SOLUTION OF ELEMENTARY PROBLEM

E 2558. Proposed by A. Torchinsky, Cornell University

Suppose that \( \sum a_n \) is a divergent series of positive terms, and let \( s_n = a_1 + \cdots + a_n \) for \( n = 1, 2, \ldots \). For which values of \( p \) does the series \( \sum a_n/s_n^p \) converge?

Solution by Elmer K. Hayashi. We prove a more general theorem from which we deduce that \( \sum a_n/s_n^p \) converges if and only if \( p > 1 \).

Theorem. Let \( f(x) \), for \( x > 0 \), be any nonnegative, continuous, monotonically decreasing, real-valued function. If \( \sum a_n \) is a divergent series of positive terms and if \( s_n = a_1 + \cdots + a_n \) for \( n = 1, 2, \ldots \), then

\[
\sum a_n f(s_n) \text{ converges if } \int_{s_1}^{\infty} f(x) \, dx < \infty,
\]

and

\[
\sum a_n f(s_{n-1}) \text{ diverges if } \int_{s_1}^{\infty} f(x) \, dx = \infty.
\]

Proof: Intuitively we reason that if \( u = s_n \), then \( du \) is analogous to \( s_n - s_{n-1} = a_n \). Hence \( \sum a_n f(s_n) \) probably behaves somewhat like \( f(f(u)) \, du \). Furthermore, if \( F(x) \) is any antiderivative of the continuous function \( f(x) \), then \( \int_{s_1}^{s_n} f(u) \, du = F(b) - F(a) \). Thus a natural series with which to compare \( \sum a_n f(s_n) \) is the telescoping series

\[
\sum_{n=2}^{\infty} \left( F(s_n) - F(s_{n-1}) \right)
\]

since

\[
\sum_{k=2}^{n} \left( F(s_k) - F(s_{k-1}) \right) = F(s_n) - F(s_1) = \int_{s_1}^{s_n} f(x) \, dx.
\]

From equation (2), it is apparent that the series (1) converges if and only if the integral, \( \int_{s_1}^{\infty} f(x) \, dx \), is convergent. Now, by the mean value theorem,

\[
F(s_k) - F(s_{k-1}) = F'(c_k)(s_k - s_{k-1}) = a_k f(c_k)
\]

for some \( c_k \) between \( s_{k-1} \) and \( s_k \). Since \( f \) is monotonically decreasing, we have for \( k = 2, 3, \ldots \),

\[
F(s_k) - F(s_{k-1}) \leq a_k f(s_{k-1})
\]

and

\[
F(s_k) - F(s_{k-1}) \geq a_k f(s_k).
\]

Using the Comparison test, we arrive at the conclusion of the theorem.

If we take \( f(x) = x^{-p}, p \geq 0 \), we conclude that \( \sum a_n/s_n^p \) converges for \( p > 1 \) and \( \sum a_n/s_n^p \) diverges for \( 0 \leq p \leq 1 \). In general, it is not true that if \( \sum a_n f(s_{n-1}) \) is divergent, then \( \sum a_n f(s_n) \) is also divergent. For example, if \( f(x) = \frac{1}{x \log x} \), \( a_1 = s_1 = 1 + e \) and