Homework #1

#1.

Let $\vec{u} \in \mathbb{R}^n$ and $\vec{v} \in \mathbb{R}^n$. Prove that $\vec{u} \cdot \vec{v}^T$ is a rank one matrix.

Solution:

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then,

$$\vec{u} \cdot \vec{v}^T = \left[u_1 \vec{v} \,\middle|\, u_2 \vec{v} \,\middle|\, \cdots \,\middle|\, u_n \vec{v} \right].$$

Therefore, $\operatorname{col}(\vec{u} \cdot \vec{v}^T) = \operatorname{span}\{\vec{v}\}$ and consequently $\vec{u} \cdot \vec{v}^T$ is rank one. ∎

#2.

If $p(x) = c_0 + c_1 x + \dots + c_{n-1} x^{n-1}$, find the matrix representation of the operator defined by:

$$\mathcal{L}(p(x)) = \int_1^x p(x)\,dx.$$

Solution:

$$\mathcal{L}(p(x)) = \int_1^x p(x)\,dx = \int_1^x \left[c_0 + c_1 x + \dots + c_{n-1} x^{n-1}\right] dx$$

$$\Rightarrow \mathcal{L}(p(x)) = c_0 x + \frac{c_1 x^2}{2} + \dots + \frac{c_{n-1} x^{n-1}}{n-1} - c_0 - \frac{c_1}{2} - \dots - \frac{c_{n-1}}{n-1}$$

The map from coefficients to coefficients is then:

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \xmapsto{\mathcal{L}} \begin{bmatrix} -c_0 - \frac{c_1}{2} - \dots - \frac{c_{n-1}}{n-1} \\ c_0 \\ c_1/2 \\ \vdots \\ c_{n-1}/n-1 \end{bmatrix} = c_0 \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1/2 \\ \vdots \\ 0 \end{bmatrix} + \dots + c_{n-1} \begin{bmatrix} -\frac{1}{n-1} \\ 0 \\ 0 \\ \vdots \\ \frac{1}{n-1} \end{bmatrix}$$

Therefore, the matrix representation is given by:

$$A = \begin{bmatrix} -1 & -1/2 & \cdots & -\frac{1}{n-1} \\ 0 & 1/2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/n-1 \end{bmatrix}$$

#3.

Give an example of a matrix $A \in \mathbb{R}^{4\times4}$ such that Range(A) = Null(A). Prove that there does not exist a matrix $A \in \mathbb{R}^{5\times5}$ such that Range(A) = Null(A).

Solution:

1. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ works.

2. If $A \in \mathbb{R}^{5\times5}$ satisfies Range(A) = Null(A) then dim(Range(A)) = dim(Null(A)). Consequently,

$$2\dim(\text{Range}(A)) = 5,$$
$$\Rightarrow \dim(\text{Range}(A)) = 5/2,$$

which is a contradiction.

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Show that if $R \in \mathbb{R}^{m\times m}$ is a nonsingular upper triangular matrix then R^{-1} is also upper triangular.

Solution:

By definition,

$$\vec{e}_j = \sum_{i=1}^{m} R^{-1}_{ij} \vec{r}_i,$$

where $\vec{r}_i$ denotes the columns of R. Therefore,

$$\vec{e}_m = \sum_{i=1}^{m} R^{-1}_{im} \vec{r}_i$$

but $\vec{e}_m \notin \text{Span}\{\vec{r}_1, \ldots, \vec{r}_{m-1}\}$ and thus $R^{-1}_{im} = 0$ for $i \le m-1$. Similarly, $\vec{e}_j \notin \text{Span}\{\vec{r}_1, \ldots, \vec{r}_{j-1}\}$ and thus $R^{-1}_{ij} = 0$ for $i \le j-1$. Thus, R^{-1} is upper triangular.

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Let $f_1, \dots, f_8$ be fonctions defined on $[1,8]$ with the property that for any numbers $d_1, \dots, d_8$ there exists coefficients $c_1, \dots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i.$$

a.) Show that $d_1, \dots, d_8$ determine $c_1, \dots, c_8$ uniquely.

b.) Let $A \in \mathbb{R}^{8\times 8}$ represent the mapping from $d_1, \dots, d_8$ to $c_1, \dots, c_8$. What is the i,j entry of A^{-1}.

Solution:

a.) Let $D_{ij} = f_j(i)$. Then,

$$\sum_{j=1}^{8} c_j f_j(i) = d_i$$

$$\Rightarrow D \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_8 \end{bmatrix}.$$

Consequently, $\operatorname{rank}(D) = 8$ which implies D is full rank, which implies D is invertible. Therefore,

$$\vec{c} = D^{-1}\vec{d}$$

and thus $\vec{d}$ determines $\vec{c}$ uniquely.

b.) From part (a) $D^{-1} = A$ which implies $A^{-1} = D$.