Discovering Fast Matrix Multiplication Algorithms via Tensor Decomposition

Grey Ballard

Wake Forest University

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Collaborators

This is joint work with...

- Austin Benson - Stanford University
- Christian Ikenmeyer - Max-Planck Institute for Informatics
- Tammy Kolda - Sandia National Laboratories
- JM Landsberg - Texas A&M University
- Kathryn Rouse - Wake Forest University
- Nick Ryder - University of California Berkeley

Fast Matrix Multiplication

Definition

Fast matrix multiplication algorithms require $o(n^3)$ arithmetic operations to multiply $n \times n$ matrices.

Examples

- Strassen [Str69]: $O(n^{\log_2 7}) = O(n^{2.81})$
- Coppersmith-Winograd [CW87]: $O(n^{2.376})$
- Le Gall [LG14]: $O(n^{2.373})$

Strassen's Algorithm

Strassen's algorithm uses 7 multiplies for 2 $\times$ 2 multiplication

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Classical Algorithm

$$\begin{aligned}
M_1 &= A_{11} \cdot B_{11} \\
M_2 &= A_{12} \cdot B_{21} \\
M_3 &= A_{11} \cdot B_{12} \\
M_4 &= A_{12} \cdot B_{22} \\
M_5 &= A_{21} \cdot B_{11} \\
M_6 &= A_{22} \cdot B_{21} \\
M_7 &= A_{21} \cdot B_{12} \\
M_8 &= A_{22} \cdot B_{22} \\
C_{11} &= M_1 + M_2 \\
C_{12} &= M_3 + M_4 \\
C_{21} &= M_5 + M_6 \\
C_{22} &= M_7 + M_8
\end{aligned}$$

Strassen's Algorithm

$$\begin{aligned}
M_1 &= (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
M_2 &= (A_{21} + A_{22}) \cdot B_{11} \\
M_3 &= A_{11} \cdot (B_{12} - B_{22}) \\
M_4 &= A_{22} \cdot (B_{21} - B_{11}) \\
M_5 &= (A_{11} + A_{12}) \cdot B_{22} \\
M_6 &= (A_{21} - A_{11}) \cdot (B_{11} + B_{12}) \\
M_7 &= (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
C_{11} &= M_1 + M_4 - M_5 + M_7 \\
C_{12} &= M_3 + M_5 \\
C_{21} &= M_2 + M_4 \\
C_{22} &= M_1 - M_2 + M_3 + M_6
\end{aligned}$$

Strassen's algorithm uses 7 multiplies for 2×2 multiplication

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

For $n \times n$ matrices, we split into quadrants and use recursion

Strassen's Algorithm

$$\begin{aligned}
M_1 &= (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
M_2 &= (A_{21} + A_{22}) \cdot B_{11} \\
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$$\begin{aligned}
C_{11} &= M_1 + M_4 - M_5 + M_7 \\
C_{12} &= M_3 + M_5 \\
C_{21} &= M_2 + M_4 \\
C_{22} &= M_1 - M_2 + M_3 + M_6
\end{aligned}$$

Flop count recurrence:

$$F(n) = 7 \cdot F(n/2) + O(n^2)$$

$$F(1) = 1$$

$$F(n) = O\left(n^{\log_2 7}\right)$$

$$\log_2 7 \approx 2.81$$

Recursion allows us to focus on base case

2×2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

multiplies	6	7	8
flop count	$O\left(n^{2.58}\right)$	$O\left(n^{2.81}\right)$	$O\left(n^{3}\right)$

Recursion allows us to focus on base case

2×2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

multiplies	$\cancel{6}$	7	8
flop count	$\cancel{O\left(n^{2.58}\right)}$	$O\left(n^{2.81}\right)$	$O\left(n^3\right)$

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$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

multiplies	$\cancel{6}$	7	8
flop count	$\cancel{O\left(n^{2.58}\right)}$	$O\left(n^{2.81}\right)$	$O\left(n^{3}\right)$

