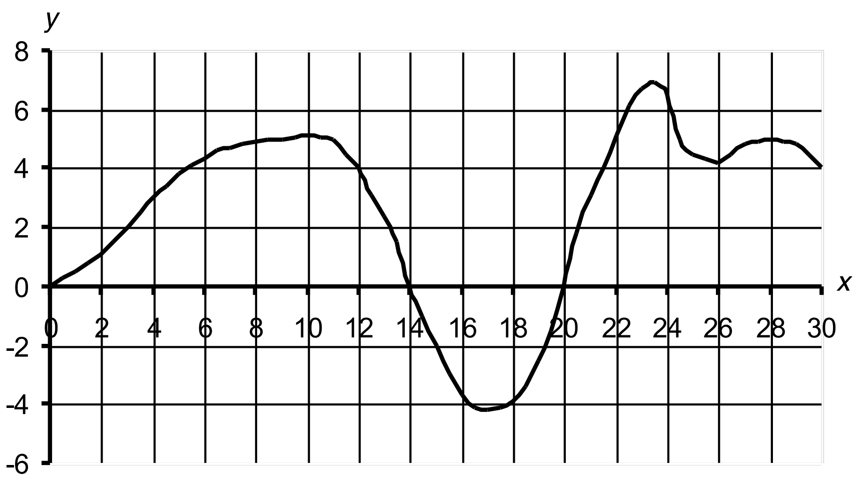
Math 111: Final Exam Review Problems

November 16, 2016

1. Below is the graph of some function $f(x)$:



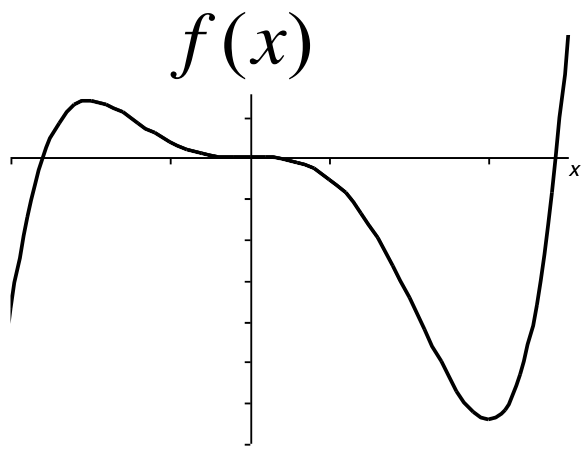
Arrange the following numbers from smallest (1) to largest (5).

- $\lim_{h \to 0} \frac{f(20+h) - f(20)}{h}$.
- The slope of $f(x)$ at $x = 10$.
- $f(16)$.
- The average rate of change of $f(x)$ from $x = 12$ to $x = 24$.
- $\left.\frac{dy}{dx}\right|_{x=28}$.

2. Let $g(2) = 3$ and $g'(2) = 1$. Find $g(-2)$ and $g'(-2)$ assuming

- $g(x)$ is an even function.
- $g(x)$ is an odd function.

3. Use the graph of $f(x)$ given below to sketch a graph of $f'(x)$.



4. Determine if the following statements are true or false.

- If $g(x)$ is continuous at $x = a$, then $g(x)$ must be differentiable at $x = a$.
- If $r''(x)$ is positive then $r'(x)$ must be increasing.
- If $t(x)$ is concave down, then $t'(x)$ must be negative.
- If $h(x)$ has a local maximum or minimum at $x = a$ then $h'(a)$ must be zero.
- $f'(a)$ is the tangent line to $f(x)$ at $x = a$.

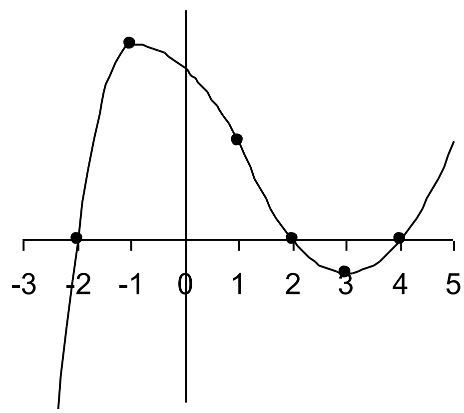
5. Sketch the graph of $f(x)$ that satisfies all of the following conditions:

- $f(x)$ is continuous and differentiable everywhere.
- The only solutions of $f(x) = 0$ are $x = -2, 2$ and 4.
- The only solutions of $f'(x) = 0$ are $x = -1$ and $x = 3$.
- The only solution of $f''(x) = 0$ is $x = 1$.

