Numerical Linear Algebra
Fall 2017
Exam 1
09/29/17

Name (Print): Key

This exam contains 8 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit**. An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	15	
4	15	
5	10	
6	10	
7	10	
8	0	
Total:	100	

1. (20 points) Let $A \in \mathbb{R}^{3\times 3}$ be given by

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) (5 points) (**Short Answer**) What is the range of A?

$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) (5 points) (**Short Answer**) What is the dimension of the null-space of A?

1.

(c) (5 points) Is A an orthogonal projection?

No, not symmetric

(d) (5 points) (**Short Answer**) What is the smallest singular values of A?

0, the matrix is not full rank.

2. (20 points) Let $Q \in \mathbb{R}^n$ be a unitary matrix.

(a) (5 points) (**Short Answer**) What do you know about Q^{-1}?

$$Q^{-1} = Q^*$$

(b) (5 points) Show for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ that $(Q\vec{x})^T(Q\vec{y}) = \vec{x}^T\vec{y}$.

$$(Q\vec{x})^T(Q\vec{y}) = \vec{x}^T Q^T Q\vec{y}$$
$$= \vec{x}^T\vec{y}$$

(c) (5 points) Show for all $\vec{x} \in \mathbb{R}^n$ that $\|Q\vec{x}\|_2 = \|\vec{x}\|_2$.

$$\|Q\vec{x}\|_2 = \left((Q\vec{x})^T Q\vec{x}\right)^{1/2} = (\vec{x}^T\vec{x})^{1/2} = \|\vec{x}\|_2$$

(d) (5 points) (**Short Answer:**) What is $\|Q\|_2$?

$$\|Q\|_2 = \max_{\|\vec{x}\|=1} \frac{\|Q\vec{x}\|}{\|\vec{x}\|} = \max_{\|\vec{x}\|=1} \frac{\|\vec{x}\|}{\|\vec{x}\|} = 1.$$

3. (15 points) Let $\vec{a}, \vec{b}, \vec{v}, \vec{w} \in \mathbb{R}^n$ be vectors with components:

$$\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{bmatrix}.$$

Let $A \in \mathbb{R}^{n \times n}$ be defined by $A = \vec{a}\vec{v}^T + \vec{b}\vec{w}^T$.

(a) (5 points) For $1 \leq j \leq n$, what is column j of A?

$$A\vec{e}_j = \vec{a} v_j + \vec{b} w_j$$

(b) (5 points) What is the rank of A?

2

(c) (5 points) What is the range of A?

span $\{\vec{a}, \vec{b}\}$.

4. (15 points) Let

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find orthonormal vectors $\vec{q}_1, \vec{q}_2 \in \mathbb{R}^3$ such that $\text{span}\{\vec{q}_1, \vec{q}_2\} = \text{span}\{\vec{a}_1, \vec{a}_2\}$.

$$\vec{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

$$\vec{a}_2 = r_{12}\vec{q}_1 + r_{22}\vec{q}_2$$

$$r_{12} = \langle \vec{q}_1, \vec{a}_2 \rangle = 1/\sqrt{2}$$

$$\Rightarrow r_{22} = \| a_2 - r_{12} q_1 \|$$

$$= \left\| \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\|$$

$$= \left\| \begin{matrix} -1/2 \\ 1/2 \\ 1 \end{matrix} \right\|$$

$$= \frac{\sqrt{6}}{2}$$

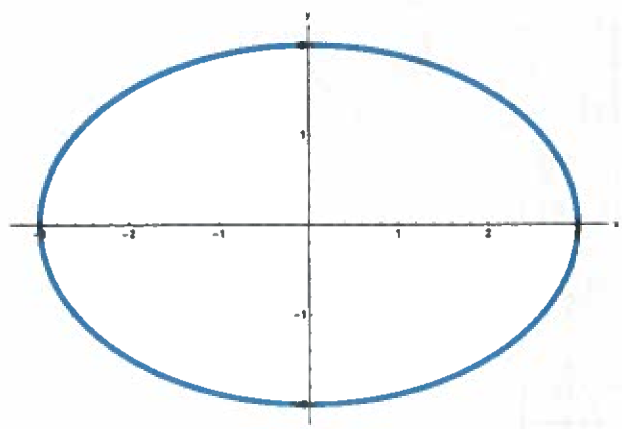
$$\Rightarrow \vec{q}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

5. (10 points) Find the SVD of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

6. (10 points) The figure below is the image of the unit ball under a matrix $A \in \mathbb{R}^2$:



If the SVD of A is given by $A = U\Sigma V^*$. Find possible matrices for U, Σ and V **or** explain why it is not possible to determine these matrices.

$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, V cannot determine.

7. (10 points) **For students who are not MST graduate students.**

Suppose $\vec{x} \in \mathbb{R}^n$. Pick one and only one of the following statements to prove. Circle the one you chose to prove.

- $\|\vec{x}\|_2 \leq \sqrt{n}\|\vec{x}\|_\infty$.
- $\|\vec{x}\|_\infty \leq \|\vec{x}\|_2$.

- $\|\vec{x}\|_2 \leq \left(\sum_{i=1}^{n} \|x\|_\infty^2\right)^{1/2} = \sqrt{n} \cdot \|\vec{x}\|_\infty$
- $\|\vec{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \geq \left(\|x\|_\infty^2\right)^{1/2} = \|\vec{x}\|_\infty$.

8. (10 points) **For MST graduate students only. Other students can do this problem for potential bonus points.**

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and define $A \in \mathbb{R}^{n\times n}$ by $A = \vec{u}\vec{v}^T$.

(a) (5 points) Prove that $\|A\vec{x}\|_2 \leq \|\vec{u}\|_2\|\vec{v}\|_2\|\vec{x}\|_2$.

$$\|A\vec{x}\|_2 = \|\vec{u}\vec{v}^T\vec{x}\|_2$$
$$= |\vec{v}^T\vec{x}| \cdot \|\vec{u}\|_2$$
$$\leq \|\vec{v}\|_2 \cdot \|\vec{x}\|_2 \cdot \|\vec{u}\|_2 .$$

(b) (5 points) Prove that $\|A\|_2 = \|\vec{u}\|_2\|\vec{v}\|_2$.

Let $\vec{x} = \frac{\vec{v}}{\|\vec{v}\|}$. Then,

$$\|A\vec{x}\|_2 = \left\|\vec{u} \cdot \frac{\vec{v}^T \cdot \vec{v}}{\|\vec{v}\|}\right\|$$
$$= \|\vec{u}\| \cdot \|v\|^2 / \|\vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\|$$

Therefore, by part a:

$$\max_{\|\vec{x}\|=1} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \|\vec{u}\| \cdot \|\vec{v}\|.$$