EXPLICIT BOUNDS FOR THE NUMBER OF p-CORE PARTITIONS

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ABSTRACT. In this article, we derive explicit bounds on $c_t(n)$, the number of t-core partitions of n. In the case when $t = p$ is prime, we express the generating function $f(z)$ as the sum

$$f(z) = e_p E(z) + \sum_i r_i g_i(z)$$

of an Eisenstein series and a sum of normalized Hecke eigenforms. We combine the Hardy-Littlewood circle method with properties of the adjoint square lifting from automorphic forms on GL(2) to GL(3) to bound $R(p) := \sum_i |r_i|$, solving a problem raised by Granville and Ono.

In the case of general t, we use a combination of techniques to bound $c_t(n)$ and as an application prove that for all $n \geq 0$, $n \neq t + 1$,

$$c_{t+1}(n) \geq c_t(n)$$

provided $4 \leq t \leq 198$, as conjectured by Stanton.

1. INTRODUCTION AND STATEMENT OF RESULTS

A partition λ is a non-increasing sequence of natural numbers whose sum is n. Partitions are represented as Ferrers-Young diagrams, where the summands in the partition are arranged in rows. For example, the Ferrers-Young diagram for $12 = 5 + 4 + 2 + 1$ is below.
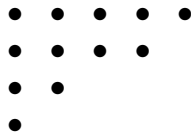
The *hook number* $h_{i,j}$ of a node (i, j) in the Ferrers-Young diagram is the number of nodes in the hook containing that node. For example, the hook numbers of the nodes in the first row above are 8, 6, 4, 3, and 1, respectively. If t is a positive integer, a partition is called t-core if none of the hook numbers are multiples of t. If $c_t(n)$ is the number of t-core partitions of n, then it is well-known [11] that

$$(1.1) \qquad \sum_{n=0}^{\infty} c_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{tn})^t}{1 - q^n}.$$

One motivation for studying t-core partitions comes from the representation theory of the symmetric group. Each partition α of n corresponds naturally to an irreducible representation

$\rho : S_n \to \mathrm{GL}_d(\mathbb{C})$. Here the dimension d is given by the Frame-Thrall-Robinson hook formula (see [21], Theorem 2.3.21)

$$(1.2) \qquad d = \frac{n!}{\prod_{i,j} h_{i,j}},$$

where the denominator is the product of the hook numbers of the partition α. Alfred Young showed that a basis can be chosen for the d-dimensional space on which S_n acts so that the image of ρ lies in $\mathrm{GL}_d(\mathbb{Z})$ (see [21], Section 3.4). As a consequence, one obtains for each partition α a representation $S_n \to \mathrm{GL}_d(\mathbb{F}_p)$, by composing ρ with the natural map $\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{F}_p)$. This resulting p-modular representation is irreducible if and only if the power of p dividing d is equal to the power of p dividing $n!$. From (1.2), this occurs if and only if the original partition is a p-core partition.

A number of papers (see [7], [11], [12]) have investigated the combinatorial properties of $c_t(n)$. Of particular note is the paper [11] of Garvan, Kim and Stanton, in which t-core partitions are used to produce cranks that combinatorially prove Ramanujan's congruences

$$\begin{aligned} p(5n + 4) &\equiv 0 \quad (\mathrm{mod}\ 5) \\ p(7n + 5) &\equiv 0 \quad (\mathrm{mod}\ 7) \\ p(11n + 6) &\equiv 0 \quad (\mathrm{mod}\ 11), \end{aligned}$$

where $p(n)$ is the number of partitions of n.

Because of the connections with representation theory, the positivity of and asymptotics for $c_t(n)$ have been extensively studied (see the papers by Ono [29, 30], Granville and Ono [14], and by Anderson [1]). In [35], Stanton stated (a slight variant) of the following conjecture.

Conjecture (Stanton's Conjecture). *If $t \geq 4$ and $n \neq t + 1$, then*

$$c_{t+1}(n) \geq c_t(n).$$

The restriction on n is necessary since $c_{t+1}(t + 1) = c_t(t + 1) - 1$. Motivated by this conjecture, Anderson [1] uses the circle method to establish asymptotics for $c_t(n)$ and to verify that Stanton's conjecture is true for a fixed t provided n is sufficiently large. In [14], Granville and Ono prove that if $t \geq 4$, then $c_t(n) > 0$ for all $n > 0$. When $t \geq 17$, Granville and Ono use an expression (due to Garvan, Kim, and Stanton) for $c_t(n)$ as the number of representations of n by a particular quadratic form to prove positivity. The previous papers [29, 30] of Ono established positivity in all other cases $t \leq 16$ with the exception of $t = 13$. To describe Granville and Ono's approach in this case, we need some notation.

Let $p \geq 5$ be prime. Let $\chi_p(n) = \left(\frac{n}{p}\right)$. The modular form

$$f(z) := \frac{\eta^p(pz)}{\eta(z)} = \sum_{n=0}^{\infty} a_p(n)q^n = \sum_{n=0}^{\infty} c_p(n)q^{n + \frac{p^2 - 1}{24}} \in M_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$$

is essentially the generating function for $c_p(n)$. Let

$$\sigma_{\frac{p-1}{2},\chi_p}(n) = \sum_{d|n} \chi_p\left(\frac{n}{d}\right) d^{\frac{p-3}{2}},$$

and

$$E_{\frac{p-1}{2}}(z) := \sum_{n=1}^{\infty} \sigma_{\frac{p-1}{2},\chi_p}(n)q^n$$

be one of the Eisenstein series of weight $\frac{p-1}{2}$ and level p. From the q-expansions of $f(z)$ and $E_{\frac{p-1}{2}}(z)$ at the cusp 0, we see that $f(z)$ can be decomposed as

$$f(z) = e_p E_{\frac{p-1}{2}}(z) + g(z)$$

where $g(z)$ is a cusp form in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ and e_p is the constant defined by

(1.3)
$$\frac{1}{e_p} = \frac{\left(\frac{p-3}{2}\right)! p^{\frac{p}{2}}}{(2\pi)^{\frac{p-1}{2}}} L\left(\frac{p-1}{2}, \chi_p\right).$$

The form $g(z)$ can be expressed as a linear combination

(1.4)
$$g(z) = \sum_{i=1}^{s} r_i g_i(z),$$

of normalized Hecke eigenforms, where $s = \dim S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. As a consequence of the Weil conjectures, Deligne proved that the nth Fourier coefficient of $g_i(z)$ is bounded by $d(n)n^{\frac{p-3}{4}}$. To compute an explicit bound on $c_p(n)$, the problem is therefore to bound the "cusp constant"

(1.5)
$$R(p) := \sum_{i=1}^{s} |r_i|.$$

In [33], the second author found asymptotics for the cusp constants of powers of $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. The problem of bounding $R(p)$ is significantly more challenging for two reasons: (i) the levels of the forms in questions are tending to infinity, and (ii) the form $f(z)$ is not a cusp form, and so we must understand the "size" of the difference between $f(z)$ and $e_p E_{\frac{p-1}{2}}(z)$. In [14], Granville and Ono explicitly calculate $R(p)$ for $p = 13$ by working in the 6-dimensional vector space $S_6(\Gamma_0(13), \chi_{13})$, and leave the remaining cases as an unsolved problem. We are able to determine an explicit upper bound on $R(p)$ valid for all primes p. As a consequence, we obtain the following explicit upper and lower bounds on $c_p(n)$.

Theorem 1.1. If $p \geq 5$ is an odd prime, $a_p(n) = c_p\left(n - \frac{p^2-1}{24}\right)$, and e_p is defined by (1.3), then

$$|a_p(n) - e_p\sigma_{\frac{p-1}{2},\chi_p}(n)| \leq \begin{cases} 98304 e^{6\pi} p^4 \log(p)\left(\frac{e^{1.5}}{8\pi}\right)^{\frac{p-1}{4}} d(n)n^{\frac{p-3}{4}} & \text{if } p \equiv 1 \pmod 4, \\ 388535 e^{6\pi} p^{\frac{9}{2}} \log(p)^{11/4}\left(\frac{e^{1.5}}{8\pi}\right)^{\frac{p-1}{4}} d(n)n^{\frac{p-3}{4}} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Noting that $e^{1.5} \approx 4.48 < 25.13 \approx 8\pi$, we immediately see the following.

Corollary 1.2. *Under the same assumptions as Theorem 1.1, $R(p) = \sum_{i=1}^{s} |r_i|$ tends to zero as p tends to infinity.*

Remark.   (1) *The bound in Theorem 1.1 is far from optimal. Numerical evidence suggests that $R(p)$ is not too far from the lower bound of about*

$$\frac{(2\pi^2/3)^{(p-3)/4}}{\left(\frac{p-3}{2}\right)!}.$$

(2) *When $p = 5$, $e_5 = 1$ and $f(z)$ is the Eisenstein series $E_2(z)$. Therefore, for all $n \geq 1$, $a_5(n) = \sigma_{2,\chi_5}(n)$.*

We briefly describe our approach to the problem. First, we derive bounds on $c_p(n)$ using the circle method. From these bounds, we derive an upper bound A on the Petersson inner product $\langle f, g_i \rangle$, defined by

$$\langle f, g_i \rangle := \frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)]} \int_{\mathbb{H}/\Gamma_0(p)} f(z)\overline{g_i(z)}y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2}.$$

It is known that the forms $\{E_{\frac{p-1}{2}}, g_1, \ldots, g_s\}$ are pairwise orthogonal, and so we have

$$\langle f, g_i \rangle = r_i \langle g_i, g_i \rangle.$$

Hence if B is a lower bound for $\langle g_i, g_i \rangle$, then $r_i \leq A/B$.

To derive a lower bound on $\langle g_i, g_i \rangle$, we use the fact that this quantity is essentially the special value at $s = 1$ of the adjoint square L-function associated to g_i. Goldfeld, Hoffstein, and Lieman showed in the appendix to [17] that this L-function has no Siegel zeroes, and we make their argument effective. An argument of Hoffstein [16] translates this zero-free region into a lower bound for the special value. In order to do this, we need to compute the local factors at p of the adjoint square and symmetric fourth powers of the g_i. This is done using the local Langlands correspondence.

As a consequence of Theorem 1.1, we obtain the following more precise version of [14, Theorem 4]. Recall that there is a bijection between the defect zero p-blocks of S_n and the p-core partitions of n.

Corollary 1.3. *Let $p \geq 5$ be an odd prime and let e_p and $R(p)$ be the constants defined by (1.3) and (1.5). Then there are more than $\frac{2e_p}{5}n^{\frac{p-3}{2}}$ p-blocks with defect zero provided $n > \left(\frac{10R(p)}{e_p}\right)^{\frac{4}{p-5}}$.*

Remark. *Note that $\left(\frac{10R(p)}{e_p}\right)^{\frac{4}{p-5}} \leq p^4$ for large primes p.*

As a second application, we will prove an inequality involving $c_p(n)$. Recently, many interesting inequalities for the number of p-core partitions have been investigated using either modular equations or modular forms (see [5], [6], and [23]). The following inequality gives an explicit version of [23, Theorem 4].

