{(\ell+1)^{k-1}} \cdot \sum_{n=\ell+1}^{\infty} e^{4\pi n \cdot 0.865} e^{-2\pi\sqrt{3}n}.$$

The sum on n is a geometric series, and we have $4\pi \cdot 0.865 - 2\pi\sqrt{3} \leq -0.01288$. This gives the bound

$$\frac{(k-2)!(12168805)^2}{(4\pi)^k} \left| \sum_{m=1}^{\ell} a(m) e^{-2\pi m \cdot 1.16} \right|^2 \cdot \frac{(7.358)^{k/6} 12^k e^{k\pi\sqrt{3}/6} e^{-0.00107k}}{k^{k-1}}.$$

Thus, we have

$$\langle G, G \rangle \leq \frac{12(k-2)!}{(4\pi)^k} \sum_{m=1}^{\ell} \frac{|a(m)|^2}{m^{k-1}} + \frac{(12168805)^2 (k-2)!}{(4\pi)^k} \left| \sum_{m=1}^{\ell} a(m) e^{-2\pi m \cdot 1.16} \right|^2 \cdot \frac{(41.41)^k}{k^{k-1}}.$$

Now, we write $G = \sum_{i=1}^{\ell} c_i g_i$, where the g_i are the normalized Hecke eigenforms. Using the lower bound on $L(\mathrm{Sym}^2 g_i, 1)$ and the relation between $L(\mathrm{Sym}^2 g_i, 1)$ and $\langle g_i, g_i \rangle$, we get

$$\begin{aligned} \langle G, G \rangle &= \sum_{i=1}^{\ell} |c_i|^2 \langle g_i, g_i \rangle \\ &\geq \sum_{i=1}^{\ell} |c_i|^2 \cdot \left(\frac{3(k-1)!}{32\pi^2 (4\pi)^k \log(k)} \right). \end{aligned}$$

This gives an upper bound on $\sum_{i=1}^{\ell} |c_i|^2$ in terms of $\langle G, G\rangle$. The Cauchy-Schwarz inequality gives

$$\begin{aligned}
\sum_{i=1}^{\ell} |c_i| \leq & \sqrt{\ell}\sqrt{\sum_{i=1}^{\ell} |c_i|^2} \\
& \leq \sqrt{\frac{k}{k-1} \cdot \frac{32\pi^2}{3} \log(k)} \cdot \sqrt{\sum_{m=1}^{\ell} \frac{|a(n)|^2}{m^{k-1}}} \\
& + \sqrt{\frac{k}{k-1} \cdot \frac{32\pi^2}{3} \cdot \log(k)} \cdot 12168805 \cdot \left| \sum_{m=1}^{\ell} a(m) e^{-7.288m} \right| \cdot \frac{(41.41)^{k/2}}{k^{(k-1)/2}} \\
& \leq \sqrt{\log(k)} \left(11 \cdot \sqrt{\sum_{m=1}^{\ell} \frac{|a(m)|^2}{m^{k-1}}} + \frac{e^{18.72}(41.41)^{k/2}}{k^{(k-1)/2}} \cdot \left| \sum_{m=1}^{\ell} a(m) e^{-7.288m} \right| \right).
\end{aligned}$$

This concludes the proof of Theorem 1.

4. Proof of Theorem 2

Write $F_{k,0}(z) = E_k(z) + h(z)$, where

$$h(z) = \sum_{n=1}^{\infty} b(n) q^n.$$

Since $F_{k,0}(z) = 1 + O(q^{\ell+1})$, we have

$$b(m) = \frac{2k}{B_k} \sigma_{k-1}(m)$$

for $1 \leq m \leq \ell$. We now apply Theorem 1, which gives that $b(n)$ is bounded by

$$\sqrt{\log(k)} \left(11 \sqrt{\sum_{m=1}^{\ell} \frac{|b(m)|^2}{m^{k-1}}} + \frac{e^{18.72}(41.41)^{k/2}}{k^{(k-1)/2}} \cdot \left| \sum_{m=1}^{\ell} b(m) e^{-7.288m} \right| \right) d(n) n^{\frac{k-1}{2}}.$$

