Numerical Linear Algebra
Fall 2017
Exam 2
11/03/17

Name (Print): Key.

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
8	0	
Total:	100	

1. (10 points) Suppose $f, g : \mathbb{R} \mapsto \mathbb{R}$ satisfy

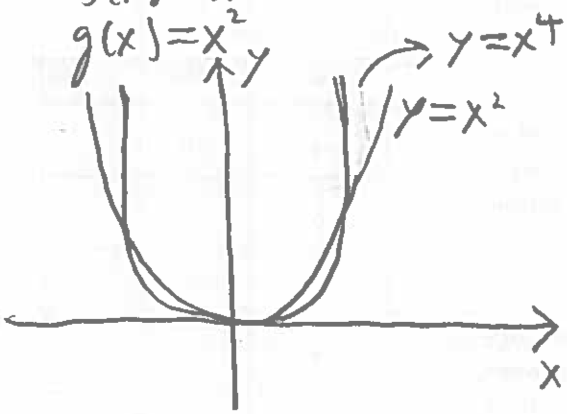
$$\lim_{x\to 0} f(x) = 0,$$
$$\lim_{x\to 0} g(x) = 0,$$
$$\lim_{x\to \infty} f(x) = \infty,$$
$$\lim_{x\to \infty} g(x) = \infty,$$
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 0,$$
$$\lim_{x\to \infty} \frac{f(x)}{g(x)} = 0.$$

(a) (5 points) Is $f(x) = O(g(x))$ as $x \to 0$? Is $g(x) = O(f(x))$ as $x \to 0$? Briefly explain.

Example:

$f(x)=x^4$

$g(x)=x^2$



$f(x)=O(g(x))$ as $x \to 0$.

$g(x) \neq O(f(x))$

(b) (5 points) Is $f(x) = O(g(x))$ as $x \to \infty$? Is $g(x) = O(f(x))$ as $x \to \infty$? Briefly explain.

Example:

$f(x)=x^2$

$g(x)=x^4$

$f(x)=O(g(x))$ as $x \to \infty$.

$g(x) \neq O(f(x))$

2. (15 points) Consider an algorithm for the problem of computing the (full) QR factorization of a matrix $A \in \mathbb{R}^{n \times n}$. The data for this problem is a matrix A, and the solution is a unitary matrix Q and an upper triangular matrix R. Can this algorithm be backward stable? Explain.

This cannot be backwards stable since $\tilde{Q}$ will have roundoff error and thus not be unitary.

3. (15 points) Consider the following Matlab code provided below. Find an asymptotic formula for the number of flops for this algorithm.

```
function [B,C] = Exam2(A)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%   Exam 2
%
%   This algorithm does something to a matrix.
%
%   Inputs:
%   1. A (nxn) matrix.
%
%   Ouputs:
%   1. B (nxn) matrix.
%   2. C (nxn) matrix.
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Extracting information from A and allocating space for matries
[¬,n]=size(A);
B=zeros(n,n);
C=zeros(n,n);

%% Main loop
for j=1:n,
    v=A(:,j);

    for i=1:j-1,
        v=B(:,i);
        aj=A(:,j);
        C(i,j)=v'*aj; → n mult., n additions
        v=v-C(i,j)*v; → n mult, n subtractions
    end

    B(:,j)=v/norm(v);
end
```

$$\text{flops} \sim \sum_{j=1}^{n}\sum_{i=1}^{j-1} 4n$$

$$\sim \sum_{j=1}^{n} 4nj$$

$$\sim 2n^3.$$

4. (15 points) Consider the following problem:

$$f(x) = 2x + 1$$

and the following algorithm for computing this problem on a computer:

$$\tilde{f}(x) = (2 \otimes \mathrm{fl}(x)) \oplus 1.$$

Is this algorithm backwards stable? Is this algorithm stable?

$$\tilde{f}(x) = (2 \otimes fl(x)) \oplus 1$$
$$= (2 \otimes x(1+\varepsilon_1)) \oplus 1$$
$$= (2x(1+\varepsilon_1)(1+\varepsilon_2)) \oplus 1$$
$$= [2x(1+\varepsilon_1)(1+\varepsilon_2)+1](1+\varepsilon_3)$$
$$= 2x(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) + \varepsilon_3 + 1$$

Let
$$\tilde{x}_1 = x(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) + \frac{\varepsilon_3}{2}$$
$$\Longrightarrow \tilde{f}(x) = f(\tilde{x}_1).$$

However,
$$\frac{\|x - \tilde{x}_1\|}{\|x\|} = \frac{\|x\varepsilon_1(1+\varepsilon_2)(1+\varepsilon_3) + \varepsilon_3/2\|}{\|x\|},$$
which diverges as $\|x\| \to 0$.

