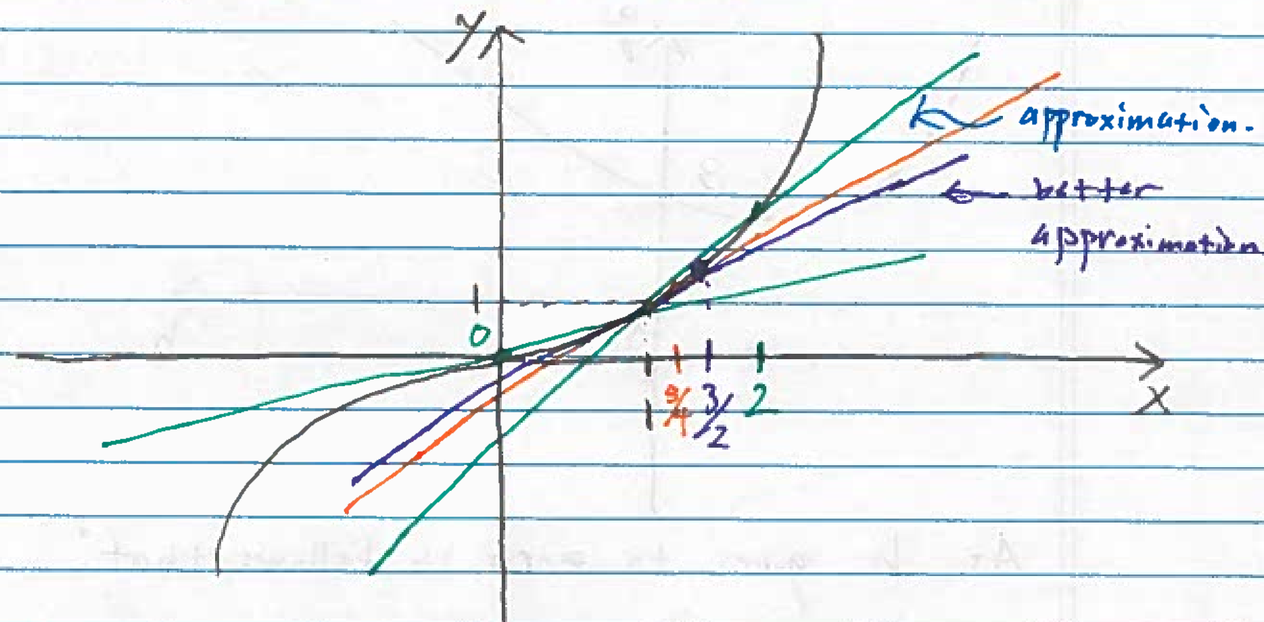
Read 1.4-1.5

Lecture 4: Tangents and Limits

Tangent line: The tangent line passing through a point $(x_0, f(x_0))$ lying on the graph of a function $f(x)$ "touches" $(x_0, f(x_0))$.

example:
Find an equation of the tangent line to the function $f(x) = x^3$ at $(1,1)$.

Solution:
The idea is to generate a sequence of approximations that get better and better.



* $\dfrac{f(1+1) - f(1)}{1} = \dfrac{\text{rise}}{\text{run}} = \dfrac{8-1}{1} = 7$

* $\dfrac{f(1+\frac{1}{2}) - f(1)}{\frac{1}{2}} = \dfrac{\text{rise}}{\text{run}} = \dfrac{2\frac{7}{8} - 1}{\frac{1}{2}} = \dfrac{19}{4} = 4.75$

* $\dfrac{f(1+\frac{1}{4}) - f(1)}{\frac{1}{4}} = \dfrac{\text{rise}}{\text{run}} = \dfrac{1\frac{25}{64} - 1}{\frac{1}{4}} = \dfrac{61}{16} = 3.81$

* $\dfrac{f(1-1) - f(1)}{-1} = \dfrac{\text{rise}}{\text{run}} = 1$

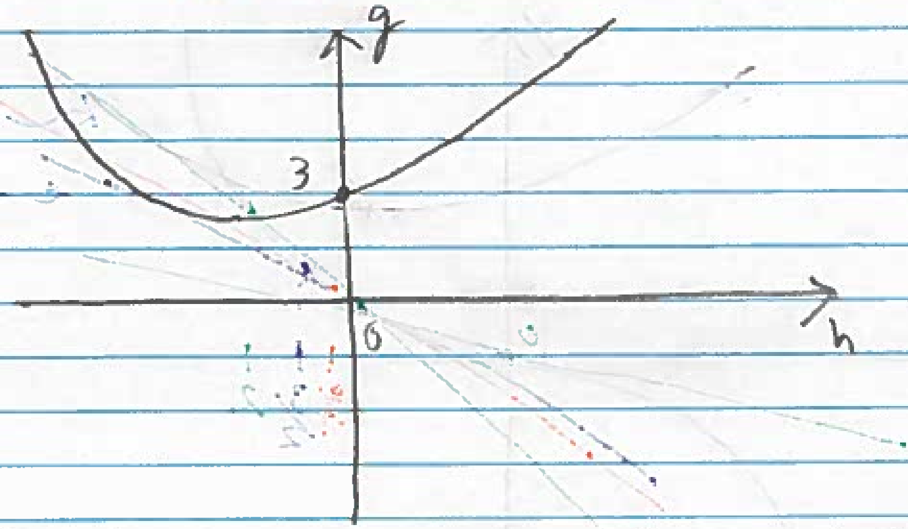
Lets make a new function which gives slope of secant lines:

