Lecture #1: Parametric Curves: (11.1-11.2)

Functions $F: \mathbb{R} \mapsto \mathbb{R}^2$

Domain ($\mathbb{R}$), Range ($\mathbb{R}^2$)

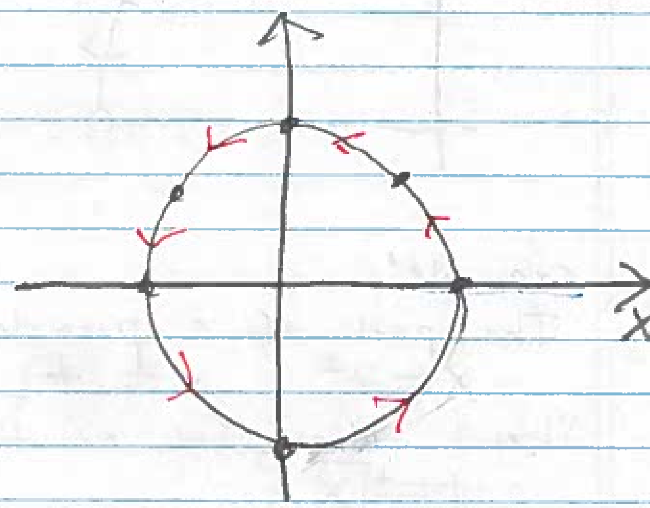
example:

$F(t) = (\cos(t), \sin(t)), \quad t \geq 0$

$x(t) = \cos(t)$

$y(t) = \sin(t)$

t	x	y
0	1	0
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/2$	0	1
$3\pi/4$	$-1/\sqrt{2}$	$1/\sqrt{2}$
π	-1	0
$3\pi/2$	0	-1
2π	1	0

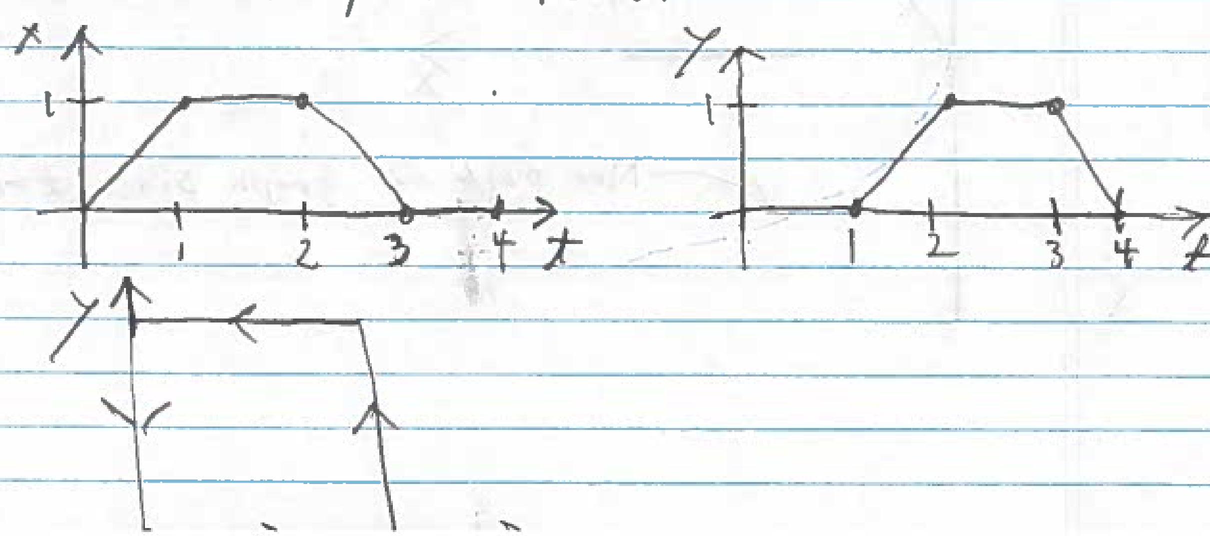


Circle?

$x^2(t) + y^2(t) = \cos^2(t) + \sin^2(t) = 1.$

example:

Describe the motion of the particles whose coordinates are plotted below.