Let $\tilde{x}_2 = x(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)$. Then,
$$\frac{\|\tilde{f}(x) - f(\tilde{x}_2)\|}{\|f(\tilde{x}_2)\|} = \frac{\|\varepsilon_3\|}{\|2x(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)+\varepsilon_3+1\|} = O(\varepsilon_m).$$

Also, $\frac{\|\tilde{x}_2 - x\|}{\|x\|} = O(\varepsilon_m)$. Therefore, the algorithm is forward stable.

5. (15 points) Suppose $A \in \mathbb{R}^3$ has the following singular value decomposition:

$$A = U \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-3} \end{bmatrix} V^*.$$

and $\vec{x} \in \mathbb{R}^3$ is given by

$$\vec{x} = \begin{bmatrix} 10^{-3} \\ 100 \\ 1 \end{bmatrix}.$$

If $A\vec{x}$ is computed using a backwards stable algorithm, what is the largest *absolute error with respect to the two norm* between the exact value $A\vec{x}$ and the vector computed by the algorithm.

$$K(A) = \frac{10}{10^{-3}} = 10^4.$$

$$\Rightarrow \frac{\|f(x) - \tilde{f}(x)\|}{\|f(x)\|} \leq 10^{-12}$$

Now, $\|f(x)\| = \|Ax\| \leq 10$

$$\Rightarrow \|f(x) - \tilde{f}(x)\| \leq 10^{-11}.$$

6. (15 points) Given the vector $\vec{x} \in \mathbb{R}^3$ defined by

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

construct a unitary matrix Q, i.e. a Householder reflector, such that

$$Q\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

$$\vec{v} = Q\vec{x} - \vec{x} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}.$$

$$\Rightarrow \frac{vv^*}{v^*v} = \frac{1}{12} \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & -2 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 4 & -4 & -4 \\ -4 & 4 & 4 \\ -4 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ -1/3 & 1/3 & 1/3 \\ -1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\Rightarrow Q = I - 2\frac{vv^*}{v^*v}$$

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

7. (15 points) **For students who are not MST graduate students.**

(a) (10 points) Suppose $A \in \mathbb{R}^n$ is nonsingular. Prove that the relative condition number $\kappa(A) = \|A\| \|A^{-1}\|$ *relative to the two norm* satisfies

$$\kappa(A) \geq 1.$$

$$K(A) = \frac{\sigma_1}{\sigma_n} \geq \frac{\sigma_1}{\sigma_1} = 1.$$

(b) (5 points) For a very simple reason the relative condition number for any problem cannot be less than 1. Explain why this must be true by assuming there exists a backwards stable algorithm for the problem.

If $K(A)$ less than 1 this would imply that a backwards stable algorithm would be more accurate than ε_m. This is impossible. In fact the definition of K is constructed precisely so that this is impossible.

8. (15 points) **For MST graduate students only. Other students can do this problem for potential bonus points.**

   (a) (10 points) Suppose $A \in \mathbb{R}^n$ is nonsingular and $p \in \mathbb{R}$ satisfies $1 \le p < \infty$. Prove that the condition number $\kappa(A) = \|A\|\|A^{-1}\|$ relative to the norm $\|\cdot\|_p$ satisfies

$$\kappa(A) \ge 1.$$

$$1 = \|I\| = \|A \cdot A^{-1}\| \le \|A\| \cdot \|A^{-1}\| = \kappa(A)$$

   (b) (5 points) For a very simple reason the relative condition number for any problem cannot be less than 1. Explain why this must be true by assuming there exists a backwards stable algorithm for the problem.

See problem #7.