6. Find $\lim_{h\to 0} \dfrac{(3+h)^{\pi} - 3^{\pi}}{h}$ by recognizing the definition of of $f'(a)$ for some value a.

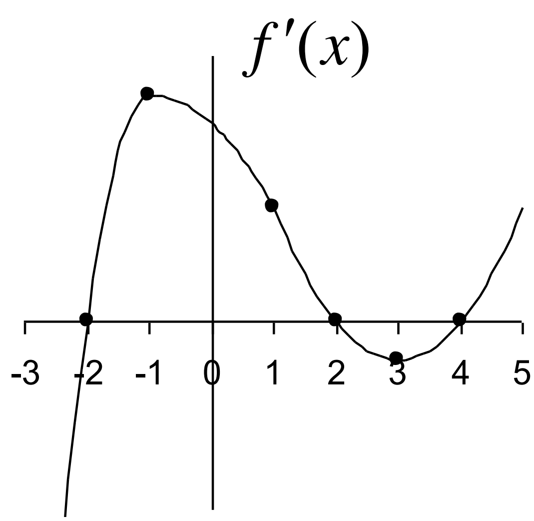
7. Use the graph of $f(x)$ below to find the values of x so that

- $f(x) = 0$
- $f'(x) = 0$
- $f''(x) = 0$



8. Use the graph of $f'(x)$ below to find intervals where

- $f(x)$ is decreasing.
- $f(x)$ is concave down.



9. Let $a > 0$ be a constant. Find $\frac{dy}{dx}$ for each of the following:
   - $y = \sin(a + x)$
   - $y = \frac{a}{a^2 + x^2}$
   - $y = \sec^3(ax)$
   - $y = \frac{1}{x^a} + x^a$
10. Let $f(x)$ be a continuous function with $f(4) = 3$, $f'(4) = 5$, and $f''(4) = -9$.
    - Find the equation of the tangent line to $h(x) = 2f(x) + 7$ at $x = 4$.
    - Is $g(x) = \frac{x^2}{f(x)}$ increasing or decreasing at $x = 4$?
    - Find $k'(2)$ where $k(x) = f(x^2)$.
    - Is $j(x) = (f(x))^2$ concave up or concave down at $x = 4$?
11. If $g(x) = x^3 - 6x^2 - 12x + 5$ and $g'(y) = 3$, find y.
12. Determine where the slope of $y(\theta) = \theta + \cos^2(3\theta)$ will equal 1 on the interval $0 \leq \theta \leq \pi$.
13. Find the indicated derivatives
    - $\frac{dm}{dv}$ for $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$
    - $\frac{d}{dz}\left(|z^2 - 9|\right)$
14. Find A and B so that $f(x)$ is continuous and differentiable on the interval $(-1, 5)$.

$$f(x) = \begin{cases} \sqrt{2x+5} & -1 < x \leq 2 \\ x^2 + A(x-2) + B & 2 < x < 5 \end{cases}$$

15. For what values of k will $f(x) = x^3 - kx^2 + kx + k$ have an inflection point at $x = 5$?
16. Consider the curve defined by $t^3 + x^2 - 5t + 9x = 7$.
    - Find $\frac{dx}{dt}$.
    - For what values of t will the tangent line be horizontal?
    - For what values of x will the tangent line be vertical?

17. A cable is made of of an insulating material in the shape of a long, thin cylinder of radius R. It has an electrical charge distributed evenly throughout it. The electric field, E, at a distance r from the center of the cable is given below. k is a positive constant.

$$E = \begin{cases} kr & r \le R \\ \frac{kR^2}{r} & r > R \end{cases}.$$

- Is E continuous at $r = R$?
- Is E differentiable at $r = R$?
- Sketch E as a function of r.
- Find $\dfrac{dE}{dr}$.

18. Let $f(t) = -\dfrac{1}{t^2} + \dfrac{2}{t^3}$ for $t \ge 2$. Find

- The critical points and determine if they are local maximum or minimum.
- The inflection points.
- The global maximum and minimum on the given interval.