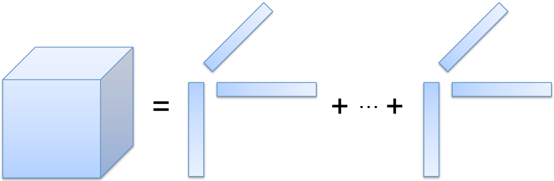
3×3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

multiplies	19	21	23	27
flop count	$O\left(n^{2.68}\right)$	$O\left(n^{2.77}\right)$	$O\left(n^{2.85}\right)$	$O\left(n^{3}\right)$

Searching for a base case algorithm

Finding a base case algorithm corresponds to computing an exact CP decomposition of a particular 3D tensor



$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{u}_r \circ \mathbf{v}_r \circ \mathbf{w}_r$$

*Note the $=$ sign: we're not looking for an approximation

2×2 matrix multiplication as a tensor operation

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \mathbf{C}$$

is equivalent to

$$\mathcal{T} \times_1 \mathbf{a} \times_2 \mathbf{b} = \mathcal{T} \times_1 \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} \times_2 \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{21} \\ c_{12} \\ c_{22} \end{pmatrix} = \mathbf{c}$$

where $\mathcal{T}$ is a $4 \times 4 \times 4$ tensor with the following slices:

$$\mathbf{T}_1 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & \\ & & & \end{pmatrix} \quad \mathbf{T}_2 = \begin{pmatrix} & & & \\ & & & \\ 1 & & & \\ & & 1 & \end{pmatrix} \quad \mathbf{T}_3 = \begin{pmatrix} & 1 & & \\ & & & 1 \\ & & & \\ & & & \end{pmatrix} \quad \mathbf{T}_4 = \begin{pmatrix} & & & \\ & & & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

2×2 matrix multiplication as a tensor operation

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \mathbf{C}$$

is equivalent to

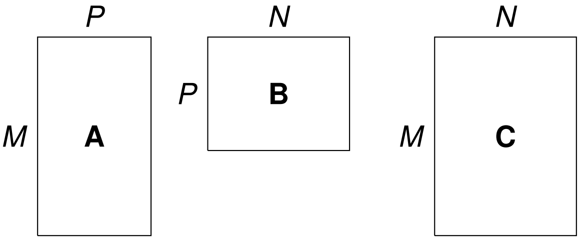
$$\mathcal{T} \times_1 \mathbf{a} \times_2 \mathbf{b} = \mathcal{T} \times_1 \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} \times_2 \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{21} \\ c_{12} \\ c_{22} \end{pmatrix} = \mathbf{c}$$

For example:

$$\mathbf{T}_2 \times_1 \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} \times_2 \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} & & & \\ & & & \\ 1 & & & \\ & & 1 & \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{pmatrix} = a_{21} b_{11} + a_{22} b_{21} = c_{21}$$

General matrix multiplication tensor

$\langle M, P, N\rangle$ means multiplying $M \times P$ by $P \times N$



Matrix multiplication tensor $\mathfrak{T}$ is $MP \times PN \times MN$
Number of 1's in $\mathfrak{T}$ is MPN

$$\langle M, P, N\rangle \equiv \langle N, M, P\rangle \equiv \langle P, N, M\rangle \equiv \langle P, M, N\rangle \equiv \langle M, N, P\rangle \equiv \langle N, P, M\rangle$$

Matrix multiplication using low-rank decomposition

Here's the matrix multiplication as tensor operation again:

$$\mathcal{T} \times_1 \mathbf{a} \times_2 \mathbf{b} = \mathbf{c}$$

Here's our low-rank decomposition:

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{u}_r \circ \mathbf{v}_r \circ \mathbf{w}_r$$

Here's an encoding of our matrix multiplication algorithm:

$$\mathcal{T} \times_1 \mathbf{a} \times_2 \mathbf{b} = \sum_{r=1}^{R} (\mathbf{u}_r \circ \mathbf{v}_r \circ \mathbf{w}_r) \times_1 \mathbf{a} \times_2 \mathbf{b} = \sum_{r=1}^{R} \left[(\mathbf{a}^T \mathbf{u}_r) \cdot (\mathbf{b}^T \mathbf{v}_r) \right] \mathbf{w}_r = \mathbf{c}$$

