RECOUNTING BINOMIAL FIBONACCI IDENTITIES

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In [4], Carlitz demonstrates

$$F_L \sum_{x_1=0}^{n} \sum_{x_2=0}^{n} \cdots \sum_{x_L=0}^{n} \binom{n-x_L}{x_1}\binom{n-x_1}{x_2}\cdots\binom{n-x_{L-1}}{x_L} = F_{(n+1)L}, \tag{1}$$

using sophisticated matrix methods and Binet's formula. Nevertheless, the presence of binomial coefficients suggests that an elementary combinatorial proof should be possible. In this paper, we present such a proof, leading to other Fibonacci identities.

Proof: Recall that for $m \geq 1$, F_m counts the number of ways to tile a length $m-1$ board with squares and dominoes (see [1],[2],[3]). Hence the right side of equation (1) counts the number of tilings of a board with length $(n+1)L-1$.

Before explaining the left side of equation (1), we first demonstrate that any such tiling can be created in a unique way using $n+1$ *supertiles* of length L. Given a tiled board of length $(n+1)L-1$, with *cells* numbered 1 through $(n+1)L-1$, we break the tiling into $n+1$ supertiles $S_1, S_2, \ldots, S_{n+1}$ by cutting the board after cells $L, 2L, 3L, \ldots, nL$. See Figure 1.

Notice that a supertile may begin or end with a *half-domino*. For instance, if a domino covers cells L and $L+1$, then S_1 ends with a half-domino, and S_2 begins with a half-domino. A supertile that begins with a half-domino is called *open* on

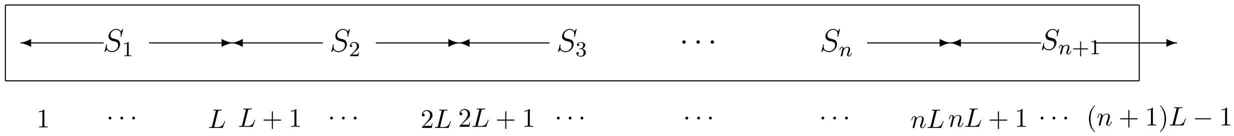

Figure 1. A board of length $(n+1)L-1$ (with a half-domino attached) can be split into $n+1$ supertiles of length L.

the left; otherwise it is *closed* on the left. Likewise a supertile is either open or closed on the right. Naturally, S_1 must be closed on the left.

For convenience, we append a half-domino to the last supertile so that S_{n+1} has length L, like all the other supertiles, and is open on the right. Notice that $S_1, \ldots, S_{n+1}$ must obey the following "following" rule:

For $1 \leq i \leq n$, S_i is open on the right iff S_{i+1} is open on the left.

Given supertiles $S_1, \ldots, S_{n+1}$, we can extract subsequences $O_1, \ldots, O_t$ and $C_1, \ldots, C_{n+1-t}$ for some $0 \leq t \leq n$, where $O_1, \ldots, O_t$ are open on the left, and $C_1, \ldots, C_{n+1-t}$ are closed on the left. By the "following" rule, there are exactly $t+1$ supertiles that are open on the right, necessarily including C_{n+1-t}. Conversely, given $0 \leq t \leq n$ and $O_1, \ldots, O_t$, $C_1, \ldots, C_{n+1-t}$, there is a unique way to reconstruct the sequence $S_1, \ldots, S_{n+1}$ that preserves the relative order of the O's and C's. Specifically, we must have $S_1 = C_1$, and for $1 \leq i \leq n$, if S_i is closed on the right then S_{i+1} is the lowest numbered unused C_j; else S_{i+1} is the lowest numbered unused O_j.

To summarize, $F_{(n+1)L}$ counts the number of ways to create, for all $0 \leq t \leq n$, length L supertiles $O_1, \ldots, O_t$, open on the left, and length L supertiles $C_1, \ldots, C_{n+1-t}$ closed on the left, where C_{n+1-t} is open on the right and exactly t of the other supertiles are open on the right. It remains to show that the left side of equation (1) counts the number of ways that such a collection of supertiles can be constructed.

Given $0 \le t \le n$, we begin by tiling C_{n+1-t}. Since it must end with a half-domino and has $L-1$ *free* cells, it can be tiled F_L ways. Now for any non-negative integers $x_1 \dots, x_{L-1}$, we prove that the remaining supertiles can be created $\binom{n-x_L}{x_1}\binom{n-x_1}{x_2}\cdots\binom{n-x_{L-1}}{x_L}$ ways, where $x_L = t$ and for $1 \le i \le L-1$, exactly x_i of these n supertiles have a domino beginning at its i-th cell.