19. Let $f(x) = x^3 - 3a^2x + 2a^4$ with constant $a > 1$. Find

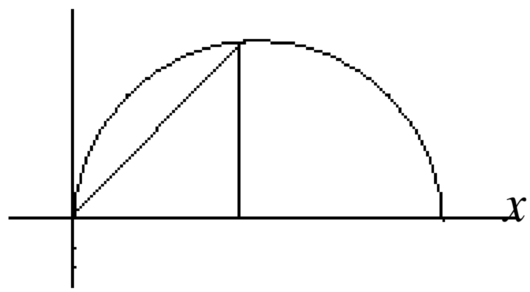
- The coordinates of the local maxima and minima.
- The coordinates of the inflection points.

20. Consider the family of functions $f(t) = \dfrac{Bt}{1 + At^2}$. Find the values of A and B so that $f(t)$ has a critical point at $(4, 1)$.

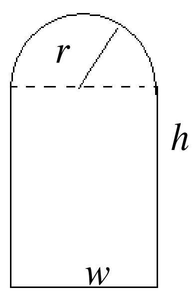
21. A closed rectangular box with a square bottom has a fixed volume V. It must be constructed from three different types of materials. The material used for the four sides costs \$1.28 for square foot; the material for the bottom costs \$3.39 per square foot, and the material for the top costs \$1.61 per square foot. Find the minimum cost for such a box in terms of V.

22. An electric current, I, measured in amps, is give by $I = \cos(\omega t) + \sqrt{3}\sin(\omega t)$ where $\omega \neq 0$ is a constant. Find the maximum and minimum valus of I. For what values of t will these occur if $0 \le t \le 2\pi$.

23. The hypotenuse of the right triangle shown below is the segment from the origin to a point on the graph of $y = \sqrt{4 - (x-2)^2}$. Find the coordinates on the graph that will maximize the area of the right triangle.



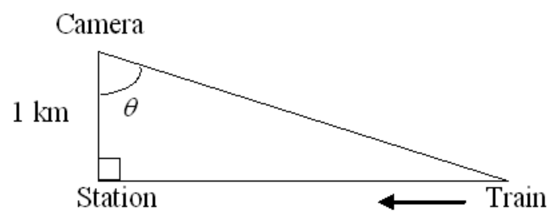
24. A stained glass window will be created as shown below. The cost of the semi-circular region will be \$10.00 per square foot and the cost of the rectangular region will be \$8.00 per square foot. Due to construction constraints the outside perimeter must be 50 feet. Find the maximum total cost of the window? What are the dimensions of the window?



25. For $b > 0$, the line $b(b^2+1)y = b - x$ forms a triangle in the first quadrant with the x-axis and the y-axis.

- Find the value of b so that area of the triangle is exactly $1/5$.
- Find the value of b so that maximizes the area of the triangle.

26. A camera is focused on a train as the train moves along a track towards a station as shown at the right. The train travels at a constant speed of 10 km/hr. How fast is the camera rotating (in radians/min) when the train is 2 km from the camera?



27. Sand is poured into a pile from above. It forms a right circular cone with a base radius that is always 3 times the height of the cone. If the sand is poured at a rate of $15 ft^3$ per minute, how fast is the height of the pile growing when the pile is 12 feet high?

28. A voltage, V, measured in volts, applied to a resistor of R ohms produces an electrical current of I amps where $V = U \cdot R$. As the current flows, the resistor heats up and its resistance fails. If 100 volts is applied to a resistor of 1000 ohms, the current is initially 0.1 amps but increases by 0.001 amps per minute. At what rate is the resistance changing if the voltage remains constant?

29. The rate of change of a population depends on the current population, P, and is given by

$$\frac{dP}{dt} = kP(L - P)$$

for some positive constant k and L.

- For what nonnegative values of P is the population increasing in time? Decreasing? For what values of P does the population remain constant?
- Find $\frac{d^2P}{dt^2}$ as a function of P. For what values of P will $\frac{d^2P}{dt^2} = 0$?

30. The mass of a circular oil slick of radius r is $M = K(r + 1 - \sqrt{r + 1})$, where K is a positive constant. What is the relationship between the rate of change of the radius with respect to time and the rate of change of the mass with respect to time?