We have that

$$\zeta(k) = \frac{(-1)^{\frac{k}{2}-1} (2\pi)^k B_k}{(k-1)! \cdot 2k}.$$

If $k \geq 12$, then $1 \leq \zeta(k) \leq \zeta(12) \leq 1.00025$. Thus, for $k \geq 12$ we have

$$0.9997 \frac{(2\pi)^k}{(k-1)!} \leq -\frac{2k}{B_k} \leq \frac{(2\pi)^k}{(k-1)!}.$$

Now, we have

$$\sigma_{k-1}(m) = \sum_{d \mid m} d^{k-1} = \sum_{d \mid m} (m/d)^{k-1} = m^{k-1} \sum_{d \mid m} \frac{1}{d^{k-1}} \leq m^{k-1} \zeta(k-1).$$

We have

$$\sqrt{\sum_{m=1}^{\ell} \frac{|b(m)|^2}{m^{k-1}}} \leq -\frac{2k\zeta(k-1)}{B_k} \sqrt{\sum_{m=1}^{\ell} m^{k-1}}.$$

Also,

$$\begin{aligned} \sum_{m=1}^{\ell} m^{k-1} &= \int_1^{\ell+1} \lfloor x \rfloor^{k-1} \, dx \leq \int_1^{\ell+1} x^{k-1} \, dx \leq \frac{(\ell+1)^k}{k} \\ &\leq \frac{\ell^k \left(1+\frac{1}{\ell}\right)^{12\ell+12}}{k} \leq \frac{e^{12} \ell^k}{k} \cdot \left(1+\frac{1}{\ell}\right)^{12}. \end{aligned}$$

Thus, the contribution from the first term in Theorem 1 is

$$\frac{(2\pi)^k}{(k-1)!} \frac{11 \cdot 1.0005 \cdot e^6 \ell^{k/2}}{\sqrt{k}} \left(1+\frac{1}{\ell}\right)^6 \sqrt{\log(k)}.$$

The function $m^{k-1} e^{-7.288m}$ always has a maximum at $m = \ell$. Thus, the second term of the bound from Theorem 1 is at most

$$\begin{aligned} &\frac{\zeta(11) e^{18.72} (41.41)^{k/2} \ell^k e^{-7.288\ell} (2\pi)^k \sqrt{\log(k)}}{(k-1)! k^{(k-1)/2}} \\ &\leq \frac{(2\pi)^k}{(k-1)!} e^{28.4657} \ell^{(k+1)/2} (1.0242382)^{k/2} \sqrt{\log(k)}. \end{aligned}$$

Adding the two contributions above, we have that

$$C(h) \leq \frac{(2\pi)^k}{(k-1)!} e^{28.466} \sqrt{\ell \log(k)} (1.0242382\ell)^{k/2},$$

and so $|b(n)| \leq C(h) d(n) n^{\frac{k-1}{2}} \leq 2C(h) n^{k/2}$. Now, we have

$$a(n) = -\frac{2k}{B_k} \sigma_{k-1}(n) + b(n) \geq 0.9997 \frac{(2\pi)^k}{(k-1)!} n^{k-1} - 2C(h) n^{k/2}.$$

The right hand side is positive if

$$\begin{aligned} n^{\frac{k}{2}-1} &\geq \frac{2e^{28.466} \sqrt{\ell \log(k)} (1.0242382\ell)^{k/2}}{0.9997} \\ n &\geq e^{58.366/(k-2)} \left(\ell^3 \log(k)\right)^{\frac{1}{k-2}} \cdot 1.0242382\ell. \end{aligned}$$

This concludes the proof of Theorem 2.

5. Proof of Corollary 3

To verify that all Fourier coefficients of $F_{k,0}(z)$ are non-negative for $k \in \{81288, 81460, 81632\}$, we use the bound from Theorem 2. This shows that any negative Fourier coefficient occurs within the first 10000. We find the unique linear combination

$$\sum_{i=0}^{k/4} c_i E_4^{k-3i} \Delta^i = 1 + O(q^{\ell+1})$$

and this form will equal $F_{k,0}(z)$. It then suffices to check the first 10000 Fourier coefficients are non-negative. These computations are performed in Magma [3], and take approximately 3 days for each weight.