Corollary 1.4. *Let $p \geq 7$ is prime, t is a positive integer ≥ 2, and $k \geq 1$. Let $\delta_p = \frac{p^2-1}{24}$, and e_p and $R(p)$ the constants defined by (1.3) and (1.5). Then for all*

$$n > \left(\frac{2\zeta \left(\frac{p-3}{2} \right)}{e_p} R(p) \left((k+1)t^{\frac{k(p-3)}{4}} + \sigma_{\frac{p-1}{2}, \chi_p}(t^k) - 1 \right) \right)^{\frac{4}{p-5}}$$

with $(n, t) = 1$, we have

(1.6) $$c_p(t^k n + \delta_p(t^k - 1)) > \left(\sigma_{\frac{p-1}{2}, \chi_p}(t^k) - 1 \right) c_p(n).$$

Remark. *For large primes p, the bound on n in Corollary 1.4 is less than or equal to p^{4+k}.*

Finally, the bounds we obtain on $c_t(n)$ using the circle method allow us to derive an explicit bound on possible counterexamples to Stanton's conjecture.

Theorem 1.5. *For all integers $t \geq 7$, if*

$$n \geq \begin{cases} \left(45503 t^{\frac{2t+1}{2}} \left(\frac{1}{2^7 \pi^3 \sqrt{e}} \right)^{\frac{t-1}{4}} \right)^{\frac{4}{t-4}}, & \text{if } t \geq 36, \\ \left(288305 t^{\frac{3t+7}{4}} \left(\frac{1}{4\pi^3 \sqrt{e}} \right)^{\frac{t-1}{4}} \right)^{\frac{4}{t-4}}, & \text{if } 7 \leq t \leq 35, \end{cases}$$

and $n \geq (t+1)^2$, then $c_{t+1}(n) > c_t(n)$.

Applying this theorem when $t \geq 12$, as well as more specialized arguments when $4 \leq t \leq 11$, we can verify Stanton's conjecture.

Corollary 1.6. *For $4 \leq t \leq 198$, Stanton's conjecture holds.*

The paper is organized as follows. In Section 2, we will review basic facts on the circle method and modular forms. In Sections 3, the circle method is used to derive explicit bounds on $c_t(n)$ and also on $\langle f, g_i \rangle$. In Section 4, the result of Goldfeld, Hoffstein and Lieman is made effective and a lower bound on $\langle g_i, g_i \rangle$ is computed. In Section 5, we will prove Theorem 1.1 and its corollaries. In Section 6, we will prove Theorem 1.5 and Corollary 1.6. In Section 7, we raise some questions for future study.

2. Preliminaries

In this section, we give a brief background on modular forms and basic tools for the circle method. For additional properties of modular forms, see [31, Chaps. 1, 2, and 3].

As usual, let $\eta(z)$ be Dedekind's eta function defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n),$$

where $q = \exp(2\pi i z)$ and z is in the complex upper half plane $\mathbb{H}$.

We define $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}$. For a meromorphic function f on $\mathbb{H}$, we define the slash operator by

$$(f|_k\gamma)(z) := (\det \gamma)^{\frac{k}{2}} (cz + d)^{-k} f(\gamma z),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. Suppose that f is a holomorphic function on $\mathbb{H}$ and χ is a Dirichlet character modulo N. We say f is a holomorphic modular (resp. cusp) form of weight k on $\Gamma_0(N)$ with character χ if f is holomorphic (resp. vanishing) at the cusps of $\Gamma_0(N)$ and $f|_k\gamma(z) = \chi(d)f(z)$ for all $\gamma \in \Gamma_0(N)$. Let $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) denote the vector space of holomorphic forms (resp. cusp forms) on $\Gamma_0(N)$ with character χ. It is well-known that for primes $p \geq 5$, we have $\frac{\eta^p(pz)}{\eta(z)} \in M_{(p-1)/2}(\Gamma_0(p), \chi_p)$.

For each prime p, the Hecke operator T_p is a linear operator on $S_k(\Gamma_0(N), \chi)$. If $f(z) \in S_k(\Gamma_0(N), \chi)$ has the Fourier expansion $f(z) = \sum_{n \geq 0} a(n)q^n$, then

$$T_p f := \sum_{n \geq 0} \left(a(pn) + \chi(p)p^{k-1}a\left(\frac{n}{p}\right) \right) q^n.$$

We say that $f(z)$ is an eigenform of T_p if there is a $\lambda_p \in \mathbb{C}$ such that $T_p f = \lambda_p f$. We call $f(z) \in M_k(\Gamma_0(N), \chi)$ a Hecke eigenform if $f(z)$ is an eigenform of T_p for all primes p. It is well-known that $S_k(\Gamma_0(p), \chi_p)$ has basis of Hecke eigenforms (since in this case the old space is trivial), and these can be normalized so that the leading Fourier coefficient is 1. With this normalization, these forms are referred to as newforms. The Atkin-Lehner involution on $M_k(\Gamma_0(p), \chi_p)$ is defined by $f|_k \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$.

Now, we turn to the basic facts about the circle method. If $f(z) := \sum_{n=0}^{\infty} a(n)q^n$, then the residue theorem implies that

$$(2.1) \qquad a(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{f(z)}{q^{n+1}} dq.$$

We choose $r = e^{\frac{-2\pi}{N^2}} := e^{-2\pi\rho}$ for a positive N to be determined later. By following the dissection given in [2, chap. 5] or [9, p.115–117] and setting $z = k(\rho - i\varphi)$ and $\tau = \frac{h+iz}{k}$, we arrive at

$$(2.2) \qquad a(n) = \sum_{\substack{1 \leq k \leq N \\ 0 \leq h \leq k \\ (h,k)=1}} \sum e^{\frac{-2\pi i n h}{k}} \int_{\xi_{h,k}} f(\tau)e^{2\pi n \rho}e^{-2\pi i n \varphi} d\varphi,$$

where $\xi_{h,k} = [-\theta'_{h,k}, \theta''_{h,k}]$, and

$$\theta'_{h,k} = \frac{h}{k} - \frac{h_0 + h}{k_0 + k}$$

$$\theta''_{h,k} = \frac{h_1 + h}{k_1 + h} - \frac{h}{k}.$$

Here $\frac{h_0}{k_0}, \frac{h}{k}, \frac{h_1}{k_1}$ are three consecutive terms of the Farey sequence of order N. Note that each θ satisfies $\frac{1}{2kN} \le \theta \le \frac{1}{kN}$.

The following transformation formulas for the Dedekind η function will play an important role in the next section. For a proof of the transformation formulas, see [4, pp. 52-61].

Theorem 2.1. *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have*

$$\eta(\gamma z) = e^{-\pi i s(d,c)} e^{\frac{\pi i (a+d)}{12c}} \sqrt{-i(cz+d)} \eta(z),$$

where $s(d,c)$ is the Dedekind sum defined by $s(d,c) = \sum_{r=1}^{c-1} \left(\frac{r}{c} - \left[\frac{r}{c} \right] - \frac{1}{2} \right) \left(\frac{dr}{c} - \left[\frac{dr}{c} \right] - \frac{1}{2} \right).$

We prove the following two lemmas by using Theorem 2.1. We omit the proofs.

Lemma 2.2. *Let h, k be integers such that $k > 0$ and $(h,k) = 1$. Let $hh' \equiv -1 \pmod{k}$ and $z \in \mathbb{H}$. If $\tau = \frac{h+iz}{k}$, then*

$$\eta \left(\frac{h' + iz^{-1}}{k} \right) = e^{-\pi i s(-h,k)} e^{\pi i \frac{h'-h}{12k}} \sqrt{z} \eta(\tau).$$

Lemma 2.3. *Let h, k be integers such that $k > 0$ and $(h,k) = 1$. Let $hh' \equiv -1 \pmod{k}$, $thh'' \equiv -(t,k) \pmod{k}$, and $\tau = \frac{h+iz}{k}$. Then*

$$\eta \left(\frac{(t,k)h''}{k} + i \frac{(t,k)^2}{ktz} \right) = e^{-\pi i s\left(-\frac{th}{(t,k)}, \frac{k}{(t,k)}\right)} e^{\pi i \frac{h'' - \frac{th}{(t,k)}}{12 \frac{k}{(t,k)}}} \sqrt{\frac{t}{(t,k)}} z \eta(t\tau).$$

We obtain the following lemma by modifying the argument in [9, Lemma 3.2].

Lemma 2.4. *Let*

$$I := \int_{\xi_{h,k}} z^{-\frac{p-1}{2}} e^{2\pi n \rho} e^{-2\pi i n \varphi} d\varphi.$$

Then we have

(2.3)
$$I = \frac{(2\pi)^{\frac{p-1}{2}}}{k^{\frac{p-1}{2}} \Gamma(\frac{p-1}{2})} n^{\frac{p-3}{2}} + E(I),$$

where $|E(I)| \le 2^{\frac{p+1}{2}} N^{\frac{p-1}{2}} \frac{e^{2\pi n \rho}}{2\pi n}.$

The following estimate will play an important role in Sections 3 and 6. Let

$$F(q) = \prod_{n=0}^{\infty}(1 - q^n)^{-1} = \sum_{n=0}^{\infty} p(n)q^n.$$

Then from the upper bound

$$p(n) < e^{\pi\sqrt{2n/3}}$$

(see Theorem 14.5 of [4, p. 316]), we have

$$(2.4) \qquad |F(q)| \leq \sum_{n=0}^{\infty} p(n)|q|^n \leq \sum_{n=0}^{\infty} e^{\pi\sqrt{\frac{2n}{3}}} e^{-2\pi yn}.$$

It is easy to see that $\pi\sqrt{2n/3} - 2\pi ny \leq -\pi ny$ if $n \geq \frac{2}{3y^2}$. It follows that

$$(2.5) \qquad |F(q)| \leq \sum_{0 \leq n < \frac{2}{3y^2}} e^{\frac{\pi}{12y}} + \sum_{n \geq \frac{2}{3y^2}} e^{-\pi yn} \leq \frac{2}{3y^2} e^{\frac{\pi}{12y}} + \frac{e^{-\frac{2\pi}{3y}}}{1 - e^{-\pi y}}.$$

We will use this estimate with $y = \frac{1}{2t}$. When t is small, we will use the estimate

$$(2.6) \qquad \sum_{n=0}^{\infty} p(n)e^{-2\pi yn} \leq \exp\left(\frac{e^{-2\pi y}}{(1 - e^{-2\pi y})^2}\right),$$

given by Chan [9, Equation (3.19)].

3. An upper bound for $|\langle f, g \rangle|$

Recall that $p \geq 7$ is prime, $f(z) = \frac{\eta^p(pz)}{\eta(z)}$, and $g(z) \in S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ is a normalized Hecke eigenform. In this section, we will get an upper bound for

$$(3.1) \qquad \langle f, g \rangle = \frac{3}{\pi} \frac{1}{[\Gamma : \Gamma_0(p)]} \sum_{j=1}^{[\Gamma:\Gamma_0(p)]} \int_F f|_{\alpha_j^{-1}}(z)\overline{g|_{\alpha_j^{-1}}(z)} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2}.$$

Here $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ is a union of right cosets $\Gamma = \bigcup_j \alpha_j \Gamma_0(p)$ and F is the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.