31. A function $f(t)$ satisfying $f'(t) > 0$ has values given in the table below.

- Find upper and lower estimate for $\int_1^{1.8} f(t)\,dt$ using 4 rectangles.
- Find $\int_{1.2}^{1.6} f'(t)dt$.

t	1.0	1.2	1.4	1.6	1.8
$f(t)$	0.1	0.2	0.4	0.7	1.1

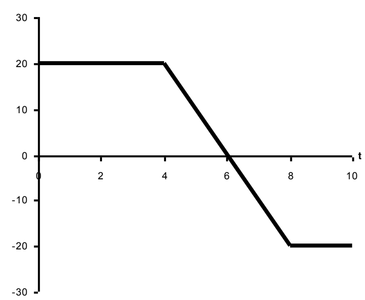
32. A function $g(t)$ is positive and decreasing everywhere. Arrange the following numbers from smallest (1) to largest (3)

- $\sum_{k=1}^{10} g(t_k)\Delta t$
- $\sum_{k=0}^{9} g(t_k)\Delta t$
- $\lim_{n\to\infty} \sum_{k=1}^{n} g(t_k)\Delta t$

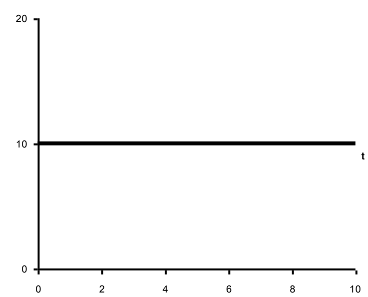
33. Several objects are moving in a straight line from time $t = 0$ to time $t = 10$ seconds. The following are graphs of the **velocities** of these objects (in cm/sec).

- Which object is farthest from the original position at the end of 10 seconds?
- Which object is closest to the original position at the end of 10 seconds?
- Which object has traveled the greatest total distance during these 10 seconds?
- Which object has traveled the least total distance during these 10 seconds?

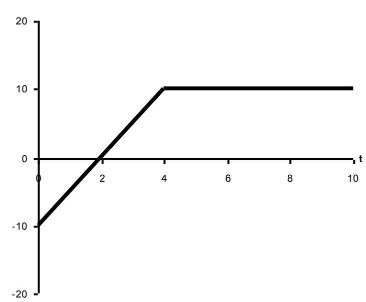
Velocity of Object A



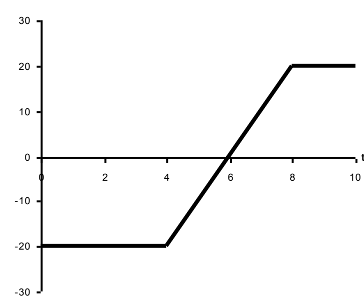
Velocity of Object B



Velocity of Object C



Velocity of Object D



34. Let b be a positive constant. Evaluate the following.

- $\int (bx^2 + 1)\, dx$
- $\int \frac{b + x^2}{x}\, dx$
- $\int \frac{x}{b + x^2}\, dx$

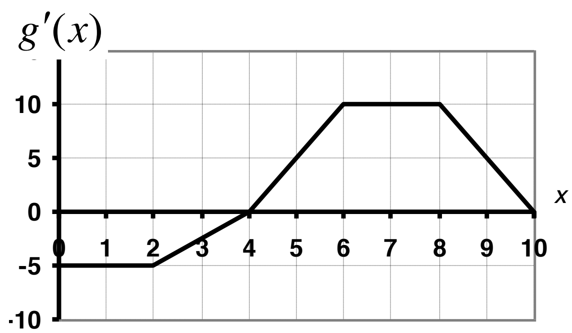
35. Find the exact area of the regions. Include a sketch of the regions.

- The region bounded between $y = x(4 - x)$ and the x-axis.
- The region bounded between $y = x + 2$ and $y = x^2 - 3x + 2$.

36. It is predicted that the population of a particular city will grow at the rate of $p(t) = 3\sqrt{t} + 2$ (measured in hundreds of people per year). How many people will added to the city in the first four years according to this model?