Connection between factor matrices and algorithm

Strassen's algorithm

$$
\begin{aligned}
M_1 &= (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
M_2 &= (A_{21} + A_{22}) \cdot B_{11} \\
M_3 &= A_{11} \cdot (B_{12} - B_{22}) \\
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\end{aligned}
$$

$$
\begin{aligned}
C_{11} &= M_1 + M_4 - M_5 + M_7 \\
C_{12} &= M_3 + M_5 \\
C_{21} &= M_2 + M_4 \\
C_{22} &= M_1 - M_2 + M_3 + M_6
\end{aligned}
$$

Strassen's factor matrices:

$$
\mathbf{U} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}
$$

$$
\mathbf{V} = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}
$$

$$
\mathbf{W} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}
$$

$\mathbf{U}, \mathbf{V}, \mathbf{W}$ matrices encode the algorithm

Strassen's algorithm

$$
\begin{aligned}
M_1 &= (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
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\end{aligned}
$$

Strassen's factor matrices:

		M_1	M_2	M_3	M_4	M_5	M_6	M_7
U	A_{11}	1		1		1	−1	
	A_{12}					1		1
	A_{21}		1				1	
	A_{22}	1	1		1			−1
V	B_{11}	1	1		−1		1	
	B_{12}			1			1	
	B_{21}				1			1
	B_{22}	1		−1		1		1
W	C_{11}	1			1	−1		1
	C_{21}		1		1			
	C_{12}			1		1		
	C_{22}	1	−1	1			1	

U, **V**, **W** matrices encode the algorithm

How do you solve it?

Problem: Find $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ such that $\mathcal{T} = \sum \mathbf{u}_r \circ \mathbf{v}_r \circ \mathbf{w}_r$

- the problem is NP-complete (for general $\mathcal{T}$)
- many combinatorial formulations of the problem
- efficient numerical methods can compute low-rank approximations
  - typical approach is "alternating least squares" (ALS)
  - pitfall: getting stuck at local minima > 0
  - pitfall: facing ill-conditioned linear least squares problems
  - pitfall: numerical solution is good only to machine precision
- we seek exact, discrete, and sparse solutions

Alternating least squares with regularization

Most successful scheme due to Smirnov [Smi13]

Repeat

1. $$\mathbf{U} = \arg\min_{\mathbf{U}} \left\|\mathbf{T}_{(U)} - \mathbf{U}(\mathbf{W} \odot \mathbf{V})^{\mathsf{T}}\right\|_F^2 + \lambda \left\|\mathbf{U} - \tilde{\mathbf{U}}\right\|_F^2$$

2. $$\mathbf{V} = \arg\min_{\mathbf{V}} \left\|\mathbf{T}_{(V)} - \mathbf{V}(\mathbf{W} \odot \mathbf{U})^{\mathsf{T}}\right\|_F^2 + \lambda \left\|\mathbf{V} - \tilde{\mathbf{V}}\right\|_F^2$$

3. $$\mathbf{W} = \arg\min_{\mathbf{W}} \left\|\mathbf{T}_{(W)} - \mathbf{W}(\mathbf{V} \odot \mathbf{U})^{\mathsf{T}}\right\|_F^2 + \lambda \left\|\mathbf{W} - \tilde{\mathbf{W}}\right\|_F^2$$

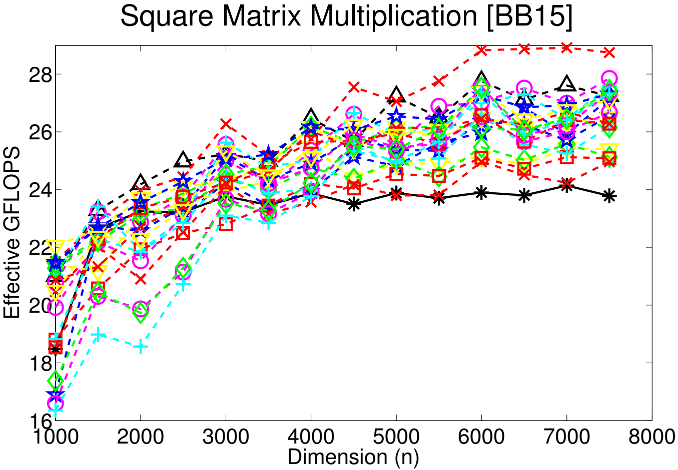
Until convergence

Art of optimization scheme in tinkering with λ, $\tilde{\mathbf{U}}$, $\tilde{\mathbf{V}}$, $\tilde{\mathbf{W}}$ (each iteration)

