Numerical Linear Algebra
Fall 2017
Final Exam
12/13/17

Name (Print): Key

This exam contains 16 pages (including this cover page) and 13 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
11	20	
12	20	
13	0	
Total:	240	

Stress Relief Page: If you get stressed out during the exam just look at these cute baby hippos.

1. (20 points) Let $A, B \in \mathbb{R}^{4\times 4}$ and suppose

$$\text{range}(A) = \text{null}(B).$$

Prove that $(AB)^2 = 0$.

$$(AB)^2\vec{e}_i = ABAB\vec{e}_i$$
$$= AB\cdot\vec{x},$$

where $\vec{x} \in \text{range}(A)$. Therefore,

$$(AB)^2\vec{e}_i = 0$$

and thus all of the columns of $(AB)^2$ are 0.

2. (20 points) Prove or give a counterexample: If $A \in \mathbb{R}^{n\times n}$ and $\|Ae_i\| = 1$ for each e_i, then A is unitary.

Counterexample: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

3. (20 points) (a) (10 points) Determine a singular value decomposition of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}.$$

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) (10 points) Determine a singular value decomposition of the matrix

$$B = \begin{bmatrix} 2 & -2 \\ 2 & 4 \\ -1 & 4 \end{bmatrix}.$$

Hint: What do you notice about the columns of B?

Let $\vec{c}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}$

$$\Rightarrow \vec{c}_1^T \vec{c}_2 = 0, \quad \|\vec{c}_1\| = 3, \quad \|c_2\| = 6$$

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

4. (20 points) Recall that for $A \in \mathbb{R}^{n\times n}$ with entries a_{ij} that the Frobenius norm is defined by

$$\|A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

(a) (10 points) Prove using the SVD that

$$\|A\|_F = (\sigma_1^2 + \dots \sigma_n^2)^{\frac{1}{2}},$$

where σ_i are the singular values of A. **Hint:** First prove that $\|A\|_F = \sqrt{\mathrm{Tr}(A^TA)}$.

The i,j entry of A^TA is given by:

$$\sum_{k=1}^{n} a_{ki}a_{kj}$$

$$\Rightarrow \mathrm{Tr}(A^TA) = \sum_{l=1}^{n}\sum_{k=1}^{n} a_{kl}a_{kl} = \|A\|_F^2.$$

Let $A = U\Sigma V^*$ be the SVD of A. Therefore,

$$\|A\|_F = \sqrt{\mathrm{Tr}(V\Sigma^*\Sigma V^*)},$$
$$= \sqrt{\mathrm{Tr}(\Sigma^*\Sigma)},$$

where I have used the fact that $V\Sigma^*\Sigma V^*$ is similar to $\Sigma^*\Sigma$.

(b) (10 points) Prove that

$$\|A\|_F \le \sqrt{\mathrm{rank}(A)}\|A\|_2.$$

$$\|A\|_F^2 = \sigma_1^2 + \dots + \sigma_n^2$$
$$= \sigma_1^2\left(1 + \left(\frac{\sigma_2}{\sigma_1}\right)^2 + \dots + \frac{\sigma_n^2}{\sigma_1^2}\right)$$
$$\le \sigma_1^2(1 + 1 + \dots + 1)$$
$$= \sigma_1^2 \,\mathrm{rank}(A)$$

$$\Rightarrow \|A\|_F \le \sqrt{\mathrm{rank}(A)}\,\|A\|_2$$

5. (20 points) Let

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find orthonormal vectors $\vec{q}_1, \vec{q}_2 \in \mathbb{R}^3$ such that $\text{span}\{\vec{q}_1, \vec{q}_2\} = \text{span}\{\vec{a}_1, \vec{a}_2\}$.

$$\vec{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\vec{a}_2^T \cdot \vec{q}_1 = 1/\sqrt{2}$$