Note that

$$(3.2) \qquad f|_{\frac{p-1}{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(z) = (-i)^{\frac{p-1}{2}} p^{\frac{-p}{2}} \frac{\eta^p(z)}{\eta(pz)} := (-i)^{\frac{p-1}{2}} p^{-\frac{p}{2}} \sum_{n=0}^{\infty} b_p(n)q_p^n.$$

Recall that $a_p(n)$ is the n-th Fourier coefficient of $f(z)$. Before calculating $|\langle f, g \rangle|$, we need to obtain an upper bound for $|a_p(n)|$ and $|b_p(n)|$.

Lemma 3.1. *For all integers $n \geq 1$ and odd primes $p \geq 7$, we have*

(3.3) $$|a_p(n)| \leq A_\infty(p) n^{\frac{p-3}{2}} + B_\infty(p) n^{\frac{p-1}{4}},$$

(3.4) $$|b_p(n)| \leq A_0(p) n^{\frac{p-3}{2}} + B_0(p) n^{\frac{p-1}{4}},$$

where $A_\infty(p)$, $A_0(p)$, $B_\infty(p)$, and $B_0(p)$ are constants (depending only on p) defined by

(3.5) $$A_\infty(p) = p^{-\frac{p}{2}} \frac{(2\pi)^{\frac{p-1}{2}} \zeta(\frac{p-3}{2})}{\Gamma(\frac{p-1}{2})},$$

(3.6) $$B_\infty(p) = e^{6\pi} \left(\frac{2e^{\frac{\pi}{p}}(C(p)-1)}{p^{\frac{p}{2}}} \left(\frac{p(p-1)}{8\pi e} \right)^{\frac{p-1}{4}} + 2.1 \left(\frac{3}{e\pi(p+1)} \right)^{\frac{p-1}{4}} + p^{-\frac{p}{2}} \frac{2^{\frac{p-1}{2}}}{\pi} \right),$$

(3.7) $$A_0(p) = \frac{(2\pi)^{\frac{p-1}{2}} \zeta(\frac{p-3}{2})}{\Gamma(\frac{p-1}{2})},$$

(3.8) $$B_0(p) = e^{6\pi} \left(2C(p)\sqrt{p} \left(\frac{3(p-1)}{e\pi(p-\frac{1}{p})} \right)^{\frac{p-1}{4}} + 2.1 \left(\frac{p-1}{8\pi e} \right)^{\frac{p-1}{4}} + \frac{2^{\frac{p-1}{2}}}{\pi} \right),$$

and $C(p) := \frac{8p^2}{3} e^{\frac{\pi p}{6}} + \frac{e^{\frac{-4\pi p}{3}}}{1-e^{\frac{-\pi}{2p}}}$.

We will prove this lemma at the end of the section.

Let $g(z) = \sum_{n=1}^{\infty} c(n)q^n \in S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. If $d(n)$ is the number of divisors of n, then Deligne's bound is

$$|c(n)| \leq d(n) n^{\frac{p-3}{4}}.$$

Note that g is an eigenform of the Atkin-Lehner involution with eigenvalue λ_p where $|\lambda_p| = 1$. Thus,

$$g|_{\frac{p-1}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (z) = \lambda_p p^{-\frac{p-1}{4}} \sum_{n=1}^{\infty} c(n)q_p^n.$$

Now we are ready to calculate an upper bound for $|\langle f, g \rangle|$. It is well known that $\alpha_j = I$ or $T^{-k}S$, where $k = 0, 1, \ldots, p-1$. When we set $\alpha_j = I$ in (3.1), we have

(3.9) $$\left| \int_F f(z)\overline{g(z)} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2} \right|$$

$$\leq \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{k=2}^{\infty} \left(\sum_{n=1}^{k-1} |a_p(k-n)||c(n)| \right) e^{-2\pi ky} y^{\frac{p-5}{2}} \, dy$$

$$\leq \frac{1}{(2\pi)^{\frac{p-3}{2}}} \int_{\pi\sqrt{3}}^{\infty} e^{-u} u^{\frac{p-5}{2}} \sum_{k\leq u} \frac{1}{k^{\frac{p-3}{2}}} \left(\sum_{n=1}^{k-1} |a_p(k-n)||c(n)| \right) du.$$

By using the summation by parts formula and Lemma 3.1, we obtain

$$\sum_{n=1}^{k-1} |a_p(k-n)||c(n)| \le A_\infty(p) \sum_{n=1}^{k} d(n)(k-n)^{\frac{p-3}{2}} n^{\frac{p-3}{4}} + B_\infty(p) \sum_{n=1}^{k} d(n)(k-n)^{\frac{p-1}{4}} n^{\frac{p-3}{4}}$$

$$\le A_\infty(p) \frac{p-3}{2} k \int_1^k D(t) t^{\frac{p-7}{4}} (k-t)^{\frac{p-5}{2}} dt$$

$$+ B_\infty(p) \frac{p-1}{4} k \int_1^k D(t) t^{\frac{p-7}{4}} (k-t)^{\frac{p-5}{4}} dt,$$

where $D(t) := \sum_{n \le t} d(n)$. Therefore, by using the Beta integral

$$\int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

and $D(t) \le 1.8t^{5/4} + 3.6t^{1/4}$, we arrive at

$$\sum_{n=1}^{k-1} |a_p(k-n)||c(n)| \le A_\infty(p) \frac{9(p-3)}{10} \left(k^{\frac{3p-4}{4}} \frac{\Gamma\left(\frac{p+2}{4}\right)\Gamma\left(\frac{p-3}{2}\right)}{\Gamma\left(\frac{3p-4}{4}\right)} + 2k^{\frac{3p-8}{4}} \frac{\Gamma\left(\frac{p-2}{4}\right)\Gamma\left(\frac{p-3}{2}\right)}{\Gamma\left(\frac{3p-8}{4}\right)} \right)$$

$$+ B_\infty(p) \frac{9(p-1)}{20} \left(k^{\frac{2p+1}{4}} \frac{\Gamma\left(\frac{p+2}{4}\right)\Gamma\left(\frac{p-1}{4}\right)}{\Gamma\left(\frac{2p+1}{4}\right)} + 2k^{\frac{2p-3}{4}} \frac{\Gamma\left(\frac{p-2}{4}\right)\Gamma\left(\frac{p-1}{4}\right)}{\Gamma\left(\frac{2p-3}{4}\right)} \right).$$

Note that for all real numbers $x \ge 1$, $\sum_{k \le u} k^x \le \frac{1}{x+1} u^{x+1} + u^x$. Applying this to (3.9), we arrive at

$$(3.10) \qquad \left| \int_F f(z)\overline{g(z)} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2} \right|$$

$$\le \frac{36 A_\infty(p)p}{5(2\pi)^{\frac{p-3}{2}}} \Gamma\left(\frac{p-2}{4}\right) \Gamma\left(\frac{p-3}{2}\right) + \frac{2B_\infty(p)p^2}{7(2\pi)^{\frac{p-3}{2}}} \Gamma\left(\frac{p-2}{4}\right) \Gamma\left(\frac{p-1}{4}\right)$$

$$:= U_\infty(p).$$

Similarly, for other α_j,

$$\left| \int_F f(z)|_{\alpha_j^{-1}} \overline{g(z)|_{\alpha_j^{-1}}} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2} \right|$$

$$\le \frac{1}{(2\pi)^{\frac{p-3}{2}}} \left(\frac{1}{p}\right)^{\frac{p+5}{4}} \int_{\frac{\pi\sqrt{3}}{p}}^{\infty} e^{-u} u^{\frac{p-5}{2}} \sum_{k \le u} \frac{1}{k^{\frac{p-3}{2}}} \left(\sum_{n=1}^{k} |b_p(k-n)||c(n)| \right) du.$$

By using a similar argument, we arrive at

$$(3.11) \qquad \left| \int_F f(z)\overline{g(z)} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2} \right|$$

$$\leq \frac{p^{-\frac{p+5}{4}}}{(2\pi)^{\frac{p-3}{2}}} \left(\frac{36A_0(p)p}{5} \Gamma\left(\tfrac{p-2}{4}\right) \Gamma\left(\tfrac{p-3}{2}\right) + \frac{2B_0(p)p^2}{7} \Gamma\left(\tfrac{p-2}{4}\right) \Gamma\left(\tfrac{p-1}{4}\right) + \Gamma\left(\tfrac{p-1}{2}\right) \right)$$

$$:= U_0(p).$$

By using Lemma 3.1, (3.10) and (3.11), we obtain the following theorem.

Theorem 3.2. *Let* $f(z) = \frac{\eta^p(pz)}{\eta(z)} \in M_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$, *and let* $g(z)$ *be a normalized newform in* $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. *Then,*

$$\frac{\pi[\Gamma : \Gamma_0(p)]}{3} |\langle f, g \rangle| \leq 161.6 \cdot e^{6\pi} \Gamma\left(\frac{p-2}{4}\right) \Gamma\left(\frac{p-1}{4}\right) p^{\frac{7}{2}} \left(\frac{e^{1.5}}{32\pi^3}\right)^{\frac{p-1}{4}}.$$

Now we will prove Lemma 3.1 by using the circle method. This is very similar to the argument of Anderson in [1].

Proof of Lemma 3.1. By (2.2), we have

$$a_p(n) = \left(\sum_{\substack{1 \leq k \leq N \\ (k,p)=1}} + \sum_{\substack{1 \leq k \leq N \\ p|k}} \right) \sum_{\substack{0 \leq h \leq k \\ (h,k)=1}} e^{\frac{-2\pi inh}{k}} \int_{\xi_{h,k}} \frac{\eta^p(p\tau)}{\eta(\tau)} e^{2\pi n\rho} e^{-2\pi in\varphi} d\varphi$$

$$=: S_1(A) + S_2(A),$$

where A is the integrand.

First, we consider $S_1(A)$. By (2.2) and (2.3), we have

$$(3.12) \qquad A = p^{-\frac{p}{2}} \omega_{h,k} z^{-\frac{p-1}{2}} \frac{\eta^t \left(\exp\left(2\pi i \frac{h''}{k} - 2\pi \frac{1}{kpz}\right) \right)}{\eta \left(\exp\left(2\pi i \frac{h'}{k} - 2\pi \frac{1}{pz}\right) \right)},$$

where $\omega_{h,k}$ is a constant depending on h and k with $|\omega_{h,k}| = 1$. Then,

$$S_1(A) = p^{-\frac{p}{2}} \sum_{\substack{1 \leq k \leq N \\ (k,p)=1}} \sum_{\substack{0 \leq h \leq k \\ (h,k)=1}} e^{\frac{-2\pi inh}{k}} \int_{\xi_{h,k}} \omega_{h,k} z^{-\frac{p-1}{2}} e^{2\pi n\rho} e^{-2\pi in\varphi} d\varphi$$

$$+ p^{-\frac{p}{2}} \sum_{\substack{1 \leq k \leq N \\ (k,p)=1}} \sum_{\substack{0 \leq h \leq k \\ (h,k)=1}} e^{\frac{-2\pi inh}{k}} \int_{\xi_{h,k}} \omega_{h,k} \times$$

$$\left(\frac{\eta^p \left(\exp\left(2\pi i \frac{h''}{k} - 2\pi \frac{1}{kpz}\right) \right)}{\eta \left(\exp\left(2\pi i \frac{h'}{k} - 2\pi \frac{1}{kz}\right) \right)} - 1 \right) z^{-\frac{p-1}{2}} e^{2\pi n\rho} e^{-2\pi in\varphi} d\varphi$$

$$:= T_1 + T_2.$$

By Lemma 2.4, we have

$$|T_1| \le p^{-\frac{p}{2}} \frac{(2\pi)^{\frac{p-1}{2}} \zeta(\frac{p-3}{2})}{\Gamma(\frac{p-1}{2})} n^{\frac{p-3}{2}} + e^{6\pi} p^{-\frac{p}{2}} n^{\frac{p-1}{4}} \frac{2^{\frac{p-1}{2}}}{\pi},$$

by setting $N = [\sqrt{n}]$, because $\frac{n}{[\sqrt{n}]^2} \le 3$ for all $n \ge 1$.