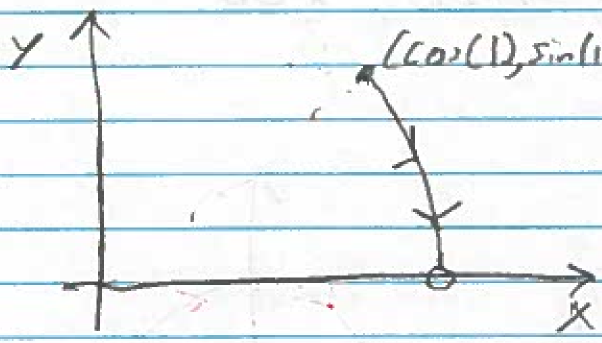
example:

Describe the motion of the particle whose x, y coordinates are given by:

$x(t) = \cos(e^{-t^2})$, $t \geq 0$

$y(t) = \sin(e^{-t^2})$

$x^2 + y^2 = 1 \Rightarrow$ lies on circle.



example:

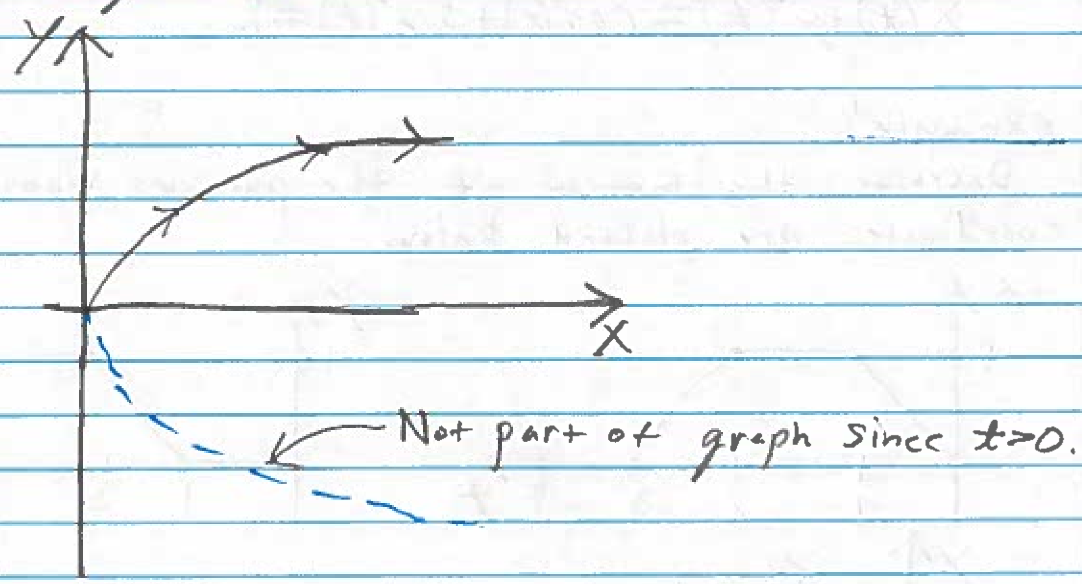
The path of a particle is given by

$x = t^2$, $y = t$, $t \geq 0$

Sketch the path of the particle.

$t = \pm\sqrt{x}$

$\Rightarrow y = \pm\sqrt{x}$



Motion along a line:

An object moves with constant speed along a line through the point (x_0, y_0). Both the x, y-coordinates have constant rate of change.

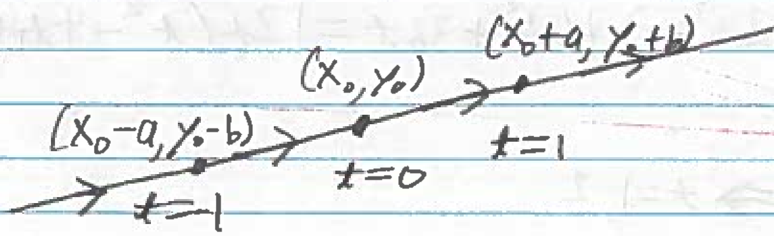
$$a = \frac{dx}{dt}, \quad \frac{dy}{dt} = b.$$

F.T.C.:

$$x(t) - x(0) = \int_0^t \frac{dx}{ds}\,ds, \quad y(t) - y(0) = \int_0^t \frac{dy}{ds}\,ds$$

$$\Rightarrow x(t) - x_0 = at, \quad y(t) - y_0 = bt$$

$$\Rightarrow \left.\begin{aligned} x(t) &= at + x_0 \\ y(t) &= bt + y_0 \end{aligned}\right\} \text{parametric equations for a line.}$$

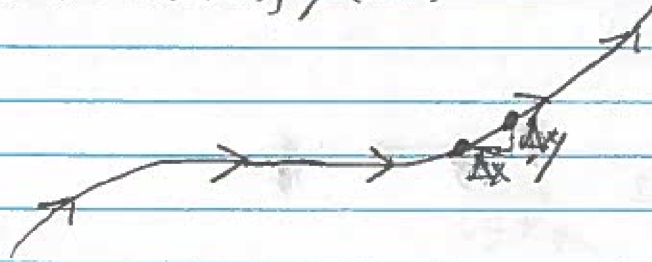


Slope is given by:

$$m = \frac{\text{rise}}{\text{run}} = \frac{b}{a}.$$

Speed and Velocity:

$$F(t) = (x(t), y(t))$$



$$\text{speed} = \frac{\text{distance}}{\text{time}} \approx \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta t} \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$$\Rightarrow v = \lim_{\Delta t \to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

example:

A particle moves in the xy-plane with

$$x = 2t^3 - 9t^2 + 12t,$$
$$y = 3t^4 - 16t^3 + 18t^2$$

where t is time.

a.) At what times is the particle stopped?

b.) At what times is the particle moving parallel to the x-axis?

$$\frac{dx}{dt} = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2)$$

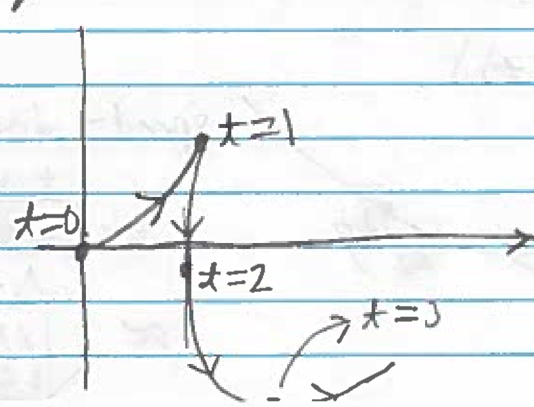
$$\frac{dy}{dt} = 12t^3 - 3 \cdot 16t^2 + 36t = 12t(t^2 - 4t + 3)$$

$$\frac{dx}{dt} = 0 \Rightarrow t = 1, 2$$

$$\frac{dy}{dt} = 0 \Rightarrow t = 0, 1, 3.$$

Stops moving at $t=1$.

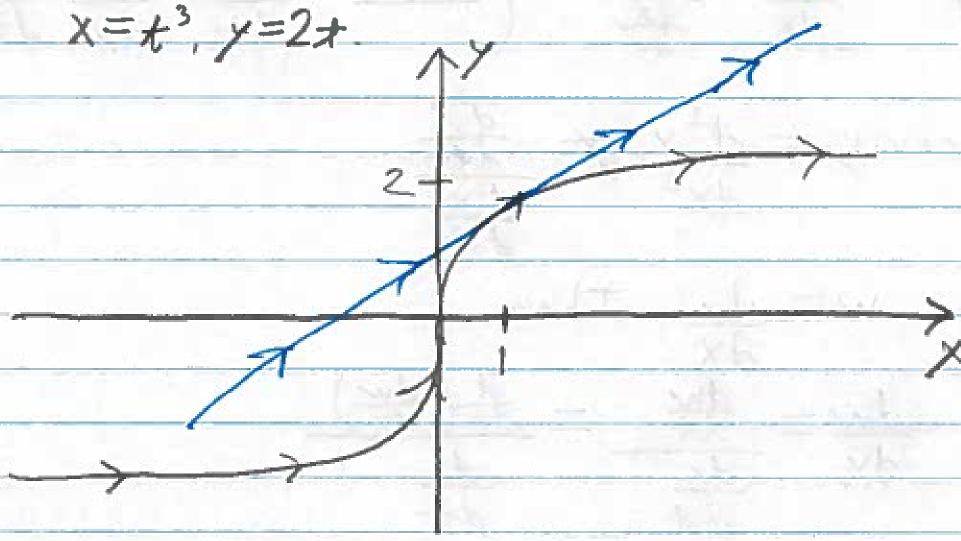
Moving parallel to x-axis $t = 0, 1, 3$.



Tangent Lines

Find the tangent line at $(1,2)$ to the curve defined by

$x = t^3,\ y = 2t.$



Two methods:

1. $\dfrac{dx}{dt} = 3t^2, \quad \dfrac{dy}{dt} = 2$

$\left.\dfrac{dx}{dt}\right|_{t=1} = 3, \quad \left.\dfrac{dy}{dt}\right|_{1} = 2$

The tangent line is:

$x(t) = 3t + 1$
$y(t) = 2t + 2$

2. $\boxed{\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3}{2}}$

$\Rightarrow y = \dfrac{3}{2}(x-1) + 2.$

Concavity:

Let $x(t)=f(t)$, $y(t)=g(t)$.

$$\text{Slope} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \left(\text{proof: } \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}\right)$$

$$\text{Concavity} = \frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$

Let $w = \frac{dy}{dx}$. Then,

$$\frac{dw}{dx} = \frac{\frac{dw}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dw}{dx}\right)}{\frac{dx}{dt}}.$$

example:

If $x(t)=\cos(t)$, $y(t)=\sin(t)$ find the slope and concavity at $t=\pi/4$.