Since t of the supertiles (excluding C_{n+1-t}) must be open on the right, $x_L = t$ of these n supertiles have half-dominoes beginning at their L-th cells. Now there are $\binom{n-t}{x_1} = \binom{n-x_L}{x_1}$ ways to pick x_1 supertiles among $\{C_1, \dots, C_{n-t}\}$ to begin with a domino. (The remaining $n-t-x_1$ C_j's (other than C_{n+1-t}) begin with a square and all of the O_j's begin with a half-domino.) Next there are $\binom{n-x_1}{x_2}$ ways to pick x_2 supertiles to have a domino covering the second and third cell among those not chosen in the last step to have a domino covering the first and second cell. The unchosen $n-x_1-x_2$ supertiles have a square on the second cell. Continuing in this fashion, there are $\binom{n-x_{i-1}}{x_i}$ ways to pick which supertiles have a domino beginning at the i-th cell for $1 \le i \le L$. Hence $O_1, \dots, O_t$ and $C_1, \dots, C_{n-t}, C_{n+1-t}$ can be created in exactly $F_L\binom{n-x_L}{x_1}\binom{n-x_1}{x_2}\cdots\binom{n-x_{L-1}}{x_L}$ ways. Summing over all values of x_i gives us the left side of equation (1). $\square$

By counting our tilings in a slightly different way, we combinatorially obtain another identity presented in [4]:

$$\sum_{i\geq 0}\sum_{j\geq 0}\binom{i+j}{i}\binom{n-j-i}{j}F_{L-1}^{i}F_{L}^{2j+1}F_{L+1}^{n-2j-i}=F_{(n+1)L}. \tag{2}$$

Proof: $F_{L(n+1)}$ counts the number of ways to create supertiles $S_1,\ldots,S_{n+1}$ subject to the same conditions as before. This time, we classify supertiles in four ways, depending on whether we are closed on the left only, right only, both, or neither. If, for some $0\leq j\leq \lfloor \frac{n}{2}\rfloor$, $S_1,\ldots,S_{n+1}$ contains exactly j supertiles $R_1,\ldots,R_j$ closed on the right only there must be exactly $j+1$ supertiles $L_1,\ldots,L_{j+1}$ closed on the left only. Subsequently, $S_1,\ldots,S_{n+1}$ has subsequence

$$L_1,R_1,L_2,R_2,\ldots,L_j,R_j,L_{j+1}.$$

For example, see Figure 2. Since each of the supertiles above has length L with one half-domino and $L-1$ free cells, this subsequence can be tiled $(F_L)^{2j+1}$ ways.

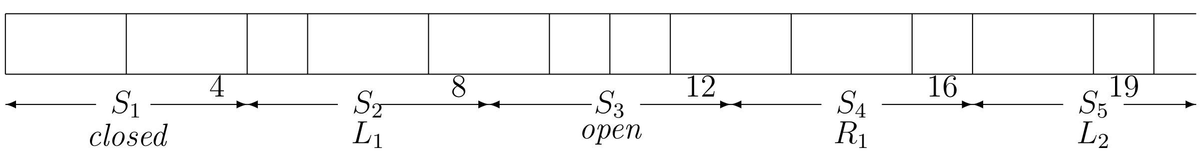

Figure 2. When this length 19 board (plus half-domino) is split after every 4 cells, we create 5 supertiles that are closed, respectively, on both sides, left side, neither side, right side, and left side.

Now suppose $S_1,\ldots,S_{n+1}$ is to have exactly i supertiles that are open at both ends, where $0\leq i\leq n-2j$. We first place these supertiles, like i identical balls to be placed in $j+1$ distinct buckets, between any L_k and R_k or after L_{j+1}. Since there are $\binom{a+b-1}{a}$ ways to place a identical balls into b distinct buckets, there are $\binom{i+j}{i}$ ways to do this. Once placed, since each has $L-2$ free cells, they can be tiled $(F_{L-1})^i$ ways.

Finally, the remaining $n-2j-i$ supertiles that are closed on both ends can be placed into $j+1$ different buckets (before L_1 or between any R_k and L_{k+1}) in $\binom{n-j-i}{n-2j-i} = \binom{n-j-i}{j}$ ways. Once placed, they can be tiled $(F_{L+1})^{n-2j-i}$ ways.

Consequently, the number of legal ways to choose supertiles $S_1, \ldots, S_{n+1}$ with exactly j supertiles closed on the right only and i supertiles open on both ends is $\binom{i+j}{i}\binom{n-j-i}{j} F_{L-1}^{i} F_{L}^{2j+1} F_{L+1}^{n-2j-i}$. (Notice that the second binomial coefficient causes this quantity to be zero whenever $n-j-i<j$, i.e., when $2j+i>n$.) Summing over all i and j proves equation (2). □

Notice that both equations (1) and (2) imply that for all $n \geq 1$, F_L divides F_{nL}. However, a more direct combinatorial proof is possible, without invoking supertiles. Specifically, we have:

$$F_L \sum_{j=1}^{n} (F_{L-1})^{j-1} F_{(n-j)L+1} = F_{nL}. \tag{3}$$

Proof: The right side counts the number of ways to tile a board of length $nL-1$. The left side of (3) counts this by conditioning on the first j, $1 \leq j \leq n$, for which the tiling has a square or domino ending at cell $jL-1$. Such a tiling consists of $j-1$ tilings of length $L-2$, each followed by a domino. This is followed by a tiling of the next $L-1$ cells (cells $(j-1)L+1$ through $jL-1$), followed by a tiling of the remaining $nL-jL$ cells. This can be accomplished $(F_{L-1})^{j-1} F_L F_{(n-j)L+1}$ ways, and the identity follows. □

Acknowledgments

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