$$g(h) = \frac{f(1+h) - f(1)}{h} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{(1+h)^3 - 1}{h}$$

$$= \frac{\cancel{1} + 3h + 3h^2 + h^3 - \cancel{1}}{h}$$

$$= 3 + h + h^2$$



As h goes to zero it follows that:

$$\lim_{h \to 0} g(h) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 3 = \text{slope of tangent line.}$$

However, $g(0) = \frac{f(1+0) - f(1)}{0} = \frac{0}{0}$ Undefined!!

$$\Rightarrow y - 1 = 3(x-1)$$

$$\Rightarrow y = 3(x-1) + 1$$ Equation of tangent line.

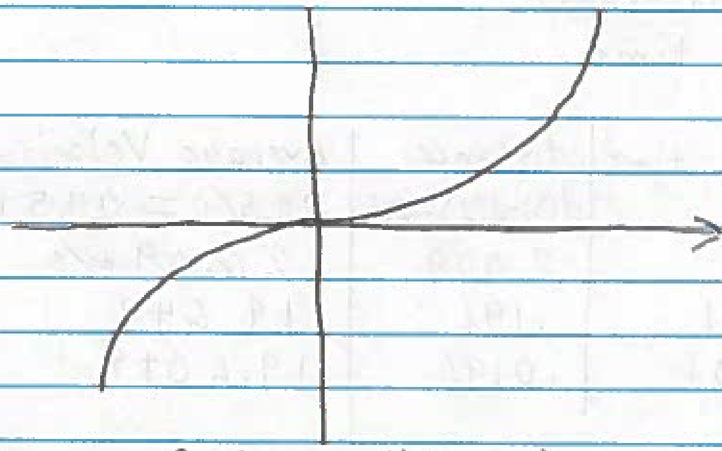
Limit: Suppose $f(x)$ is defined when x is near a number a. Then we write

$$\lim_{x\to a} f(x)=L$$

if $f(x)$ gets arbitrarily close to L by restricting x to be sufficiently close to a.

example:

1. $\lim_{x\to 2} 3x^3 = 3\cdot 8 = 24$



2. $\lim_{x\to 3} \frac{x^2-9}{x-3}$, "dumb" thing $\frac{3^2-9}{3-3} = \frac{0}{0} \to$ undefined. Do more work.

$\lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \lim_{x\to 3} x+3 = 6.$

3. $\lim_{t\to 0} \frac{\sqrt{t^2+4}-3}{t^2}$, "dumb" thing $\frac{\sqrt{9}-3}{0} = \frac{0}{0} \to$ undefined! Do more work.

$\lim_{t\to 0} \frac{\sqrt{t^2+4}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+4}+3} = \lim_{t\to 0} \frac{\cancel{t^2}+\cancel{9}-\cancel{9}}{\cancel{t^2}(\sqrt{t^2+4}+3)} = \lim_{t\to 0} \frac{1}{(\sqrt{t^2+4}+3)} = \frac{1}{6}$

Velocity:

The distance of an object in free fall is given by:

$$d(t) = 4.9t^2 \text{ (meters)}.$$

How fast is the object falling after 2 seconds?

rate x time = distance

$\Rightarrow$ rate $= \frac{\text{distance}}{\text{time}}$

Interval of time	distance	Average Velocity
$2 \le t \le 3$	$d(3)-d(2)=24.5$	$24.5/1 = 24.5$ m/s
$2 \le t \le 2.1$	2.009	20.09 m/s
$2 \le t \le 2.01$	.196	19.649
$2 \le t \le 2.001$	.0196	19.6049

Instantaneous Velocity ≈ 19.6 m/s.

The exact result is:

$$v(2) = \lim_{\Delta t \to 0} \frac{d(2+\Delta t) - d(2)}{\Delta t}$$

↑ velocity at time $t=2$

$$= \lim_{\Delta t \to 0} \frac{4.9(2+\Delta t)^2 - 4.9 \cdot 2^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{4.9(4 + 2\Delta t + \Delta t^2) - 4.9 \cdot 4}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 9.8 \cdot 2 + 4.9\Delta = 19.6 \text{ m/s}.$$

Harder Limits:

1. What is $\lim_{x\to 0} \frac{1-\cos(x)}{x}$?

"Dumb" thing $\frac{1-\cos(0)}{0} = \frac{0}{0}$, undefined, do more work!!