Recall that

$$F_{k,0}(z) = \sum_{n=0}^{\infty} a(n) q^n.$$

We will show that $a(\ell+2) < 0$ for k sufficiently large (depending on $k \bmod 12$), making effective the work of Mallows, Odlyzko, and Sloane. Write

$$E_4^{-k/4} = \sum_{n=0}^{\infty} A(n) j^{-n}$$

where j is the usual j-function. Bürmann's theorem gives that

$$A(n) = \left(-\frac{k}{4n}\right) \cdot \text{ the coefficient of } q^{n-1} \text{ in } \left(\frac{dE_4}{dq} \frac{E_4^{3n-k/4-1} q^n}{\Delta^n}\right). \tag{2}$$

Mallows, Odlyzko, and Sloane show (see [10], pg. 73) that

$$\begin{aligned} a(\ell+1) &= -A(\ell+1) > 0 \\ a(\ell+2) &= -A(\ell+2) + A(\ell+1)\left(24\ell - 240\nu + 744\right). \end{aligned}$$

We write

$$\begin{aligned} A(\ell+1) &= -\frac{k}{4(\ell+1)} \int_{-1/2}^{1/2} \theta(E_4) E_4^{2-\nu} \frac{1}{\Delta^{\ell+1}}\, dx \\ A(\ell+2) &= -\frac{k}{4(\ell+2)} \int_{-1/2}^{1/2} \theta(E_4) E_4^{5-\nu} \frac{1}{\Delta^{\ell+2}}\, dx \end{aligned}$$

where $\theta\left(\sum a_n q^n\right) = \sum n a_n q^n$, and the integrals are over the line segment $x + iy$, $-1/2 \le x \le 1/2$ where y is fixed. We wish to find an upper bound on $|A(\ell+2)|$ and a lower bound on $|A(\ell+1)|$.

We choose y so that $\frac{\Delta'(iy)}{\Delta(iy)} = 0$ (so $y \approx 0.52352$). We write the integrals above in the form
$$\int_{-1/2}^{1/2} H_j(x+iy)e^{-(\ell+j)\ln(\Delta(x+iy))}\,dx$$
where $H_1(x+iy) = \theta(E_4)(x+iy)E_4(x+iy)^{2-\nu}$ and $H_2(x+iy) = \theta(E_4)(x+iy)E_4(x+iy)^{5-\nu}$.

If $B(x) = -\ln(\Delta(x+iy))$, then $|B(x)| \leq B(0) \approx 4.23579$. Moreover, the choice of y gives that $B'(0) = 0$. We use Taylor's theorem with the Lagrange form of the remainder to write
$$B(x) = B(0) + \frac{1}{2}x^2\mathrm{Re}(B)''(z_1) + \frac{i}{2}x^2\mathrm{Im}(B)''(z_2) := B(0) + x^2C_1(x) + ix^2C_2(x).$$
for some z_1 and z_2 between 0 and x. We bound from above and below the second derivatives of the real and imaginary parts of B. We derive similar bounds on $H_1(x+iy)$ and $H_2(x+iy)$.

We then have
$$e^{-(\ell+j)B(x)} = e^{-(\ell+j)B(0)} \cdot e^{C_1(x)x^2}\left(\cos\left((\ell+j)C_2(x)x^2\right) + i\sin\left((\ell+j)C_2(x)x^2\right)\right).$$
Since the integrals we are studying are both real, we wish to approximate the real part of the integrand. The main contribution comes in an interval of length about $\frac{1}{\sqrt{\ell}}$ in a neighborhood of $x = 0$, chosen so that $\cos((\ell+j)C_2(x)x^2)$ is positive. We bound the contribution of the remaining part of $-1/2 \leq x \leq 1/2$ trivially.

The bounds we obtain from this method show that $a(\ell+2) < 0$ if $k \geq 84636$, $k \geq 83332$, and $k \geq 82532$ if $\nu = 0$, $\nu = 1$, or $\nu = 2$, respectively. We use (2) to compute the coefficient $a(\ell+2)$ for all k between the bounds given in Corollary 3 and the bounds above. This concludes the proof.

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