Discovered algorithms (a subset)

Algorithm base case	Multiplies (fast)	Multiplies (classical)	Speedup per recursive step	ω_0
$\langle 2,2,3\rangle$ [BB15]	11	12	9%	2.89
$\langle 2,2,5\rangle$ [BB15]	18	20	11%	2.89
$\langle 2,2,2\rangle$ [Str69]	7	8	14%	2.81
$\langle 2,2,4\rangle$ [BB15]	14	16	14%	2.85
$\langle 3,3,3\rangle$ [BB15]	23	27	17%	2.85
$\langle 2,3,3\rangle$ [BB15]	15	18	20%	2.81
$\langle 2,3,4\rangle$ [BB15]	20	24	20%	2.83
$\langle 2,4,4\rangle$ [BB15]	26	32	23%	2.82
$\langle 3,3,4\rangle$ [BB15]	29	36	24%	2.82
$\langle 3,4,4\rangle$ [Smi17]	38	48	26%	2.82
$\langle 3,3,6\rangle$ [Smi13]	40	54	35%	2.77
$\langle 2,2,3\rangle^*$ [BCRL79]	10	12	20%	2.78
$\langle 3,3,3\rangle^*$ [Sch81]	21	27	29%	2.77

Are these algorithms faster in practice?



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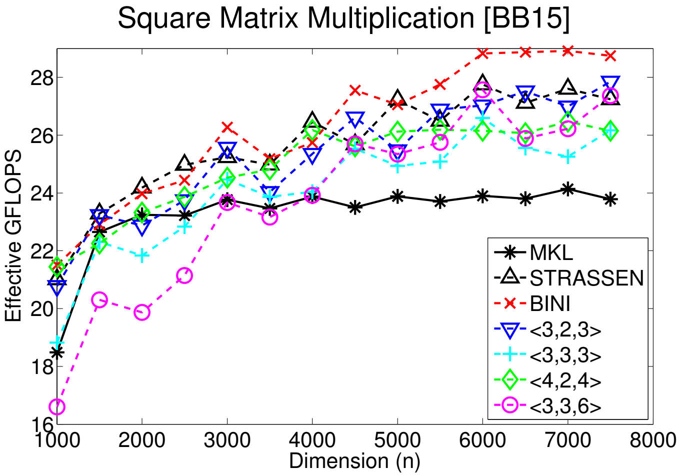
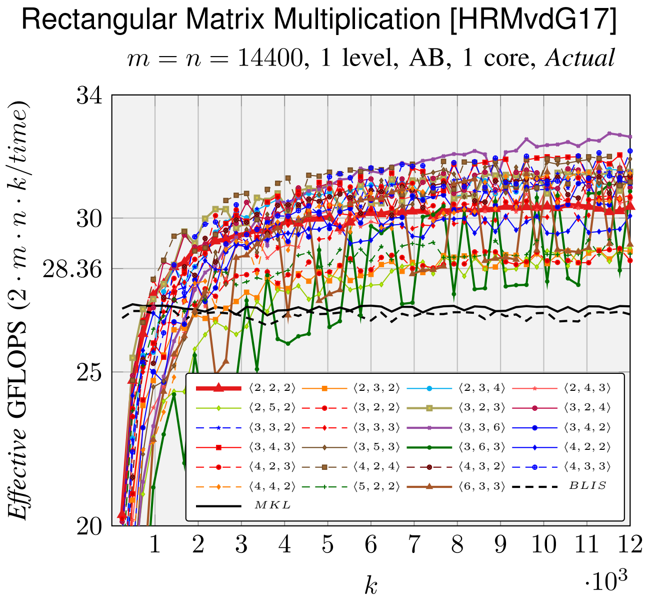


Are these algorithms faster in practice?