Let $\vec{v} = \vec{a}_2 - (\vec{a}_2^T \cdot \vec{q}_1)\vec{q}_1$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1/\sqrt{2} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \|\vec{v}\| = \sqrt{6}/2$$

$$\Rightarrow \vec{q}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

6. (20 points) Consider the following Matlab code provided below. What is this algorithm computing? Find an asymptotic formula for the number of flops in this algorithm. **Feel free to use a separate piece of paper.**

```
function [y] = FinalExam(A,x)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%   This algorithm outputs a vector y.
%
%   Inputs:
%   1. A (nxn) full rank matrix.
%   2. x (nx1) vector.
%
%   Ouputs:
%   1. y (nx1) vector.
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Extracting information from A and allocating space for matries
[¬,n]=size(A);
B=zeros(n,n);
C=zeros(n,n);

for j=1:n,
    v=A(:,j);

    for i=1:j-1,
        bi=B(:,i);
        aj=A(:,j);
        C(i,j)=bi'*aj; ~2n flops
        v=v-C(i,j)*bi; ~2n flops.
    end

    C(j,j)=norm(v);
    bj=v/C(j,j);
    B(:,j)=bj;
end

for j=n:-1:1,
    y(j)=x(j);

    for k=j+1:n,
        y(j)=y(j)-C(j,k)*y(k);
    end
    y(j)=y(j)/C(j,j);
end

end
```

Solving $Ay = x$ by QR factorization followed by back Substitution.

$$\text{flops} \sim \sum_{j=1}^{n} \sum_{i=1}^{j-1} 4n = \sum_{j=1}^{n} 4n(j-1) \sim \sum_{j=1}^{n} 4nj = 2n^3.$$

7. (20 points) (a) (5 points) What does it mean for an algorithm to be backwards stable?

An algorithm is backwards stable if $\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ satisfying

$$\frac{\|x-\tilde{x}\|}{\|x\|} = O(\varepsilon_{machine}).$$

(b) (15 points) Consider the following problem:

$$f(x) = x^2$$

and the following algorithm for computing this problem on a computer:

$$\tilde{f}(x) = \text{fl}(x) \otimes \text{fl}(x).$$

Is this algorithm backwards stable? Is this algorithm stable?

$$\begin{aligned}\tilde{f}(x) &= (x(1+\varepsilon_1)) \otimes (x(1+\varepsilon_1)) \\ &= x^2(1+\varepsilon_1)^2 \cdot (1+\varepsilon_2) \\ &= f(\tilde{x}),\end{aligned}$$

where,

$$\tilde{x} = x(1+\varepsilon_1)(1+\varepsilon_2)^{1/2}$$

$$\begin{aligned}\Rightarrow \tilde{x} - x &= x[(1+\varepsilon_1)(1+\varepsilon_2)^{1/2} - 1] \\ &= x[(1+\varepsilon_1)(1+\tfrac{1}{2}\varepsilon_2) - 1 + O(\varepsilon_2^2)] \\ &= x[\tfrac{1}{2}\varepsilon_2 + \varepsilon_1 + O(\varepsilon_{machine}^2)]\end{aligned}$$

$$\Rightarrow \frac{\|\tilde{x}-x\|}{\|x\|} = O(\varepsilon_{machine}).$$

Therefore, $\tilde{f}$ is backwards stable and hence stable.

8. (20 points) Let $A, \delta A \in \mathbb{R}^{n\times n}$ be full rank and $b, x, \delta x \in \mathbb{R}^n$. Prove that if

$$Ax = b \text{ and } (A + \delta A)(x + \delta x) = b,$$

then

$$\frac{\|\delta x\|}{\|x + \delta x\|} \leq \kappa(A)\frac{\|\delta A\|}{\|A\|},$$

where $\kappa(A)$ is the condition number of A. **Note:** The specific norm for this problem doesn't matter.

$$Ax + \delta A \cdot x + A\delta x + \delta A \delta x = b$$

$$\delta A x + A\delta x + \delta A \delta x = 0$$

$$\delta A(x + \delta x) = -A \cdot \delta x$$

$$-A^{-1}\delta A(x + \delta x) = \delta x$$

$$\Rightarrow \|A^{-1} \cdot \delta A \cdot (x + \delta x)\| = \|\delta x\|$$

$$\Rightarrow \|\delta x\| \leq \|A^{-1}\| \cdot \|\delta A\| \cdot \|x + \delta x\|$$

$$\Rightarrow \frac{\|\delta x\|}{\|x + \delta x\|} \leq \|A^{-1}\| \cdot \|\delta A\|$$

$$\Rightarrow \frac{\|\delta x\|}{\|x + \delta x\|} \leq \kappa(A) \cdot \frac{\|\delta A\|}{\|A\|}.$$

9. (20 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

C I Let $A \in \mathbb{R}^{n\times n}$. The eigenvectors of A are orthogonal.

