Linear momentum of a system of particles

The linear momentum of a system of particles is:

$$ \vec{p}_{\text{tot}} = \sum_i m_i \vec{v}_i $$

A vector sum!

This can be expressed in terms of the center of mass:

$$ \vec{p}_{\text{tot}} = M \vec{v}_{\text{com}} $$

The linear momentum of a system of particles equals their total mass times the velocity of the center of mass!

Conservation of Linear Momentum

In an isolated system, $F_{\text{net}}=0$, which means:

$$ \frac{d \vec{p}}{dt} = \vec{F} $$

$$ \frac{d \vec{p}}{dt} = 0 \Rightarrow \vec{P} = \text{constant} $$

If no external forces acts on a system, the total linear momentum is a constant!

(Of course momentum can be transferred within the system)

Elastic versus Inelastic collisions

Momentum is conserved (unless there is an external force)

Kinetic Energy is conserved only in an elastic collision

Elastic:

$$ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} $$

$$ m_1 \vec{v}_{1i}^2 + m_2 \vec{v}_{2i}^2 = m_1 \vec{v}_{1f}^2 + m_2 \vec{v}_{2f}^2 $$
Rotational motion with **constant** rotational acceleration, $\alpha$

\[
\omega_f = \omega_i + \alpha t \\
\theta_f = \theta_i + \omega_i t + \frac{1}{2} \alpha t^2 \\
\omega_f^2 = \omega_i^2 + 2\alpha (\theta_f - \theta_i)
\]

Linear motion with **constant** linear acceleration, $a$

\[
v_{xf} = v_{xi} + a_x t \\
x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2 \\
v_{xf}^2 = v_{xi}^2 + 2a_i (x_f - x_i)
\]

Rotational motion with **constant** rotational acceleration, $\alpha$

\[
\omega_f = \omega_i + \alpha t \\
\theta_f = \theta_i + \omega_i t + \frac{1}{2} \alpha t^2 \\
\omega_f^2 = \omega_i^2 + 2\alpha (\theta_f - \theta_i)
\]

**Angular and linear quantities**

<table>
<thead>
<tr>
<th>Linear motion</th>
<th>Rotational motion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Kinetic Energy:</strong> $K = \frac{1}{2} m v^2$</td>
<td><strong>Kinetic Energy:</strong> $K_x = \frac{1}{2} I \omega^2$</td>
</tr>
<tr>
<td><strong>Force:</strong> $F = ma$</td>
<td><strong>Torque:</strong> $\tau = I\alpha$</td>
</tr>
<tr>
<td><strong>Momentum:</strong> $p = mv$</td>
<td><strong>Angular Momentum:</strong> $L = I\omega$</td>
</tr>
<tr>
<td><strong>Work:</strong> $W = \vec{F} \cdot \vec{s}$</td>
<td><strong>Work:</strong> $W = \tau \cdot \theta$</td>
</tr>
</tbody>
</table>
Torque

A force $F$ is acting at an angle $\phi$ on a lever that is rotating around a pivot point. $r$ is the distance between $F$ and the pivot point.
This force-lever pair results in a torque $\tau$ on the lever

$$\tau = r \cdot F \cdot \sin \phi$$

Torque $\tau$ and angular acceleration $\alpha$.

Particle of mass $m$ rotating in a circle with radius $r$.
Radial force $F_r$ keeps particle on circular path.
Tangential force $F_t$ accelerates particle along tangent.

$$F_t = ma$$

Torque acting on particle is proportional to angular acceleration $\alpha$.

$$\tau = I \alpha$$

Angular momentum of a particle

Definition:

$$\vec{L} = \vec{r} \times \vec{p}$$

$L$: angular momentum
$r$: distance from the origin
$p$: linear momentum of particle

$L$ is perpendicular to $r$ and $p$
$L$ has magnitude $L = r \cdot p \cdot \sin \Phi$
Newton’s Second law: Again

Recall:

\[ \vec{F} = \frac{d\vec{p}}{dt} \]

In Angular Form:

\[ \vec{\tau} = \frac{d\vec{L}}{dt} \]

Conservation of angular momentum

The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.

\[ \vec{L} = \text{constant} \]

If the system undergoes an internal “rearrangement”, then

\[ \vec{L}_i = \vec{L}_f = \text{constant} \]

If the object is rotating about a fixed axis (say z-axis), then:

\[ I_i\omega_i = I_f\omega_f = \text{constant} \]

Newton’s Law of Universal Gravitation

Every particle in the Universe attracts every other particle with a force of:

\[ \vec{F}_{12} = -G \frac{m_1 \cdot m_2}{r^2} \cdot \hat{r} \]

- \( G \) : Gravitational constant \( G = 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \)
- \( m_1, m_2 \) : masses of particles 1 and 2
- \( r \) : distance separating these particles
- \( \hat{r} \) : unit vector in \( r \) direction
Energy in Orbits

\[ E = K + U = \frac{mv^2}{2} - \frac{GMm}{r} \]

Suppose that the smaller object is moving in a circular orbit, then

\[ F = ma = m \frac{v^2}{r} \]

The gravitational force is the force constraining the object to move in a circle and so,

\[ F = m \frac{v^2}{r} = \frac{GMm}{r^2} \Rightarrow \frac{mv^2}{r} = \frac{GMm}{r} \]

\[ E = -\frac{GMm}{2r} \]

Kepler’s Second law

Kepler’s Second law:
The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.

This is a consequence of angular momentum and the nature of gravity. The gravitational force is parallel to \( r \), and so there is no torque from gravity on an orbiting planet. Therefore angular momentum is conserved!

\[ \vec{L} = \vec{r} \times \vec{p} = M \vec{r} \times \vec{v} = \text{constant} \]

Ponder the following: does this depend on the inverse-square law?

Kepler’s Third law: Circular Orbits

Kepler’s Third law:
The square of the orbital period of a planet is proportional to the cube of the semimajor axis.

Let us consider this for circular orbits.

\[ \frac{GMm}{r^2} = \frac{mv^2}{r} \]

\[ v = \frac{2\pi r}{T} \]

\[ \frac{GMm}{r^2} = \frac{(2\pi r)^2}{r} \]

\[ T^2 \propto r^3 \]