How big can we go?

- Current numerical techniques are hitting their limits
  - tensor size grows like N^6 if $M = P = N$
  - number of variables grows faster than $3N^4$ if $M = P = N$
- Nothing new has been found for $\langle 4, 4, 4 \rangle$
  - Strassen's $\langle 2, 2, 2 \rangle$ algorithm can be used twice
- Can we exploit properties particular to matrix multiplication?

Cyclic symmetry of square matrix multiplication

Let $\mathcal{M}$ be the matrix multiplication tensor for $M = P = N$

$\mathcal{M}$ has cyclic symmetry:

$$m_{ijk} = m_{kij} = m_{jki}$$

Cyclic symmetry of square matrix multiplication

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This means if $\mathbf{U}, \mathbf{V}, \mathbf{W}$ is a solution,
then so are $\mathbf{W}, \mathbf{U}, \mathbf{V}$ and $\mathbf{V}, \mathbf{W}, \mathbf{U}$

Cyclic symmetry of square matrix multiplication

Let $\mathcal{M}$ be the matrix multiplication tensor for $M = P = N$

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$$m_{ijk} = m_{kij} = m_{jki}$$

This means if $\mathbf{U}, \mathbf{V}, \mathbf{W}$ is a solution,
then so are $\mathbf{W}, \mathbf{U}, \mathbf{V}$ and $\mathbf{V}, \mathbf{W}, \mathbf{U}$

Is this property reflected in the low-rank decomposition?

$$\sum_r \mathbf{u}_r \circ \mathbf{v}_r \circ \mathbf{w}_r \equiv \sum_r \mathbf{w}_r \circ \mathbf{u}_r \circ \mathbf{v}_r \equiv \sum_r \mathbf{v}_r \circ \mathbf{w}_r \circ \mathbf{u}_r ?$$

Cyclic invariance of Strassen's algorithm

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{W} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Cyclic invariance of Strassen's algorithm

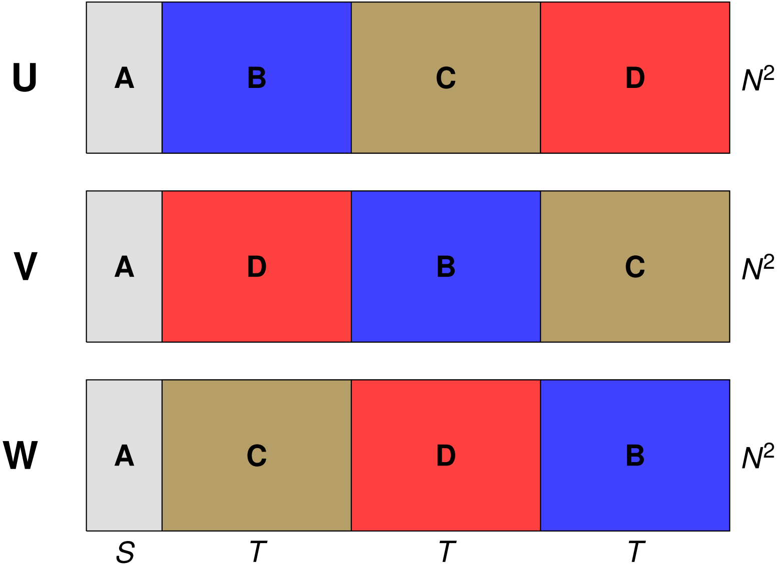
$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{W} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

If you cyclically permute **U**, **V**, **W**,
you get the same rank-one components in a different order

Searching for cyclic-invariant solutions



$$R = S + 3T$$

Cyclic-invariant decompositions

Decomposition with no constraints

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{u}_r \circ \mathbf{v}_r \circ \mathbf{w}_r$$

Decomposition with cyclic-invariant constraint

$$\mathcal{T} = \sum_{s=1}^{S} \mathbf{a}_s \circ \mathbf{a}_s \circ \mathbf{a}_s + \sum_{t=1}^{T} (\mathbf{b}_t \circ \mathbf{c}_t \circ \mathbf{d}_t + \mathbf{d}_t \circ \mathbf{b}_t \circ \mathbf{c}_t + \mathbf{c}_t \circ \mathbf{d}_t \circ \mathbf{b}_t)$$