C I Let $A \in \mathbb{R}^{n\times n}$. The eigenvalues of A are real.

C I If two matrices are similar then they have the same eigenvalues

C I If two matrices are similar then they have the same eigenvectors.

C I Let $A \in \mathbb{R}^{n\times n}$. If all of the eigenvalues of A are zero then $A = 0$.

C I Let $A \in \mathbb{R}^{n\times n}$. If all of the singular values of A are zero then $A = 0$.

C I Let $A \in \mathbb{R}^{n\times n}$ be symmetric and λ an eigenvalue of A. $|\lambda|$ is a singular value of A.

C I Let $A \in \mathbb{R}^{n\times n}$ with all real, positive and distinct eigenvalues. The largest eigenvalue of A is equal to the largest singular value of A.

C I Let $A \in \mathbb{R}^{n\times n}$. If all of the eigenvectors of A have geometric multiplicity one then the eigenvalues of A are distinct.

C I Let $A \in \mathbb{R}^{n\times n}$ with eigenvalue λ. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors corresponding to the eigenvalue λ, then $\mathbf{v}_1 + \mathbf{v}_2$ is also an eigenvector of A.

10. (20 points) Suppose $A \in \mathbb{R}^{n\times n}$.

(a) (10 points) As illustrated below, show that A can be bi-diagonalized by an appropriate sequence of multiplications on the left and right by Householder reflectors:

$$A = \begin{bmatrix} \times & \times & \times & \cdots & x & x \\ \times & \times & \times & \cdots & x & x \\ \times & \times & \times & \cdots & x & x \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \times & \times & \times & \cdots & x & x \\ \times & \times & \times & \cdots & x & x \end{bmatrix} \longrightarrow \begin{bmatrix} \times & \times & 0 & \cdots & 0 & 0 \\ 0 & \times & \times & \cdots & 0 & 0 \\ 0 & 0 & \times & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & x \\ 0 & 0 & 0 & \cdots & 0 & x \end{bmatrix}$$

$$\begin{bmatrix} X & X & X & \cdots & X \\ 0 & X & X & \cdots & X \\ 0 & X & X & \cdots & X \\ \vdots & \vdots & & & \vdots \\ 0 & X & X & \cdots & X \end{bmatrix} \longrightarrow \begin{bmatrix} X & X & 0 & \cdots & 0 \\ 0 & X & X & \cdots & X \\ 0 & X & X & \cdots & X \\ \vdots & \vdots & \vdots & & \\ 0 & X & X & \cdots & X \end{bmatrix}$$

Apply the same operations to this submatrix. The result follows by induction.

(b) (5 points) Does this transformation preserve the eigenvalues of A? Explain.

No, different unitary operations are used on the left and right.

(c) (5 points) Does this transformation preserve the singular values of A? Explain.

Yes, unitary operations preserve length.

11. (20 points) In class we discussed the QR algorithm and simultaneous iteration. For $A \in \mathbb{R}^{n\times n}$ a symmetric matrix, these algorithms are given below:

Simultaneous Iteration:

$$\begin{aligned} \bar{Q}^{(0)} &= I, \\ Z &= A\bar{Q}^{(k-1)}, \\ Z &= \bar{Q}^{(k)}R^{(k)}, \\ A^{(k)} &= (\bar{Q}^{(k)})^T A\bar{Q}^{(k)}, \\ \bar{R}^{(k)} &= R^{(k)}R^{(k-1)}\dots R^{(1)}. \end{aligned}$$

QR Algorithm:

$$\begin{aligned} A^{(0)} &= A, \\ A^{(k-1)} &= Q^{(k)}R^{(k)}, \\ A^{(k)} &= R^{(k)}Q^{(k)}, \\ \bar{Q}^{(k)} &= Q^{(1)}Q^{(2)}\dots Q^{(k)}, \\ \bar{R}^{(k)} &= R^{(k)}R^{(k-1)}\dots R^{(1)}. \end{aligned}$$

(a) (5 points) Explain what simultaneous iteration is computing.

The eigenvectors.