For T_2, note that we can set $h' = th''$. Thus, if we set $\alpha = \frac{h''}{k} + 2\pi i \frac{1}{kpz}$, then we have, by (2.4) and (2.5),

$$\left| \frac{\eta^p \left(\exp \left(2\pi i \frac{h''}{k} - 2\pi \frac{1}{kpz} \right) \right)}{\eta \left(\exp \left(2\pi i \frac{h'}{k} - 2\pi \frac{1}{kz} \right) \right)} - 1 \right| \le \left| \prod_{n=1}^{\infty} \frac{(1-q^n)^p}{1-q^{pn}} - 1 \right| \le \sum_{n=0}^{\infty} aa(n)|q|^n - 1$$

$$\le |q| \sum_{n=1}^{\infty} p(n)|q|^{n-1} \le e^{-\frac{2\pi}{k} \operatorname{Re} \frac{1}{pz}} e^{\frac{\pi}{p}} \left(C(p) - 1 \right).$$

Here, $q = \exp 2\pi i \alpha$ and $aa(n)$ is the number of partitions of n such that parts which are not multiples of p can be repeated up to p times and parts which are a multiple of p can be repeated at most $p-1$ times.

Therefore, we have

$$\left| \int_{\xi_{h,k}} \omega_{h,k} \left(\frac{\eta^p \left(\exp \left(2\pi i \frac{h''}{k} - 2\pi \frac{1}{kpz} \right) \right)}{\eta \left(\exp \left(2\pi i \frac{h'}{k} - 2\pi \frac{1}{kz} \right) \right)} - 1 \right) z^{-\frac{p-1}{2}} e^{2\pi n\rho} d\varphi \right|$$

$$\le e^{\frac{\pi}{p}} \left(C(p) - 1 \right) \int_{\xi_{h,k}} |z|^{-\frac{p-1}{2}} e^{-\frac{2\pi}{k} \operatorname{Re} \frac{1}{pz}} e^{2\pi n\rho} d\varphi$$

$$= e^{\frac{\pi}{p}} \left(C(p) - 1 \right) \int_{\xi_{h,k}} \left(\frac{p}{2\pi\rho} \right)^{\frac{p-1}{4}} \left(\frac{2\pi\rho}{pk^2(\rho^2 + \varphi^2)} \right)^{\frac{p-1}{4}} \exp \left(\frac{-2\pi\rho}{pk^2(\rho^2 + \varphi^2)} \right) e^{2\pi n\rho} d\varphi$$

$$\le e^{\frac{\pi}{p}} \left(C(p) - 1 \right) \left(\frac{p(p-1)}{8\pi e} \right)^{\frac{p-1}{4}} n^{\frac{p-1}{4}} e^{2\pi n\rho} \frac{2}{kN},$$

where for the last inequality, we used the fact that the maximum of $x^{\frac{p-1}{4}} e^{-x}$ on $[0, \infty)$ is $\left(\frac{p-1}{4e} \right)^{\frac{p-1}{4}}$ and the length of path is at most $\frac{2}{kN}$. Thus, by setting $N = [\sqrt{n}]$, we arrive at

$$|T_2| \le \frac{2e^{\frac{\pi}{p}+6\pi} \left(C(p) - 1 \right)}{p^{\frac{p}{2}}} \left(\frac{p(p-1)}{8\pi e} \right)^{\frac{p-1}{4}} n^{\frac{p-1}{4}}.$$

Similarly, we obtain the following upper bound for $|S_2(A)|$:

$$|S_2(A)| \le 2.1 e^{6\pi} \left(\frac{3}{e\pi(p+1)} \right)^{\frac{p-1}{4}} n^{\frac{p-1}{4}}.$$

In summary, we have deduced that

$$A_\infty(p) = p^{-\frac{p}{2}} \frac{(2\pi)^{\frac{p-1}{2}} \zeta(\frac{p-3}{2})}{\Gamma(\frac{p-1}{2})}, \quad \text{and}$$

$$B_\infty(p) = e^{6\pi} \left(\frac{2e^{\frac{\pi}{p}} (C(p)-1)}{p^{\frac{p}{2}}} \left(\frac{p(p-1)}{8\pi e} \right)^{\frac{p-1}{4}} + 2.1 \left(\frac{3}{e\pi(p+1)} \right)^{\frac{p-1}{4}} + p^{-\frac{p}{2}} \frac{2^{\frac{p-1}{2}}}{\pi} \right),$$

as desired. The calculation of $A_0(p)$ and $B_0(p)$ is analogous, so we omit it. $\square$

4. A LOWER BOUND FOR $\langle g, g \rangle$

In this section, we will derive a lower bound for $\langle g, g \rangle$, where $g \in S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ is a normalized Hecke eigenform. Our approach is to use that the number $\langle g, g \rangle$ arises in a formula for the special value at $s = 1$ of the adjoint square L-function $L(s, \mathrm{Ad}^2(g))$. In the appendix to [17], Goldfeld, Hoffstein and Lieman proved that this L-function has no zeroes close to $s = 1$. We make their argument effective, and use this to derive a lower bound on the special value at $s = 1$. In this section, we state all of our results at the beginning and provide proofs later in the section.

Write

$$g(z) = \sum_{n=1}^{\infty} a(n) q^n$$

and for primes q, define $\alpha_q, \beta_q \in \mathbb{C}$ by

$$\alpha_q + \beta_q = a(q)/q^{\frac{p-3}{4}}, \quad \alpha_q \beta_q = \chi_p(q).$$

Define the adjoint square L-function by

$$L(s, \mathrm{Ad}^2(g)) = \prod_q (1 - \alpha_q^2 \chi_p(q) q^{-s})^{-1} (1 - q^{-s})^{-1} (1 - \beta_q^2 \chi_p(q) q^{-s})^{-1},$$

and define the completed L-function by

$$\Lambda(s, \mathrm{Ad}^2(g)) = p^s \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k}{2}\right) L(s, \mathrm{Ad}^2(g)).$$

In [13], Gelbart and Jacquet show that $L(s, \mathrm{Ad}^2(g))$ is the L-function of an automorphic form on $\mathrm{GL}(3)$, and hence that it has an analytic continuation and functional equation of the usual type. However, it is not immediately clear that the local factors at p and ∞ of Gelbart and Jacquet's match the definition given above. The content of the next theorem is a computation of these local factors using the local Langlands correspondence.

Theorem 4.1. *Assume the notation above. Then, $L(s, \mathrm{Ad}^2(g))$ has an analytic continuation to all of $\mathbb{C}$ and satisfies the functional equation*

$$\Lambda(s, \mathrm{Ad}^2(g)) = \Lambda(1 - s, \mathrm{Ad}^2(g)).$$

The fact that $L(s, g \otimes g) = \zeta(s) L(s, \mathrm{Ad}^2(g))$ and the classical Rankin-Selberg theory (see Chapter 13 of [20]) implies the following special value formula. Recall that

$$\langle g, g \rangle = \frac{3}{\pi [\Gamma : \Gamma_0(p)]} \int_{\mathbb{H}/\Gamma_0(p)} |g(x + iy)|^2 y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2}.$$

Then

(4.1) $$L(1, \mathrm{Ad}^2(g)) = \frac{\pi}{2} \left(1 + \frac{1}{p} \right) \frac{(4\pi)^{\frac{p-1}{2}}}{\left(\frac{p-3}{2} \right)!} \langle g, g \rangle.$$

We say that a modular form $g(z) = \sum_{n=1}^{\infty} a(n) q^n$ of weight $k \geq 2$ has complex multiplication (or CM) if there is a Hecke character ξ associated to a quadratic field K so that

$$g(z) = \sum_{\mathfrak{a} \subseteq O_K} \xi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

Equivalently, $g(z)$ has CM if and only if there is a discriminant D so that $a(p) = 0$ whenever $\left(\frac{D}{p} \right) = -1$.

In order to apply Goldfeld, Hoffstein, and Lieman's argument, we need information about the symmetric fourth power L-function attached to g. It is defined by

$$L(s, \mathrm{Sym}^4(g)) = \prod_q (1 - \alpha_q^4 q^{-s})^{-1} (1 - \alpha_q^2 \chi_p(q) q^{-s})^{-1} (1 - q^{-s})^{-1} (1 - \alpha_q^{-2} \chi_p(q) q^{-s})^{-1} (1 - \alpha_q^{-4} q^{-s})^{-1},$$

and the completed L-function is given by

$$\Lambda(s, \mathrm{Sym}^4(g)) := p^s \pi^{-3s/2} \Gamma\left(\frac{s}{2} \right) \Gamma\left(\frac{s + \frac{p-3}{2}}{2} \right) \Gamma\left(\frac{s + \frac{p-1}{2}}{2} \right) \cdot$$

$$\Gamma\left(\frac{s + p - 3}{2} \right) \Gamma\left(\frac{s + p - 1}{2} \right) L(s, \mathrm{Sym}^4(g)).$$

In [24], H. Kim established the connection between this L-function and an automorphic form on GL(5). As a consequence, the symmetric fourth power L-function has the desired analytic properties. Again, we must compute the local factors at p and ∞ using the local Langlands correspondence.

Theorem 4.2. *Assume the notation above. Then $L(s, \mathrm{Sym}^4(g))$ has a meromorphic continuation to all of $\mathbb{C}$ and satisfies the functional equation*

$$\Lambda(s, \mathrm{Sym}^4(g)) = \Lambda(1 - s, \mathrm{Sym}^4(g)).$$

Moreover, if g does not have CM, then $L(s, \mathrm{Sym}^4(g))$ is entire.

Remark. *When g does have CM and corresponds to a Hecke character ξ, we have*

$$L(s, \operatorname{Sym}^4(g)) = \zeta(s) L(s, \xi^2) L(s, \xi^4).$$

Consequently, $L(s, \operatorname{Sym}^4(g))$ has a pole at $s = 1$.

The next result is an explicit version of the result of Goldfeld, Hoffstein, and Lieman.

Theorem 4.3. *Assume the notation above. If g does not have CM, then*

$$L(s, \operatorname{Ad}^2(g)) \neq 0$$

for s real with

$$s > 1 - \frac{7 - 4\sqrt{3}}{9 \log(p)}.$$

We next translate this zero-free region into a lower bound on $L(1, \operatorname{Ad}^2(g))$.