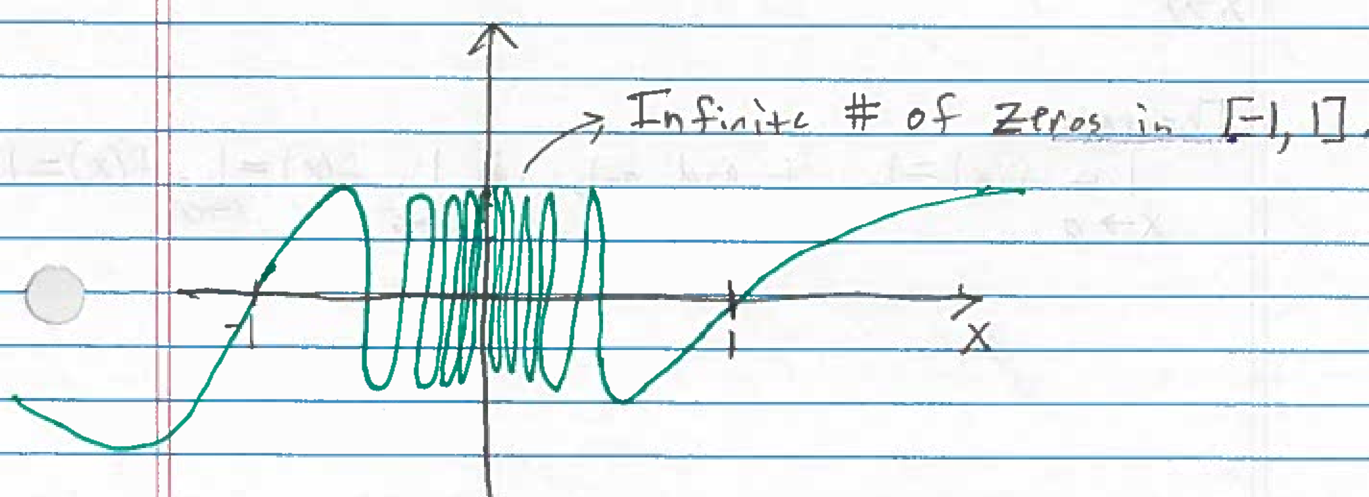
$f(x) = \frac{1-\cos(x)}{x}$	x
.45968	1
-.45968	-1
.0499583	.1
-.0499583	-.1
.00599	.01
-.005	-.01

$\lim_{x\to 0} f(x) = 0$

$\lim_{x\to \pi} f(x) = \frac{2}{\pi}$

$\lim_{x\to 2\pi} f(x) = 0$

2. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ = Does not exist.

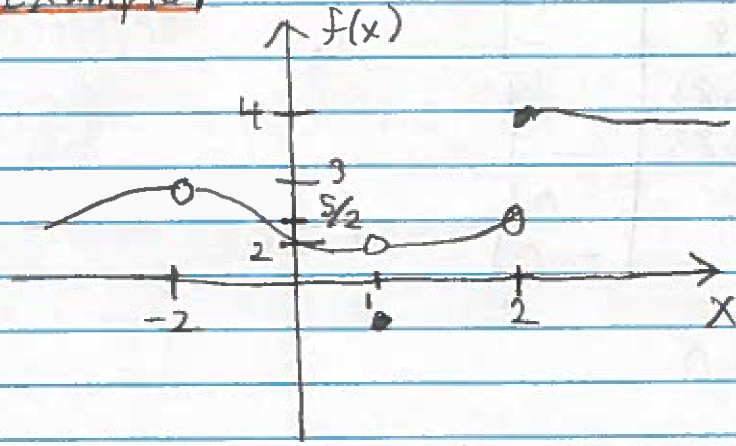


One Sided Limits: We write

$$\lim_{x \to a^-} f(x) = L \quad \text{or} \quad \lim_{x \to a^+} f(x) = L$$

and say the left hand limit (or right hand limit) as x approaches a is equal to L if f(x) gets arbitrarily close to L by taking x sufficiently close to a with x less (greater) than a.

example:



$\lim_{x \to -2} f(x) = 3$, $\lim_{x \to 2} f(x) =$ Does not exist.

$\lim_{x \to 1} f(x) = 2$, $\lim_{x \to 2^-} f(x) = 5/2$

$\lim_{x \to 2^+} f(x) = 4$

Theorem –

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Infinite Limits:

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty$$

means the values of $f(x)$ can be made arbitrarily large (or arbitrarily large negative) by taking x sufficiently close to a.