37. At time $t = 0$ water is pumped into a tank at a constant rate of 75 gallons per hour. After 2 hours, the rate decreases until the flow of water is zero according to $r(t) = -3(t - 2)^2 + 75$, gallons per hour. Find the total gallons of water pumped into the tank.

38. Use the graph of $g'(x)$ given below to sketch a graph of $g(x)$ so that $g(0) = 3$.



39. A car going 80 ft/sec brakes to a stop in 5 seconds. Assume the deceleration is constant.

- Find an equation for $v(t)$, the velocity function. Sketch the graph of $v(t)$.
- Find the total distance traveled from the time the brakes were applied until the car came to a stop. Illustrate this quantity on the graph of $v(t)$.
- Find an equation for $s(t)$, the position function. Sketch the graph of $s(t)$.

40. Consider the following function:

$$F(x) = \int_0^x \sqrt{1 + x^4}\, dx.$$

- Find $F(0)$.
- Find $F'(x)$.
- Is $F(x)$ increasing or decreasing for $x \geq 0$?
- Is $F(x)$ concave up or concave down for $x \geq 0$?

41. The average value of f from a to b is defined as

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

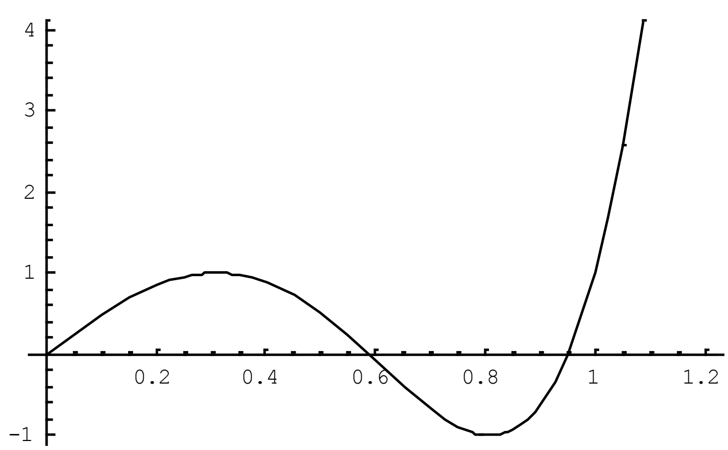
Find the average value of $f(x) = \dfrac{3}{\cos^2(x)}$ over the interval $0 \le x \le \frac{\pi}{4}$.

42. Suppose $g(x) = f(5 + 4\cos(x))$ and $\displaystyle\int_0^\pi g(x)\,dx = \pi$.

- Show that $g(x)$ is an even function.
- Find $\displaystyle\int_{-\pi}^{\pi} g(x)\,dx$.
- Find $\displaystyle\int_0^{\pi/2} g(2x)\,dx$.

43. Use the graph of $f(x)$ below to determine whether the following inequalities are true or false.

- $\displaystyle\int_0^{.1} f(x)\,dx \le \int_0^{.2} f(x)\,dx$
- $\displaystyle\int_0^{.4} f(x)\,dx \le \int_0^{.5} f(x)\,dx$
- $\displaystyle\int_0^{.1} f(x)\,dx \le \int_0^{.1} (f(x))^2\,dx$
- $\displaystyle\int_0^{1} f(x)\,dx \ge 0$
- $\displaystyle\int_0^{1} |f(x)|\,dx \ge 1$

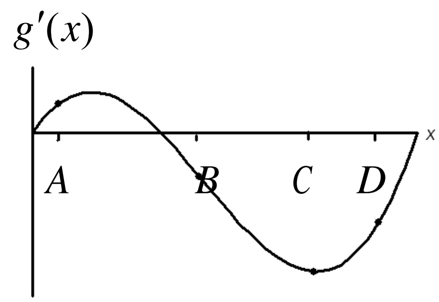


44. Assume all functions below are continuous everywhere.

- If $\int_2^5 6f(x)\,dx = 17$ find $\int_2^5 f(x)\,dx$.
- If $g(x)$ is an odd function and $\int_{-2}^3 g(x)\,dx = 20$, find $\int_2^3 g(x)\,dx$.
- If $h(x)$ is an even function and $\int_{-2}^2 (h(x)-3)\,dx = 25$, find $\int_0^2 h(x)\,dx$.
- If $\int_a^b f(t)\,dx = M$, find $\int_{a+5}^{b+5} f(t-5)\,dt$.