Theorem 4.4. *Suppose that $g \in S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ is a normalized newform. If g has CM, then*

$$L(1, \operatorname{Ad}^2(g)) \geq \frac{1}{332\sqrt{p} \log(p)^{11/4}}.$$

If g does not have CM, then

$$L(1, \operatorname{Ad}^2(g)) \geq \frac{1}{84 \log(p)}.$$

Remark. *There are CM forms in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ if and only if $p \equiv 3 \pmod{4}$.*

Proof of Theorem 4.1. The newform g in the statement of the theorem corresponds to an irreducible cuspidal automorphic representation π of $\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}})$, where $\mathbb{A}_{\mathbb{Q}}$ is the adele ring of $\mathbb{Q}$ (for details about this correspondence, see [8], Chapter 7). The representation π admits a factorization

$$\pi = \oplus_{q \leq \infty} \pi_q,$$

where each π_q is a representation of the group $\operatorname{GL}_2(\mathbb{Q}_q)$. In [13], Gelbart and Jacquet prove that there is an automorphic representation $\operatorname{Ad}^2(\pi)$ of $\operatorname{GL}_3(\mathbb{A}_{\mathbb{Q}})$ so that

$$\operatorname{Ad}^2(\pi) = \oplus_{q \leq \infty} \operatorname{Ad}^2(\pi_q).$$

The L-function $L(s, \operatorname{Ad}^2(\pi))$ is defined by $\prod_{q \leq \infty} L(s, \operatorname{Ad}^2(\pi_q))$. Let $\psi : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \mathbb{C}^\times$ be a global additive character. The ϵ-factor is given by

$$\epsilon(s, \operatorname{Ad}^2(\pi), \psi) = \prod_{q \leq \infty} \epsilon(s, \operatorname{Ad}^2(\pi_q), \psi_q).$$

The above definition does not depend on the choice of ψ. Finally, the functional equation takes the form

$$L(s, \operatorname{Ad}^2(\pi)) = \epsilon(s, \pi) L(1 - s, \operatorname{Ad}^2(\pi)),$$

since $\operatorname{Ad}^2(\pi)$ is self-contragredient.

The definition of $\text{Ad}^2(\pi_q)$ is given by the local Langlands correspondence. If F is a local field, the local Langlands correspondence gives a bijection between the set of smooth, irreducible representations of $\text{GL}_n(F)$, and the set of admissible degree n complex representations of W'_F, the Weil-Deligne group of F. For an introduction to the local Langlands correspondence, see [26], and Section 10.3 of [8]. The representations of W'_F that we consider will all be representations of the Weil group W_F, which is a quotient of W'_F.

The representation π_q corresponds to a representation $\rho_q : W'_F \to \text{GL}_2(\mathbb{C})$. Using the embedding $\text{Ad}^2 : \text{GL}_2(\mathbb{C}) \to \text{GL}_3(\mathbb{C})$, one constructs $\text{Ad}^2(\rho_q) : W'_F \to \text{GL}_3(\mathbb{C})$. The local Langlands correspondence for GL_3 associates to $\text{Ad}^2(\rho_q)$ a representation $\text{Ad}^2(\pi_q)$. We shall now compute this in the cases $q = \infty$ and $q = p$.

When $q = \infty$, π_q is the discrete series of weight $k = \frac{p-1}{2}$ (we follow the normalization of Cogdell [10]). This corresponds by the local Langlands correspondence to a representation of the Weil group of $\mathbb{R}$. This is the group $\mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for z in $\mathbb{C}^\times$. The representation in question is

$$\rho_k(re^{i\theta}) = \begin{bmatrix} e^{i(k-1)\theta} & 0 \\ 0 & e^{-i(k-1)\theta} \end{bmatrix}, \quad \rho_k(j) = \begin{bmatrix} 0 & (-1)^{k-1} \\ 1 & 0 \end{bmatrix}.$$

The adjoint square lift of ρ is

$$\text{Ad}^2(\rho_k)(re^{i\theta}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i(2k-2)\theta} & 0 \\ 0 & 0 & e^{-i(2k-2)\theta} \end{bmatrix}, \quad \text{Ad}^2(\rho_k(j)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & (-1)^{k-1} \\ 0 & (-1)^{k-1} & 0 \end{bmatrix}.$$

One can see that $\text{Ad}^2(\rho_q) = \rho_0^- \oplus \rho_{2k-1}$. Here ρ_0^- is the 1-dimensional representation given by $\rho_0^-(z) = 1$ and $\rho_0^-(j) = -1$. We have $L(s, D_0^-) = \pi^{-s/2}\Gamma\left(\frac{s+1}{2}\right)$, and $L(s, D_{2k-1}) = \pi^{-s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)$. The L and ϵ factors are defined so that they are inductive. In particular, $L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1)L(s, \rho_2)$, and $\epsilon(s, \rho_1 \oplus \rho_2, \psi_q) = \epsilon(s, \rho_1, \psi_q)\epsilon(s, \rho_2, \psi_q)$. It follows that

$$L(s, \text{Ad}^2(\rho_\infty)) = L(s, D_0^-)L(s, D_{2k-1}).$$

The local root number is $\epsilon(\frac{1}{2}, D_0^-, \psi)\epsilon(\frac{1}{2}, D_{2k-1}, \psi) = i \cdot i^{2k-1} = (-1)^k$. Here, $\psi(x) = e^{2\pi i x}$ is the standard additive character.

When $q = p$, the local representation π_p has central character χ_p (the usual Dirichlet character thought of as a character of $\mathbb{Q}_p^\times$). The conductor of π_p is the power of $p^{s/2}$ that occurs in the functional equation for $L(s, \pi)$, and since the newform g has level p, the conductor of π_p is one. This can be determined from $\epsilon(s, \pi_p, \psi_p)$, and also from more intrinsic representation-theoretic data. For any representation σ, we will denote its conductor by $c(\sigma)$.

In Schmidt [34], a list of possibilities for local representations π together with their conductors is given. A simple calculation shows that the only possibility for a representation with conductor one and central character χ_p is a principal series $\pi(\chi_1, \chi_2)$, where χ_1 is unramified, and $\chi_2 = \chi_1^{-1}\chi_p$.

The Weil group $W_{\mathbb{Q}_p}$ can be taken to be the subgroup of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ consisting of all elements restricting to some power of the Frobenius on $\overline{\mathbb{F}}_p$ (see [37]). Under the local Langlands

correspondence, $\pi(\chi_1, \chi_2)$ corresponds to a two-dimensional representation of $W_{\mathbb{Q}_p}$ which is a direct sum of two characters. These characters ρ_1 and ρ_2 of $W_{\mathbb{Q}_p}$ are constructed so that

$$\rho_i(\sigma) = \chi_i(r(\sigma)),$$

where $r : \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is the reciprocity law isomorphism of local class field theory, normalized so that $r(\mathrm{Frob}_p) \in p^{-1}\mathbb{Z}_p$.

One can easily compute that $\mathrm{Ad}^2(\rho_1 \oplus \rho_2) = 1 \oplus \rho_1\rho_2^{-1} \oplus \rho_1^{-1}\rho_2$ (here 1 denotes the trivial character). Since $\rho_1\rho_2^{-1}$ and $\rho_1^{-1}\rho_2$ both have conductor 1, it follows that $c(\mathrm{Ad}^2(\pi_p)) = 2$. From the usual definition of the L-factors, and the compatibility with the local Langlands correspondence, we see that

$$L(s, \mathrm{Ad}^2(\pi_p)) = L(s, \mathrm{Ad}^2(\rho_1 \oplus \rho_2)) = (1 - p^{-s})^{-1}.$$

Moreover, we have

$$\epsilon\left(\frac{1}{2}, \mathrm{Ad}^2(\pi_p), \psi_p\right) = \epsilon\left(\frac{1}{2}, 1, \psi_p\right) \epsilon\left(\frac{1}{2}, \chi_1\chi_2^{-1}, \psi_p\right) \epsilon\left(\frac{1}{2}, \chi_1^{-1}\chi_2, \psi_p\right).$$

Equation 4 on page 117 of [34] states that if χ and ψ_p are unramified, then $\epsilon\left(\frac{1}{2}, \chi, \psi_p\right) = 1$. Equation 7 on page 118 of [34] implies that for any character χ, we have $\epsilon\left(\frac{1}{2}, \chi, \psi_p\right) \epsilon\left(\frac{1}{2}, \chi^{-1}, \psi_p\right) = \chi(-1)$. It follows that the local root number of $\mathrm{Ad}^2(\pi_p)$ is $\chi_1\chi_2^{-1}(-1)$. Since χ_1 is unramified, $\chi_1(-1) = 1$, while $\chi_2^{-1}(-1) = \left(\frac{-1}{p}\right)$.

The global conductor of $\mathrm{Ad}^2(\rho)$ is therefore p^2 and the global root number is $(-1)^k \cdot \left(\frac{-1}{p}\right)$. Since $k = \frac{p-1}{2}$, the global root number is 1. These facts, combined with the meromorphic continuation and functional equation for L-functions of automorphic representations yield the desired result. If g does not have CM, then $\pi \otimes \chi \not\cong \pi$ for any character χ of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$, and then Theorem 9.3 of [13] implies that $\mathrm{Ad}^2(\pi)$ is cuspidal, which implies that $L(s, \mathrm{Ad}^2(\pi))$ is entire. If g does have CM, then g corresponds to a Hecke character ξ, and one can check that

$$L(s, \mathrm{Ad}^2(\pi)) = L(s, \chi_p)L(s, \xi^2),$$

which is again entire. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 4.2. This is entirely analogous to the case of the adjoint square lifting, thanks to the deep result of Henry Kim on the functoriality of the symmetric fourth power lifting [24]. The local factor at infinity is worked out in [10], with the desired result, and with the local root number equal to $(-1)^k$.

At $q = p$, $\mathrm{Sym}^4(\rho_1 \oplus \rho_2)$ is $\rho_1^4 \oplus \rho_1^3\rho_2 \oplus \rho_1^2\rho_2^2 \oplus \rho_1\rho_2^3 \oplus \rho_2^4$. Note that ρ_2 is ramified, but ρ_2^2 is not. Thus, the local L-factor has degree 3 and is given by

$$(1 - \alpha_p^4 p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \alpha_p^{-4}p^{-s})^{-1},$$

where $\alpha_p = a(p)/p^{\frac{p-1}{4}}$. Similar to the above case, the conductor of $\mathrm{Sym}^4(\rho_1 \oplus \rho_2)$ is 2, and the local root number is $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = (-1)^k$. Thus, the global conductor is p^2 and the global root number is 1. Finally, we must show that under the stated hypotheses, $\mathrm{Sym}^4(\pi)$ is cuspidal. The main result of [25] is that $\mathrm{Sym}^4(\pi)$ is cuspidal unless π is monomial (equivalently g has

CM), or π is of tetrahedral or octahedral type. This means that π arises from a representation of the global Weil group $W_{\mathbb{Q}}$, but this cannot be the case if the weight of g is greater than 1. The only case when the weight can be one is when $p = 3$. However in this case, any nonzero $g \in S_1(\Gamma_0(3), \chi_3)$ has $f^2 \in S_2(\Gamma_0(3))$, but since $\dim S_2(\Gamma_0(3)) = 0$, no such g exists. Thus, $\mathrm{Sym}^4(\pi)$ is cuspidal and $L(s, \mathrm{Sym}^4(\pi)) = L(s, \mathrm{Sym}^4(g))$ is entire. $\qquad\square$