(b) (5 points) Explain what the QR algorithm is computing.

The eigenvalues.

(c) (10 points) Prove the following theorem:

Theorem: For simultaneous iteration and QR factorization the following properties hold:

$$A^k = \bar{Q}^{(k)}\bar{R}^{(k)}$$
$$A^{(k)} = \left(\bar{Q}^{(k)}\right)^T A\bar{Q}^{(k)}.$$

Proof by induction:

k≥1, Simultaneous

$$\begin{aligned} A^k &= A \cdot A^{k-1} \\ &= A\,\bar{Q}^{(k-1)}\bar{R}^{(k-1)} \\ &= Z\,\bar{R}^{(k-1)} \\ &= \bar{Q}^{(k)} R^{(k)} \bar{R}^{(k-1)} \\ &= \bar{Q}^{(k)}\bar{R}^{(k)}. \end{aligned}$$

k≥1, QR

1. $$\begin{aligned} A^k &= A \cdot A^{k-1} \\ &= A\,\bar{Q}^{(k-1)}\bar{R}^{(k-1)} \\ &= \bar{Q}^{(k-1)} A^{(k-1)} \bar{R}^{(k-1)} \\ &= \bar{Q}^{(k-1)} Q^{(k)} R^{(k)} \bar{R}^{(k-1)} \\ &= \bar{Q}^{(k)}\bar{R}^{(k)}. \end{aligned}$$

2. $$A^{(k-1)} = Q^{(k)}R^{(k)}$$
$$\Rightarrow (Q^{(k)})^T A^{(k-1)} = R^{(k)}$$

Therefore,

$$\begin{aligned} A^{(k)} &= R^{(k)}Q^{(k)} \\ &= (Q^{(k)})^T A^{(k-1)} \cdot Q^{(k)} \\ &= (\bar{Q}^{(k)})^T (\bar{Q}^{k-1})^T \cdot A\,\bar{Q}^{(k-1)} Q^{(k)} \\ &= (\bar{Q}^{(k)})^T A\,\bar{Q}^{(k)} \end{aligned}$$

12. (20 points) **For students who are not MST graduate students.**

Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix with all of its eigenvalues λ_i satisfying $|\lambda_i| < 1$. Prove that

$$\lim_{n\to\infty} \|A^n\|_2 = 0.$$

Since A is symmetric it is unitarily diagonalizable:

$$A = Q \Lambda Q^*$$

$$\Rightarrow \|A^n\|_2 = \|Q \Lambda^n Q^*\|_2$$

$$\leq \|\Lambda^n\|_2$$

$$= \max |\lambda_i|^n$$

Therefore,

$$\lim_{n\to\infty} \|A^n\|_2 = 0.$$

13. (20 points) **For MST graduate students only. Other students can do this problem for potential bonus points.**

Prove that diagonalizable matrices are dense in $\mathbb{R}^{n\times n}$ with respect to the norm $\|\cdot\|_2$. That is, prove that if $A \in \mathbb{R}^{n\times n}$ then for all $\varepsilon > 0$ there exists diagonlizable $A_\varepsilon \in \mathbb{R}^{n\times n}$ such that

$$\|A - A_\varepsilon\|_2 < \varepsilon.$$

Let Q, T be unitary and triangular matrices so that the Schur factorization of A is given by:

$$A = QTQ^*.$$

Let T_ε be a perturbation of T satisfying:

$$\begin{cases} T_{\varepsilon,ij} = T_{ij} & \text{if } i \neq j \\ T_{\varepsilon,ij} = T_{ij} + \varepsilon_i & \text{if } i = j \end{cases},$$

where $|\varepsilon_i| < \varepsilon$ and for all i and j $T_{ii} + \varepsilon_i \neq T_{jj} + \varepsilon_j$. The algebraic multiplicity of $T_{\varepsilon,ij}$ is 1 for all eigenvalues and thus is diagonalizable. Moreover,

$$\|T - T_\varepsilon\| = \max_i |\varepsilon_i| < \varepsilon.$$

$$\Rightarrow \|A - QT_\varepsilon Q^*\|_2 = \|Q(T - T_\varepsilon)Q^*\|_2$$
$$\leq \|T - T_\varepsilon\|_2$$
$$\leq \varepsilon.$$