Answers to Math Review

- Differentiation

(i) $\frac{d}{dx}\frac{a}{(x-b)} = a\frac{d}{dx}\frac{1}{(x-b)} = -\frac{a}{(x-b)^2}$ //

(ii) $\frac{d}{dx}\ln(x-a) = \frac{1}{x-a}$ //

(iii) $\frac{d}{dx}\ln[(x-a)^2] = \frac{1}{(x-a)^2}\cdot 2(x-a) = \frac{2}{(x-a)}$ //

(iv) $\frac{d}{dx}e^{3x^2} = e^{3x^2}\cdot 3\cdot 2x = 6x\,e^{3x^2}$ //

(v) $\frac{d}{dx}\sin(3x+a) = \cos(3x+a)\cdot 3 = 3\cos(3x+a)$ //

(vi) $\frac{d}{dx}(\sin^2 x) = 2\cdot\sin x\cos x = \sin 2x$ //

(vii) $\frac{d}{dx}a^x$. This can be done as follows:

Let $a = e^b$. Then $b = \ln a$

$a^x = e^{bx}$

Now. $\frac{d}{dx}e^{bx} = b\,e^{bx} = \ln a\cdot e^{bx} = \ln a\cdot a^x$ //

2 Integrals

(i) $\int \frac{dv}{V} = \ln V + c$ ← constant.

(ii) $\int \frac{dv}{V-a}$. Let $x = V - a$. Then $dx = dV$.

In terms of x, the integral is:

$$\int \frac{dx}{x} = \ln x + c = \ln(V-a) + c$$

(iii) $$\int_{V_1}^{V_2} \frac{RT\,dV}{V} = RT \ln V \Big|_{V_1}^{V_2}$$

$$= RT\left[\ln V_2 - \ln V_1\right]$$

$$= RT \ln \frac{V_2}{V_1}$$

(iv) $\int p^2 dp = \frac{p^3}{3} + c$ $\left[\int x^n dx = \frac{x^{n+1}}{n+1}\right]$

(v) $\int e^{ax} dx = \frac{e^{ax}}{a} + c$

(vi) $\int \frac{dT}{T^2} = \frac{T^{-2+1}}{(-2+1)} + c = -\frac{1}{T} + c$

(vii) $\int x e^{ax} dx$

This can be integrated by parts:

$$\int v\,du = uv - \int u\,dv$$

Let $du = e^{ax} dx$ $\quad v = x$. Then $dv = dx$

and $u = \int e^{ax} dx = \frac{e^{ax}}{a}$

$$\therefore \int x e^{ax} dx = \frac{e^{ax}}{a} \cdot x - \int \frac{e^{ax}}{a} dx$$

$$= \frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2} + C$$

$$= \frac{e^{ax}}{a^2}(xa - 1) + C$$ } This form given in tables.

See next page for (viii).

(ix) ~~(viii)~~ $\int \frac{dx}{\sqrt{2x+5}}$. Let $y = 2x+5$.

Then $dy = 2dx$

The integral can now be written as:

$$\int \frac{dx}{\sqrt{y}} = \int \frac{dy/2}{\sqrt{y}} = \frac{1}{2}\int \frac{dy}{\sqrt{y}}$$

$$= \frac{1}{2} \cdot \frac{y^{-\frac{1}{2}+1}}{(-\frac{1}{2}+1)} = \frac{1}{2}\left(\frac{\sqrt{y}}{1/2}\right)$$

$$= \sqrt{y} = \sqrt{2x+5}$$

(IX) ~~$\int \cos ax\, dx = \frac{\sin ax}{a} + c$~~

(VIII) $\int \frac{dx}{x^2-1}$ This can be done using partial fractions.

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{a}{x+1} + \frac{b}{(x-1)}$$

Now a and b should be such that

$$a(x-1) + b(x+1) = 1$$

Easy to see that $a = -\frac{1}{2}$ $b = \frac{1}{2}$

$$\therefore \frac{1}{x^2-1} = \frac{-1/2}{x+1} + \frac{1/2}{x-1}$$

$$\therefore \int \frac{dx}{x^2-1} = -\frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$$

$$= -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + c$$

$$= \ln\frac{(x-1)^{1/2}}{(x+1)^{1/2}} + c = \ln\sqrt{\frac{x-1}{x+1}} + c \; //$$

$$= \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + c \; //$$

Can show this integral also equals : $-\coth^{-1} x$ //

Partial Derivatives

(i) $U(x,y) = x^2 - 2x^3y + y^3$

$$\left(\frac{\partial u}{\partial x}\right) = 2x - 6x^2y$$

$$\left(\frac{\partial u}{\partial y}\right) = 0 - 2x^3 + 3y^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(-2x^3+3y^2) = -6x^2$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(2x-6x^2y) = -6x^2$$

(ii) $U(x,y) = \frac{xy}{c}$

$$\left(\frac{\partial u}{\partial x}\right) = \frac{y}{c} \qquad \left(\frac{\partial u}{\partial y}\right) = \frac{x}{c}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{1}{c} \qquad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{1}{c}$$

(iii) $u(x,y) = 2x^3y^2 + \frac{x}{y}\ln c$

$$\left(\frac{\partial u}{\partial x}\right) = 6x^2y^2 + \frac{\ln c}{y} \qquad \left(\frac{\partial u}{\partial y}\right) = 4x^3y - \frac{x\ln c}{y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 12x^2y - \frac{\ln c}{y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 12x^2y - \frac{\ln c}{y^2}.$$