$$\left.\frac{dy}{dx}\right|_{\frac{1}{\sqrt{2}}} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=\pi/4} = \left.\frac{\cos(t)}{-\sin(t)}\right|_{t=\pi/4} = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{dt}{dx}\cdot\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{dt}{dx}\cdot\frac{d}{dt}\left(\frac{\cos(t)}{-\sin(t)}\right)$$

$$= \frac{\frac{d}{dt}\left(-\cot(t)\right)}{\frac{dx}{dt}}$$

$$= \frac{-1}{\sin^3(t)}.$$

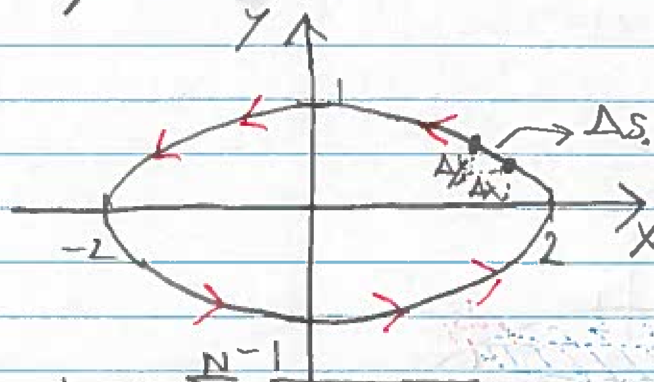
$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=} = \left.\frac{-1}{\sin^3(t)}\right|_{t=\pi/4} = \frac{-1}{(1/\sqrt{2})^3} = -2\sqrt{2}.$$

Arc Length

Find the circumference of the ellipse

$x(t) = 2\cos(t)$, $0 \leq t \leq 2\pi$

$y(t) = \sin(t)$



$$L \approx \sum_{i=1}^{N-1} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$= \sum_{i=1}^{N} \sqrt{\frac{\Delta x_i^2}{\Delta t^2} + \frac{\Delta y_i^2}{\Delta t^2}}\,\Delta t$$

$$\rightarrow L = \lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{\frac{\Delta x_i^2}{\Delta t^2} + \frac{\Delta y_i^2}{\Delta t^2}}\,\Delta t \qquad (\Delta t \to 0)$$

$$= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{4\cos^2(t) + \sin^2(t)}\,dt$$

$$= 9.69$$

The quantity:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt,$$

is known as the arc-length element

Surface Area:

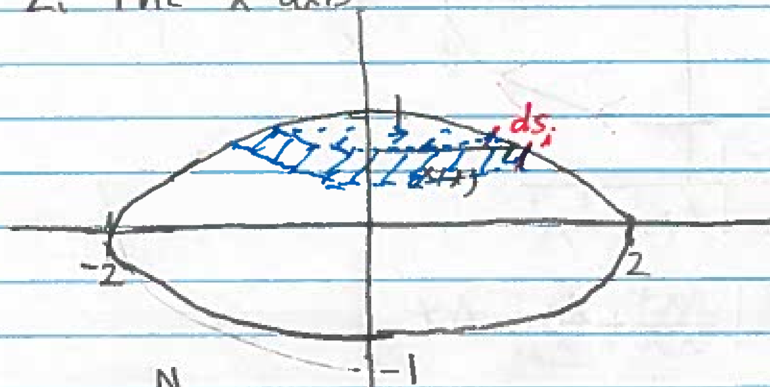
Find the surface area of the ellipse

$x(t) = 2\cos(t)$,

$y(t) = \sin(t)$

revolved around ~~the y-axis~~

1. The y-axis.
2. The x-axis.



$S \approx \sum_{i=1}^{N} 2\pi x(t_i) \cdot \Delta s_i$

$\Rightarrow S = \lim_{N \to \infty} \sum_{i=1}^{N} 2\pi x(t_i) \Delta s_i$

$= \int_{-\pi/2}^{\pi/2} 2\pi \cdot 2\cos(t) \sqrt{4\cos^2(t) + \sin^2(t)}\, dt$

$= 8\int_0^{\pi/2} \cos(t) \sqrt{4\cos^2(t) + \sin^2(t)}\, dt$

$u = \sin(t)$, $\sin^2(t) + \cos^2(t) = 1$

$du = \cos(t)$ $\Rightarrow \cos^2(t) = 1 - u^2$

$\Rightarrow S = 8\int_0^1 \sqrt{4 - 3u^2}\, du$

$= 8 \cdot \left(\frac{1}{2} + \frac{2}{9}\sqrt{3}\pi\right)$

About the y-axis:

$$S = 2\int_0^{\pi/2} \sin(t)\sqrt{4\cos^2(t)+\sin^2(t)}\,dt$$

Let $u = \cos(t)$, $\sin^2(t)+\cos^2(t)=1$

$du = -\sin(t)\,dt$

$$S = -2\int_1^0 \sqrt{4u^2+1-u^2}\,du$$

$$= 2\int_0^1 \sqrt{3u^2+1}\,du$$

$$= \frac{1}{3}\left(6+\sqrt{3}\sinh^{-1}(\sqrt{3})\right)$$