Proof of Theorem 4.3. Let

$$L(s) = \zeta(s)^2 L(s, \mathrm{Ad}^2(g))^3 L(s, \mathrm{Sym}^4(g)).$$

Let $k = \frac{p-1}{2}$ and

$$G(s) = p^{4s} \pi^{-8s} \Gamma\left(\tfrac{s}{2}\right)^3 \Gamma\left(\tfrac{s+1}{2}\right)^3 \Gamma\left(\tfrac{s+k-1}{2}\right)^4 \Gamma\left(\tfrac{s+k}{2}\right)^4 \Gamma\left(\tfrac{s+2k-2}{2}\right) \Gamma\left(\tfrac{s+2k-1}{2}\right).$$

If $\Lambda(s) = s^2(1-s)^2 G(s)L(s)$, then $\Lambda(s)$ is entire and $\Lambda(s) = \Lambda(1-s)$. One may verify from Theorems 4.1 and 4.2 that if $L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, then $b(n) \geq 0$ for all n. For the remainder of the proof, we will take s real and greater than 1. In this region, one has $L(s) > 0$ and $L'(s) = \sum_{n=2}^{\infty} \frac{-b(n)\log(n)}{n^s} < 0$. The function $\Lambda(s)$ is an entire function of order 1, and so admits a product expansion

$$\Lambda(s) = e^{A+Bs} \prod_{\rho}\left(1 - \frac{s}{\rho}\right) e^{s/\rho}.$$

Taking the logarithmic derivative gives

$$\sum_{\rho} \frac{1}{s-\rho} + \frac{1}{\rho} = \frac{2}{s} - \frac{2}{1-s} + \frac{L'(s)}{L(s)} + \frac{G'(s)}{G(s)} - B.$$

Taking the real part of both sides and using the relation $\mathrm{Re}\,(B) = -\sum_{\rho} \mathrm{Re}\left(\frac{1}{\rho}\right)$ (see Theorem 5.6, part 3 of [19]), we obtain

$$\sum_{\rho} \mathrm{Re}\left(\frac{1}{s-\rho}\right) \leq \frac{2}{s} + \frac{2}{s-1} + \frac{G'(s)}{G(s)}.$$

Suppose that β is a real zero of $L(s, \mathrm{Ad}^2(g))$. Then we get

$$\frac{3}{s-\beta} \leq \frac{2}{s-1} + 2 + \frac{G'(s)}{G(s)}.$$

We have

$$\frac{G'(s)}{G(s)} = 4\log(p) - 8\log(\pi) + 3/2\psi(s/2) + (3/2)\psi((s+1)/2) + 2\psi((s+k-1)/2)$$

$$+ 2\psi((s+k)/2) + \frac{1}{2}\psi((s+2k-2)/2) + \frac{1}{2}\psi((s+2k-1)/2),$$

where $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$. The formula $\psi(s) = \log(s) - \frac{1}{2s} - \int_0^\infty \frac{2t\,dt}{(s^2+t^2)(e^{2\pi t}-1)}$ (see for example [3], Exercise 1.43(b)) implies that $\psi(s)$ is increasing as a function of s, and that $\psi(s) \leq \log(s) - \frac{1}{2s}$. It follows that for $1 < s \leq 1.1$, $\frac{G'(s)}{G(s)} \leq 9\log(p) - 2$.

Now, set $s = 1 + \alpha$, where $0 < \alpha \leq 0.1$. We obtain $\frac{3}{1+\alpha-\beta} \leq \frac{2}{\alpha} + 9\log(p)$. Solving for β and making the optimal choice of α gives $\alpha = \frac{\sqrt{6}-2}{9\log(p)}$, which is always less than 0.1. This yields

$$\beta \leq 1 - \frac{5 - 2\sqrt{6}}{9\log(p)}.$$

Note that $5 - 2\sqrt{6} > 7 - 4\sqrt{3}$, and so the desired result holds. $\qquad\square$

Proof of Theorem 4.4. First, assume that g is a CM form corresponding to the Hecke character ξ. In this case, $L(s, \mathrm{Ad}^2(g)) = L(s, \chi_p)L(s, \xi^2)$. We derive a lower bound on $L(1, \xi^2)$ and apply the Dirichlet class number formula to bound $L(1, \chi_p)$. We wish to bound

$$\log L(1, \xi^2) = \int_1^\infty -\frac{L'(s, \xi^2)}{L(s, \xi^2)}.$$

We have the trivial bound

(4.2) $$\left| \frac{L'(s, \xi^2)}{L(s, \xi^2)} \right| \leq -\frac{2\zeta'(s)}{\zeta(s)}.$$

Also, a virtually identical argument to that in the proof of Theorem 4.3 establishes a zero-free region for $L(s, \xi^2)$ and gives that

(4.3) $$\sum_\rho \mathrm{Re}\left(\frac{1}{s - \rho} \right) \leq \frac{3}{s - 1} + 9\log(p)$$

provided $1 \leq s \leq 1.1$, where the sum is over non-trivial zeroes of $\zeta(s)^3 L(s, \xi^2)^4 L(s, \xi^4)$. It follows from this, and the equation

(4.4) $$\sum_\rho \frac{1}{s - \rho} = \frac{L'(s, \xi^2)}{L(s, \xi^2)} + \frac{G'(s)}{G(s)},$$

where $G(s) = p^{s/2}(2\pi)^{-s}\Gamma\left(s + \frac{p-3}{2}\right)$, that

(4.5) $$\left| \frac{L'(s, \xi^2)}{L(s, \xi^2)} \right| \leq \frac{3}{4(s - 1)} + \frac{15}{4}\log(p)$$

for $1 \leq s \leq 1.1$. Finally, we must derive a bound on L'/L near $s = 1$.

To do this, we use (4.4) with $s = 2$ to derive the bound

$$\sum_\rho \mathrm{Re}\left(\frac{1}{2 - \rho} \right) \leq \frac{3}{2}\log(p).$$

By pairing ρ with $1 - \rho$, we see that

$$\sum_{\substack{\rho \\ \gamma \geq \sqrt{3}/2}} \frac{1}{4 + \gamma^2} + \frac{1/2}{1 + \gamma^2} \leq \frac{3}{2} \log(p).$$

The equation (4.3) also implies that $L(s, \xi^2)$ has no zeroes in the region $\{\sigma + it : \sigma \geq \beta_0, |t| \leq s_0 - \beta_0\}$, where $s_0 = 1 + \frac{2\sqrt{3}-3}{9\log(p)}$ and $\beta_0 = 1 - \frac{7-4\sqrt{3}}{9\log(p)}$. Plugging this into (4.4) and using the bounds on sums over zeroes derived above, we obtain

(4.6)
$$\left| \frac{L'(s, \xi^2)}{L(s, \xi^2)} \right| \leq \frac{19}{6} \log(p) + \frac{30}{7 - 4\sqrt{3}} \log^2(p).$$

We apply (4.6) for $1 \leq s \leq 1 + \frac{7-4\sqrt{3}}{40 \log^2(p)}$, (4.5) for s up to $1 + \frac{1}{3\log(p)}$ and (4.2) for the remaining s to derive a bound on $L(1, \xi^2)$. Combining this bound with the bound $L(1, \chi_p) \geq \frac{3\pi}{\sqrt{p}}$ when $p > 163$, we obtain

$$L(1, \mathrm{Ad}^2(g)) \geq \frac{1}{332\sqrt{p}\log(p)^{11/4}}.$$

One can verify that this bound is satisfied with $p \leq 163$ as well.

Now, we assume that g does not have CM. We mimic the argument of Lemma 3 of [33], which is in turn based on Hoffstein's argument for Dirichlet L-functions from [16]. Assume that $p \geq 17$ and set

$$L(s, g \otimes g) = \zeta(s)L(s, \mathrm{Ad}^2(g)) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

A careful inspection of the Euler factors given in Theorem 4.1 shows that $b(n) \geq 0$ for all n, and also that $b(n^2) \geq 1$ for all n. Let $\beta = 1 - \frac{7-4\sqrt{3}}{9\log(p)}$ and note that $3/4 < \beta < 1$. We set $x = p^A$ and choose A at the end of the proof (we will choose it to be equal to $16/5$). Consider

$$I = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{L(s + \beta, g \otimes g)x^s \, ds}{s \prod_{r=2}^{10}(s+r)}.$$

We use the fact that

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s \, ds}{s \prod_{r=2}^{10}(s+r)} = \begin{cases} \frac{(x+9)(x-1)^9}{10!x^{10}} & \text{if } x > 1 \\ 0 & \text{if } x < 1. \end{cases}$$

This gives

$$I = \sum_{n \leq x} \frac{b(n)(x/n + 9)(x/n - 1)^9}{10!n^\beta(x/n)^{10}}.$$

We consider only those terms where $x/n \geq 44$. This gives

$$I \geq \frac{1}{10!} \frac{(44+9)(44-1)^9}{44^{10}} \sum_{n \leq \sqrt{x/44}} \frac{1}{n^2} \geq 1.54354$$

for $p \geq 17$. We move the contour in I to $\operatorname{Re}(s) = \alpha := -3/2 - \beta$ and pick up poles at $s = 1 - \beta$, $s = 0$ and $s = -2$. This gives

$$I = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{L(s + \beta, g \otimes g)x^s \, ds}{s \prod_{r=2}^{10}(s + r)} + \frac{L(1, \operatorname{Ad}^2(g))x^{1-\beta}}{(1 - \beta) \prod_{r=2}^{10}(1 - \beta + r)}$$
$$+ \frac{L(\beta, g \otimes g)}{10!} - \frac{L(-2 + \beta, g \otimes g)x^{-2}}{2 \cdot 8!}.$$

Since g does not have CM, Theorem 4.3 implies that $L(s, \operatorname{Ad}^2(g))$ has no real zeroes to the right of β. Therefore, we have $L(\beta, \operatorname{Ad}^2(g)) \geq 0$ and since $\zeta(\beta) < 0$, $L(\beta, g \otimes g) < 0$. Since $\beta < 1$, we have $-2 + \beta < -1$ and so $L(s, \operatorname{Ad}^2(g)) < 0$. Since $\zeta(-2 + \beta) < 0$, it follows that $L(-2 + \beta, g \otimes g) > 0$. This gives

$$I - I_2 \leq \frac{L(1, \operatorname{Ad}^2(g))x^{1-\beta}}{(1 - \beta) \prod_{r=2}^{10}(1 - \beta + r)},$$

where

$$I_2 = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{L(s + \beta, g \otimes g)x^s \, ds}{s \prod_{r=2}^{10}(s + r)}.$$

It suffices to bound I_2 in the above inequality. Using the functional equation for $L(s, g \otimes g)$, we have

$$|L(-3/2 + it, g \otimes g)| = p^4 \pi^{-8} |1/4 + it/2|^2 |3/4 + it/2|^2 |k/2 - 1/4 - it/2||k/2 - 5/4 - it/2|$$
$$|k/2 + 1/4 - it/2||k/2 - 3/4 - it/2||L(5/2 - it, g \otimes g)|.$$

We have $L(5/2 - it, g \otimes g)| \leq \zeta(5/2)^4$, and $|x^s| = p^{A(-3/2-\beta)}$. Note that

$$\frac{1}{|-3/2 - \beta + it| \prod_{r=2}^{10} |r - 3/2 - \beta + it|} \leq \frac{1}{|9/4 + it||1/4 + it| \prod_{r=3}^{10} |r - 5/2 + it|}.$$