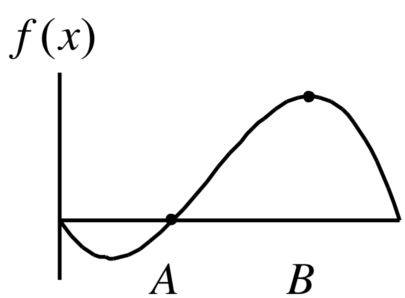
45. Use the graph of $g'(x)$ below to determine which quantity is larger.

- $g(C)$ or $g(D)$
- $g'(B)$ or $g'(C)$
- $g''(A)$ or $g''(B)$

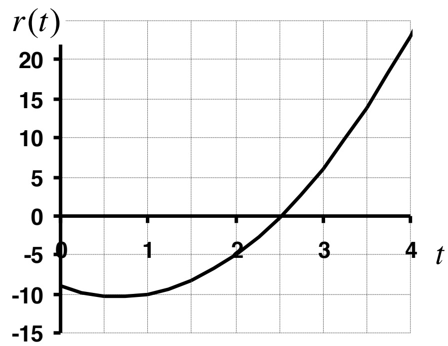


46. Let $g(x) = \int_0^x f(t)\,dt$. In each case explain what graphical feature of $f(x)$ you used to determine the answer.

- What is the sign of $g(B)$?
- What is the sign of $g'\left(\frac{A}{2}\right)$?
- What is the sign of $g''(A)$?



47. The graph below shows the rate, $r(t)$, in hundreds of algae per hour, at which a population of algae is growing as a function of the number of hours. Estimate the total change in the population over the first three hours.

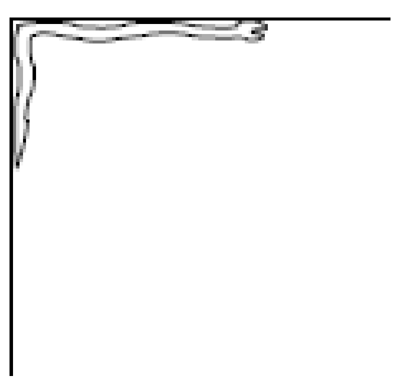


48. Find the value of K so that the total area bounded by $f(x) = K\sqrt{x}$ and the x-axis over the interval $[0, 9]$ is 7.

49. Suppose $\displaystyle\int_1^3 g(t)\,dt = 12$.

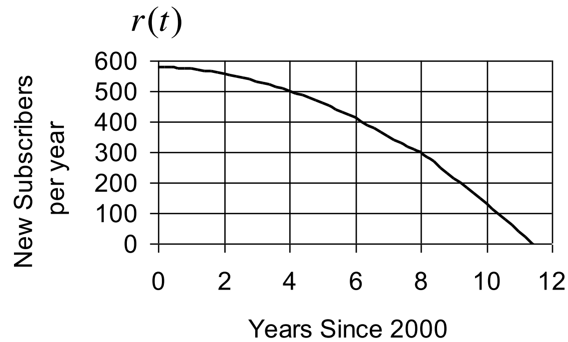
- Find $\displaystyle\int_5^{15} g\left(\frac{x}{5}\right)\,dx$.
- Find $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} g(2-3t)\,dt$.

50. A 6 foot long snake is crawling along the corner of a room. It is moving at a constant 2/3 feet per second while staying tucked snugly against the corner of the room. At the moment that the snake's head is 4 feet from the corner of the room, how fast is the distance between the head and the tail changing?



51. An internet provider modeled their rate of new subscribers to an old internet service package by $r(t)$ as shown below.

- Use the left hand sum rule with 2 rectangles to estimate the total number of new subscribers between 2002 and 2010.
- Rank the following from smallest to largest:
  - $\int_0^5 r(t)\,dt$
  - Left hand sum with 20 rectangles.
  - Right hand sum with 20 rectangles.
- What is the sign of $r'(3)$?
- Estimate $\int_4^8 r'(t)\,dt$.



52. Find the local linearization of $2\pi\sqrt{\frac{x}{g}}$ near $x = 100$.

53. Set up the integrals needed to find the area of the shaded region below.