Number of variables reduced by factor of 3,
but expression is no longer multilinear (not linear in **A**)

New rank-23 cyclic-invariant solutions for $\langle 3,3,3\rangle$

A					B						C						D					
	1										-1	-1		1		-1						
										-1	1	1		-1					-1		-1	
					1			-1			1	1	-1	-1					-1			
									1				1				-1			1		1
1													-1		-1		1	1		-1		
		-1		1								1	-1				1	1		-1		
							1		-1		-1	-1				-1				-1		
			1	-1	-1	1				-1			1		1		-1	-1		1		
		1	1	-1													-1	-1		1		

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \mathbf{D} & \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{A} & \mathbf{C} & \mathbf{D} & \mathbf{B} \end{pmatrix}$$

Rank-23 is the best known exact rank for $\langle 3,3,3\rangle$;
many previous solutions exist but none are cyclic invariant

We computed cyclic-invariant solutions with $S = 2, 5, 11$

What about $\langle 4, 4, 4 \rangle$?

- Performing two steps of Strassen's algorithm yields rank-49 cyclic-invariant solution

- No known exact decomposition of rank < 49
  - cyclic invariant or otherwise

What about $\langle 4, 4, 4 \rangle$?

- Performing two steps of Strassen's algorithm yields rank-49 cyclic-invariant solution
- No known exact decomposition of rank < 49
  - cyclic invariant or otherwise
- No success yet in computing cyclic-invariant solutions
  - but the truth is out there

Summary

- Discovering fast matrix multiplication algorithms corresponds to computing an exact CP decomposition
- Any new algorithm can be implemented efficiently
- Exploiting symmetry of matrix multiplication can reduce the size of the problem, want to tackle $\langle 4, 4, 4 \rangle$
- Still exploring how to use more structure to scale up to larger dimensions

Thank You!

Discovering Fast Matrix Multiplication Algorithms via Tensor Decomposition

Grey Ballard

WAKE FOREST UNIVERSITY

ballard@wfu.edu

More structure...

Transposition symmetry

$$t_{jik} = t'_{ijk}$$

where

$$\mathcal{T}' = \mathcal{T} \times_1 \mathbf{P} \times_2 \mathbf{P} \times_3 \mathbf{P}$$

and **P** is the "vec-permutation" matrix

Transposition symmetry

$$t_{jik} = t'_{ijk}$$

where

$$\mathcal{T}' = \mathcal{T} \times_1 \mathbf{P} \times_2 \mathbf{P} \times_3 \mathbf{P}$$

and **P** is the “vec-permutation” matrix

This is derived from the fact that

$$\mathbf{AB} = \mathbf{C}$$

implies

$$\mathbf{B}^T\mathbf{A}^T = \mathbf{C}^T$$

More structure...

Multiplication invariance

$$\mathfrak{T} = \mathfrak{T} \times_1 \left(\mathbf{Y}^{-T} \otimes \mathbf{X}\right) \times_2 \left(\mathbf{Z}^{-T} \otimes \mathbf{Y}\right) \times_3 \left(\mathbf{Z}^{-T} \otimes \mathbf{X}\right)$$

where $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are nonsingular matrices

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This is derived from the fact that

$$\mathbf{AB} = \mathbf{C}$$

implies

$$\left(\mathbf{XAY}^{-1}\right)\left(\mathbf{YBZ}^{-1}\right) = \left(\mathbf{XCZ}^{-1}\right)$$

Example algorithm: $\langle 4, 2, 4 \rangle$

Partition matrices like this:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \\ A_{41} & A_{42} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

1. Take 26 linear combos of A_{ij}'s according to **U** (68 adds)
2. Take 26 linear combos of B_{ij}'s according to **V** (52 adds)
3. Perform 26 multiplies (recursively)
4. Take linear combos of outputs to form C_{ij}'s acc. to **W** (69 adds)

Classical algorithm performs 32 multiplies yielding a possible speedup of 23% per step

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