Putting these estimates together, we get

$$|I_2| \leq \frac{\zeta(5/2)^4 p^{8+A(-3/2-\beta)}}{2^{13}\pi^9} \cdot \int_{-\infty}^{\infty} \frac{|1/2 + it|^2 |3/2 + it|^2 |1 + it|^4}{|1/4 + it||9/4 + it| \prod_{r=3}^{10} |r - 5/2 + it|} \, dt$$
$$\leq \frac{0.011322 p^{8+A(-3/2-\beta)}}{10!}.$$

Thus, we have

$$L(1, \operatorname{Ad}^2(g)) \geq (1 - \beta) \left(1.54354 p^{A(\beta-1)} - 0.011322 p^{8-5A/2}\right).$$

Setting $A = 16/5$, we obtain

$$L(1, \operatorname{Ad}^2(g)) \geq \frac{1}{84 \log(p)},$$

Numerically, we evaluate $L(1, \operatorname{Ad}^2(g))$ for all newforms in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ for $p < 17$ and check that the same relation holds. This completes the proof. $\square$

5. Proof of Theorem 1.1 and its corollaries

Recall that

$$f(z) = \frac{\eta^p(pz)}{\eta(z)} = \sum_{n=0}^{\infty} c_p(n) q^{n + \frac{p^2 - 1}{24}}.$$

We decompose

$$f(z) = e_p E_{\frac{p-1}{2}}(z) + \sum_{i=1}^{s} r_i g_i(z)$$

where $g_i(z)$ are the normalized Hecke eigenforms in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. To bound $R(p) = \sum_{i=1}^{s} |r_i|$, we use that

$$r_i = \frac{\langle f, g_i \rangle}{\langle g_i, g_i \rangle}.$$

We derived an upper bound on the numerator in Section 3 and a lower bound on the denominator in Section 4. Now we prove Theorem 1.1.

Proof of Theorem 1.1. Define

$$(5.1) \qquad L(p) := \begin{cases} \frac{2}{\pi} \left(1 + \frac{1}{p}\right)^{-1} \frac{\left(\frac{p-3}{2}\right)!}{(4\pi)^{\frac{p-1}{2}}} \frac{1}{84 \log(p)}, & \text{if } p \equiv 1 \pmod 4, \\ \frac{2}{\pi} \left(1 + \frac{1}{p}\right)^{-1} \frac{\left(\frac{p-3}{2}\right)!}{(4\pi)^{\frac{p-1}{2}}} \frac{1}{332 \sqrt{p} \log(p)^{11/4}}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Then, Theorem 4.4 states that $\langle g_i, g_i \rangle \geq L(p)$ for all i. Sturm's theorem [36] states that a modular form $f \in M_k(\Gamma_0(N), \chi)$ is determined by its first $\frac{k}{12}[\Gamma : \Gamma_0(N)]$ Fourier coefficients. It follows that the dimension of $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi) \leq \frac{p-1}{24}[\Gamma : \Gamma_0(p)]$. In summary, we have

$$\sum_{i=1}^{s} |r_i| \leq \frac{p-1}{24}[\Gamma : \Gamma_0(p)] \frac{A}{L(p)},$$

where A is an upper bound on $|\langle f, g \rangle|$.

Therefore, by Theorem 3.2 and (5.1) , we arrive at

$$\sum_{i=1}^{s} |r_i| \leq \frac{(p-1)\pi}{8} \frac{U_\infty(p) + pU_0(p)}{L(p)}$$

$$\leq \begin{cases} 98304 \cdot e^{6\pi} p^4 \log p \left(\frac{e^{1.5}}{8\pi}\right)^{\frac{p-1}{4}}, & \text{if } p \equiv 1 \pmod 4, \\ 388535 \cdot e^{6\pi} p^{\frac{9}{2}} (\log p)^{\frac{11}{4}} \left(\frac{e^{1.5}}{8\pi}\right)^{\frac{p-1}{4}}, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where we have used the inequality $\frac{x^{x-\gamma}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-\frac{1}{2}}}{e^{x-1}}$ from [27], and γ is the Euler constant. $\square$

Before we prove our corollaries, note that for $p > 5$, we have

(5.2)
$$\sigma_{\frac{p-1}{2},\chi_p}(n) \geq n^{\frac{p-3}{2}}\zeta\left(\frac{p-3}{2}\right)^{-1},$$

where $\zeta(s)$ is the Riemann zeta function.

Proof of Corollary 1.3. By Theorem 1.1,

$$|a_p(n)| \geq \frac{e_p}{\zeta\left(\frac{p-3}{2}\right)}n^{\frac{p-3}{2}} - R(p)d(n)n^{\frac{p-3}{4}}$$

$$\geq e_p n^{\frac{p-3}{2}}\left(\frac{1}{\zeta\left(\frac{p-3}{2}\right)} - \frac{2R(p)}{e_p n^{\frac{p-5}{4}}}\right),$$

where we have used the fact that $d(n) \leq 2\sqrt{n}$. Since $\zeta(2) - \frac{1}{5} > \frac{2}{5}$, we arrive at $a_p(n) > \frac{2e_p}{5}n^{\frac{p-3}{2}}$ once $n \geq \left(\frac{10R(p)}{e_p}\right)^{\frac{4}{p-5}}$, as desired. $\square$

Proof of Corollary 1.4. Since $\sigma_{\frac{p-1}{2},\chi_p}$ is multiplicative and n is coprime to t, we have

$$a_p(t^k n) - (\sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1)a_p(n)$$

$$\geq e_p \sigma_{\frac{p-1}{2},\chi_p}(n) - R(p)d(n)n^{\frac{p-3}{4}}\left((k+1)t^{\frac{k(p-3)}{4}} + \sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1\right)$$

$$\geq n^{\frac{p-1}{4}}\left(e_p\zeta\left(\frac{p-3}{2}\right)^{-1}n^{\frac{p-5}{4}} - 2R(p)\left((k+1)t^{\frac{k(p-3)}{4}} + \sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1\right)\right).$$

Thus, for

$$n > \left(\frac{\zeta(\frac{p-3}{2})}{e_p}2R(p)\left((k+1)t^{\frac{k(p-3)}{4}} + \sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1\right)\right)^{\frac{4}{p-5}},$$

we have

$$a_p(t^k n) - (\sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1)a_p(n) > 0.$$

$\square$

6. Proof of Theorem 1.5

Recall that $F(q) = \frac{q^{1/24}}{\eta(z)}$ and that the generating function $F_t(q)$ for the number of t-core partitions is $F_t(q) := \frac{F(q)}{F(q^t)^t}$. By using the transformation formula for the Dedekind eta function (Theorem 2.1) we can easily derive the transformation formula for $F(q)$:

$$F(e^{2\pi i\tau}) = e^{\pi i(\tau - \gamma\tau)/12}e^{-\pi is(d,c)}e^{\pi i(a+d)/12c}\sqrt{-i(c\tau+d)}F(e^{2\pi i\gamma\tau}), \text{ for } \gamma \in \Gamma.$$

Using this, we can derive a similar transformation formula for $F_t(q)$. By using this transformation formula and [1, Proposition 6], we can prove the following lemma.

Lemma 6.1. *Let $A(t)$ and $B(t)$ be the constants (depending only on t) defined by*

$$
A(t) = \begin{cases}
\dfrac{0.05 \cdot (2\pi)^{\frac{t-1}{2}}}{\Gamma\left(\frac{t-1}{2}\right) t^{\frac{t}{2}}}, & \text{if } t = 6, \\[4mm]
\dfrac{(2\pi)^{\frac{t-1}{2}}}{\Gamma\left(\frac{t-1}{2}\right) t^{\frac{t}{2}}} \left(2 - \zeta\left(\frac{t-3}{2}\right)\right), & \text{if } t \geq 7,
\end{cases}
$$

and

$$
B(t) = \frac{(2\pi)^{\frac{t-1}{2}}}{\Gamma\left(\frac{t-1}{2}\right) t^{\frac{t}{2}}} \zeta\left(\frac{t-3}{2}\right).
$$

Define $e^{-2\pi\left(1 + \frac{2}{t}\right) - \frac{\pi}{12}\left(1 - \frac{1}{t^2}\right)} E(t)$ by

$$
\frac{2 e^{\frac{\pi}{t}} (C(t) - 1)}{t^{\frac{t}{2}}} \left(\frac{t(t-1)}{8\pi e}\right)^{\frac{t-1}{4}} + \left(\sum_{2 \leq d \mid t} \left(\frac{1}{d^2 - 1}\right)^{\frac{t-1}{4}} d^{\frac{t}{2}}\right) \frac{2.1 C(t)}{t^{\frac{t}{2}}} \left(\frac{3(t-1)}{e\pi}\right)^{\frac{t-1}{4}} + \frac{2^{\frac{t-1}{2}}}{\pi t^{\frac{t}{2}}}.
$$

Then for all integers $n \geq t^2$ and $t \geq 6$, we have

$$
A(t) \left(n + \frac{t^2 - 1}{24}\right)^{\frac{t-3}{2}} - E(t) n^{\frac{t-1}{4}} \leq c_t(n) \leq B(t) \left(n + \frac{t^2 - 1}{24}\right)^{\frac{t-3}{2}} + E(t) n^{\frac{t-1}{4}}.
$$

Since the proof of this lemma is identical to Lemma 3.1 (except for the estimate of S_2), we omit it. Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. By Lemma 6.1, for all $n \geq (t+1)^2$, we have

$$
c_{t+1}(n) - c_t(n)
$$

$$
\geq A(t+1) \left(n + \frac{t^2 + 2t}{24}\right)^{\frac{t-2}{2}} - B(t) \left(n + \frac{t^2 - 1}{24}\right)^{\frac{t-3}{2}} - E(t+1) n^{\frac{t}{4}} - E(t) n^{\frac{t-1}{4}}
$$

$$
\geq n^{\frac{t-3}{2}} \left(A(t+1)\sqrt{n} - B(t)\right) - n^{\frac{t}{4}} \left(E(t+1) + \frac{E(t)}{\sqrt{t}}\right).
$$

Take n_1 such that if $n > n_1$, then $A(t+1)\sqrt{n} > 2B(t)$. We note that

$$
\frac{2B(t)}{A(t+1)} \leq \frac{\zeta\left(\frac{t-3}{2}\right)}{\sqrt{\pi}\left(2 - \zeta\left(\frac{t-3}{2}\right)\right)} t \left(1 + \frac{1}{t}\right)^{\frac{t}{2}}.
$$

Therefore, we can choose $n_1 = 0.3 \cdot t^2$. Since $n > (t+1)^2$, we always have

$$
c_{t+1}(n) - c_t(n) \geq B(t) n^{\frac{t-2}{2}} - n^{\frac{t}{4}} \left(E(t+1) + \frac{E(t)}{\sqrt{t}}\right).
$$

We estimate $E(t)$ as follows

$$e^{-2\pi(1+\frac{2}{t})-\frac{\pi}{12}(1-\frac{1}{t^2})}E(t) \leq \begin{cases} 6t^{\frac{3}{2}}e^{\frac{\pi}{t}}\left(\frac{e^{1.5}}{8\pi}\right)^{\frac{t-1}{4}} & \text{, if } t \geq 36, \\ 9t^{\frac{5}{2}}\left(\frac{4e^{1.5}}{t\pi}\right)^{\frac{t-1}{4}} & \text{, if } 7 \leq t \leq 36. \end{cases}$$

Therefore, we have deduced that if

$$n \geq \begin{cases} \left(45503t^{\frac{2t+1}{2}}\left(\frac{1}{2^7\pi^3\sqrt{e}}\right)^{\frac{t-1}{4}}\right)^{\frac{4}{t-4}} & \text{, if } t \geq 36, \\ \left(288305t^{\frac{3t+7}{4}}\left(\frac{1}{4\pi^3\sqrt{e}}\right)^{\frac{t-1}{4}}\right)^{\frac{4}{t-4}} & \text{, if } 7 \leq t \leq 35, \end{cases}$$

and $n \geq (t+1)^2$, then $c_{t+1}(n) > c_t(n)$, as desired. $\square$

Now we will prove Stanton's conjecture in the cases where $t \leq 198$. Since the bound in Theorem 1.5 is quite big for $t \leq 12$, we need to get sharper estimates for $c_t(n)$ for $4 \leq t \leq 13$. We will achieve this goal by using various arguments. For $t = 4$, we will use the result of Ono and Sze [32], which relates $c_4(n)$ to the class number of an imaginary quadratic field. For $t = 5$ and $t = 7$ we use that $R(5) = 0$ and $R(7) = 1/8$. For $t = 6$, we use that the generating function for $c_6(n)$ is a weight $5/2$ modular form. For $8 \leq t \leq 13$, we will use the circle method as in Lemma 6.1, but we will set $N = \lceil\sqrt{2\pi n}\rceil$ and estimate $C(t)$ by (2.6) instead of (2.5) if $t \leq 11$.

For the $t = 4$ case, we first need to find an upper bound for class numbers.

Proposition 6.2. *For any discriminant $-D < 0$, we have*

$$h(-D) \leq \frac{w_{-D}}{\pi}\sqrt{D}\log(D).$$

Here w_{-D} is half the number of units in the imaginary quadratic order of discriminant $-D$. (Note that $w_{-D} = 1$ if $-D > 4$).

Proof. One can use the Dirichlet class number formula together with the elementary bound on the sum of a Dirichlet character mod q

$$\left|\sum_{n \leq x} \chi(n)\right| \leq \min(x \bmod q, q - x \bmod q)$$

to prove this result. $\square$

Now, from Ono and Sze [32], we have

$$c_4(n) = \frac{1}{2}\sum_{d^2|8n+5} h\left(\frac{-32n-20}{d^2}\right).$$

Note that $(-32n - 20)/d^2$ cannot be equal to -3 or -4, since it is always greater than or equal to 4, and $d^2 \neq 8n + 5$ since $d^2 \equiv 0, 1, 4 \pmod{8}$. Thus, we have

$$c_4(n) \leq \frac{1}{2\pi} \sum_{d^2|8n+5} \sqrt{(32n + 20)/d^2} \log((32n + 20)/d^2)$$

$$\leq \frac{1}{2\pi} \sqrt{32n + 20} \log(32n + 20) \sum_{d^2|8n+5} \frac{1}{d}.$$

If $sq(8n + 5)$ is the largest positive integer so that $sq(8n + 5)^2 | 8n + 5$, then we have

$$\sum_{d^2|8n+5} \frac{1}{d} = \sum_{d|sq(8n+5)} \frac{1}{d} = \frac{\sigma(sq(8n + 5))}{sq(8n + 5)}.$$

Combining this with the result of Ivić [18] that $\sigma(n) < 2.59n \log(\log(n))$ for $n \geq 7$, we see that

$$c_4(n) \leq \frac{2.59}{\pi} \sqrt{8n + 5} \log(32n + 20) \log(\log(8n + 5)).$$

Now, we have $c_5(n) = \sigma_{2,\chi}(n + 1) \geq n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \phi(n)$. Since $\frac{6}{\pi^2} < \frac{\phi(n)\sigma(n)}{n^2}$, we have

$$c_5(n) > \frac{6}{2.59\pi^2} \frac{n + 1}{\log(\log(n + 1))}$$

for $n > 6$. It follows from these inequalities that $c_4(n) < c_5(n)$ provided $n \geq 1750513$. A computation verifies that the desired inequality holds if $n < 1750513$.

Now, we turn to the $t = 5$ case. Arguing as above, we have that

$$\sigma_{2,\chi}(n) \leq \frac{n^2}{\phi(n)} < \frac{\pi^2}{6} \sigma(n) \leq \frac{2.59}{6} \pi^2 (n + 1) \log(\log(n + 1))$$

provided $n \geq 7$.

Now we will estimate $c_6(n)$. Let

$$F(z) = \eta(z)^2 \eta(2z)^2 \eta(3z)^2 \eta(6z)^2 = \sum_{n=1}^{\infty} a(n)q^n \in S_4(\Gamma_0(6))$$

be the unique weight 4 newform of level 6. If D is a fundamental discriminant, define

$$L(F \otimes \chi_D, s) = \sum_{n=1}^{\infty} \frac{a(n)\chi_D(n)}{n^{s+3/2}}.$$

Proposition 6.3. *Let n be a positive integer, and write $72n + 105 = Df^2$, where D is a fundamental discriminant and $f \geq 1$. Then,*

$$c_6(n) = \frac{D^{3/2}L(2, \chi_D)}{240\pi^2} \sum_{d|f} \mu(d)\chi_D(d)d\sigma_3(f/d)$$

$$\pm \frac{(D/33)^{3/4}}{10} \sqrt{\frac{L(F \otimes \chi_D, 1/2)}{L(F \otimes \chi_{33}, 1/2)}} \sum_{d|f} \mu(d)d\chi_D(d)a(f/d).$$

Proof. This follows from writing

$$\frac{\eta(144z)^6}{\eta(24z)} = q^{24} \sum_{n=0}^{\infty} c_6(n)q^{24n+11},$$

a modular form of weight $5/2$ on $\Gamma_0(576)$ with character χ_{12}, as the sum of an Eisenstein series and a cusp form. Cohen has shown that the coefficients of the Eisenstein series involve the values at 2 of Dirichlet L-functions, and Waldspurger has shown that the cusp form coefficients are essentially the square root of the twisted L-value $L(F \otimes \chi_D, 1/2)$. Combining these two results, we get the stated formula. □

A simple estimate shows that the first term above is bounded below by $\frac{1}{40\pi^4}(72n + 105)^{3/2}$, and that

$$\left| \sum_{d|f} \mu(d)d\chi_D(d)a(f/d) \right| \leq d(f)f^{3/2} \prod_{p|f} \left(1 + \frac{1}{\sqrt{p}}\right).$$

Next, we need an upper bound on $L(F \otimes \chi_D, 1/2)$. A variant of the standard convexity bound (see [15], Theorem F.4.1.9 for example) gives the following result.

Lemma 6.4. *Assume the notation above. Suppose that g is a newform in $S_k(\Gamma_0(N))$, then*

$$L(g, 1/2) \leq e^{1/2} \left(\frac{N}{2\pi}\right)^{1/4} \frac{\Gamma\left(\frac{k+1}{2} + \frac{1}{2\alpha}\right)}{\Gamma\left(\frac{k}{2}\right)}(1 + 2\alpha)^2,$$

where $\alpha = \log\left(\frac{N}{2\pi}\right)$.

Specializing to the case at hand, we have that $k = 4$ and the conductor of $F \otimes \chi_D$ is bounded by $2D^2$. From this we get

$$|L(F \otimes \chi_D, 1/2)| \leq 5.9(2D^2)^{1/4} \log^2(2D^2).$$

Combining this bound with the elementary bound $d(n) \leq \left(\frac{1536}{35}\right)^{1/3} n^{1/3}$, we obtain an upper bound on the second term in Proposition 6.3 of

$$0.744(72n + 105) \log(72n + 105).$$

We see from these bounds that $c_6(n) > c_5(n)$ provided $n \geq 58000548$. We refine this estimate by using the bounds $L(2, \chi_D) \geq \frac{6}{\pi^2}$, $L(F \otimes \chi_D, 1/2) \leq 5.9(2D^2)^{1/4} \log^2(2D^2)$, and computing the

rest of the terms in Proposition 6.3 exactly. This requires knowing the first 12000 coefficients of $F(z)$, and shows that $c_6(n) > c_5(n)$ for $n > 110868$. It is easy to check up to this bound and verify Stanton's conjecture in this case.

For $t = 7$, one can compute that

$$\frac{\eta^7(7z)}{\eta(z)} = \frac{1}{8}E_3(z) - \frac{1}{8}\eta(z)^3\eta(7z)^3.$$

The latter form is a Hecke eigenform and from this it follows that

$$\frac{3}{4\pi^2}(n+2)^2 - \frac{1}{8}d(n)n \le c_7(n) \le \frac{\pi^2}{48}(n+2)^2 + \frac{1}{8}d(n)n.$$

This bound makes it easy to check Stanton's conjecture.

For $8 \le t \le 13$, as we mentioned above, we can get sharper estimates than Lemma 6.1 by setting $N = \sqrt{2\pi n}$. For $n \ge t^2$,

$$A(t)\left(n + \frac{t^2-1}{24}\right)^{\frac{t-3}{2}} - E'(t)\lceil\sqrt{2\pi n}\rceil^{\frac{t-1}{2}} \le c_t(n) \le B(t)\left(n + \frac{t^2-1}{24}\right)^{\frac{t-3}{2}} + E'(t)\lceil\sqrt{2\pi n}\rceil^{\frac{t-1}{2}},$$

where the $A(t)$ and $B(t)$ are the constants defined in Lemma 6.1 and $E'(t) := e^{\frac{25}{24}}e^{-2\pi(1+\frac{2}{t})-\frac{\pi}{12}(1-\frac{1}{t^2})}E(t)$.

By using the bounds mentioned above together with MAGMA, we can verify Stanton's conjecture for $t \le 198$.

7. Concluding Remarks

For simplicity, we have considered the problem of bounding the coefficients of $R(p)$. Our work raises two natural questions: (i) Can the bound on $R(p)$ in Theorem 1.1 be sharpened?, and (ii) What bound can one derive on the constant $R(t)$, for a general odd integer t? The two primary difficulties that arise in studying this latter question are the presence of old forms, and the representation theoretic issues involved in determining explicit bounds on $L(1, \mathrm{Ad}^2(g))$.

The bound given in Theorem 1.5 shows that Stanton's conjecture is true provided n is at least of size about t^4. It would be of interest to see if combinatorial methods could prove Stanton's conjecture in the range $0 \le n \le t^4$.

The same method applied in this paper can be applied to give bounds on the Fourier coefficients of any holomorphic modular form (see for example [22] for the case of level 1 cusp forms). Cases in which partition generating functions are holomorphic modular forms, such as t-core partitions, seem to be the exception rather than the rule. However, further connections between partitions and modular forms can be made using the hook length formula due to Nekrasov and Okounkov or its generalizations (see [28], formula (6.12)). To state this formula, let $\mathcal{P}$ be the set of partitions, and for a partition λ of positive integer k, let $|\lambda| = k$, and let $H(\lambda)$ be the set of hook lengths of λ. Then for any $z \in \mathbb{C}$,

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in H(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{n=1}^{\infty}(1 - q^n)^{z-1}.$$

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